Commentationes Mathematicae Universitatis Carolinae

Jiří Fiala

Representations of generalized measures by integrals

Commentationes Mathematicae Universitatis Carolinae, Vol. 4 (1963), No. 4, 153--156

Persistent URL: http://dml.cz/dmlcz/104947

Commentationes Mathematicae Universitatis Carolinae
4, 4 (1963)

REPRESENTATIONS OF GENERALIZED MEASURES BY INTEGRALS

Jiří FIALA, Praha

This note contains a generalization of classical Riesz's results (cf. 1 Ch.II. §36) of representation of functions by indefinite integrals of functions in L^p . We shall prove a necessary and sufficient condition for a generalized measure on a certain space with measure will be representable by an integral of a function in Orlicz's class L_{Φ} .

Let $\Phi(u)$ be an N-function, i.e. let

$$\Phi(u) = \int_0^{|u|} p(t)dt$$

where $p(t)$ is positive for $t > 0$, right continuous for $t \geqq 0$ and nondecreasing function which satisfies the conditions:

$$p(0) = 0 \ , \quad p(+\infty) = \lim_{t \to \infty} p(t) = +\infty \ .$$

We use these properties of Φ (see[2]):

$\Phi(u)$ is continuous and increases for $u > 0$, and

$$\text{(1)} \qquad \lim_{u \to +\infty} \frac{\Phi(u)}{u} = +\infty \ .$$

Let (X, S, μ) be a space with fully finite continuous measure. Under the continuity we understand the following: there exists a decreasing sequence $E_1 \supseteq E_2 \supseteq \dots \supseteq E_n \supseteq \dots$ of sets with positive measures for which

$$\bigcap_{n=1}^{\infty} E_n = \emptyset \quad \text{and} \quad \lim_{n \to +\infty} \mu(E_n) = 0$$

If μ is continuous in the sense [2] p.76 (for every set E there exists a subset with the measure $\frac{1}{2}\mu(E)$), then it is continuous in the sense above.

We denote $L_\Phi(X, S, \mu)$ the Orlicz's class, i.e. the set of all real functions on X, for which

$$\int_X \Phi(f(x))\, d\mu(x) < +\infty .$$

Next we use the Jensen's integral inequality: If $f \in L_\Phi$, then

$$\Phi\left(\frac{\int_E f d\mu}{\mu(E)}\right) \leq \frac{\int_E \Phi(f) d\mu}{\mu(E)} .$$

The proof in [2] substantially uses the fact that X is a subset of an n-dimensional euclidean space. Generally, let first f be an elementary function

$$(2) \qquad f(x) = \sum_{k=1}^{n} \alpha_k \chi_{E_k}(x), \qquad \bigcup_{k=1}^{n} E_k = X .$$

We have

$$\Phi\left(\frac{\int_E f d\mu}{\mu(E)}\right) = \Phi\left(\frac{\sum_{k=1}^{n} \alpha_k \mu(E_k \cap E)}{\mu(E)}\right) \leq$$

$$\leq \sum_{k=1}^{n} \frac{\Phi(\alpha_k)\mu(E_k \cap E)}{\mu(E)} = \frac{\int_E \Phi(f) d\mu}{\mu(E)} ,$$

by elementary Jensen's inequality. There exists a sequence of elementary functions $\{f_n\}$, for arbitrary f, which converges to $|f|$. By Beppo-Levi's theorem, we can write

$$\Phi\left(\frac{\int_E f d\mu}{\mu(E)}\right) \leq \lim_{n\to+\infty} \Phi\left(\frac{\int_E f_n d\mu}{\mu(E)}\right) \leq \lim_{n\to+\infty} \frac{\int_E \Phi(f_n) d\mu}{\mu(E)} = \frac{\int_E \Phi(f) d\mu}{\mu(E)} .$$

<u>Theorem</u>: Let (X, S, μ) be a space with fully finite continuous measure μ, ν a σ-finite generalized measure on S, and Φ an N-function. A necessary and sufficient condition for

$$(3) \qquad \nu(E) = \int_E f d\mu , \quad f \in L_\Phi ,$$

is that there exists a constant C such that, for arbitrary finite decomposition of X

$$X = E_1 \cup \dots \cup E_n , \quad \mu(E_i) > 0 , \tag{4}$$

the following holds:

$$\sum_{k=1}^{n} \mu(E_k) \, \Phi\left(\frac{\nu(E_k)}{\mu(E_k)}\right) < C . \tag{5}$$

Moreover,

$$\sup \sum_{k=1}^{n} \mu(E_k) \, \Phi\left(\frac{\nu(E_k)}{\mu(E_k)}\right) = \int_X \Phi(f) d\mu ,$$

where the supremum is taken over all decompositions (4).

Proof: First, let (3). Then in virtue of Jensen's integral inequality, we have

$$\sum_{k=1}^{n} \mu(E_k) \, \Phi\left(\frac{\nu(E_k)}{\mu(E_k)}\right) = \sum_{k=1}^{n} \mu(E_k) \, \Phi\left(\frac{\left|\int_{E_k} f d\mu\right|}{\mu(E_k)}\right) \leqslant$$

$$\leqslant \sum_{k=1}^{n} \mu(E_k) \, \Phi\left(\frac{\int_{E_k} |f| d\mu}{\mu(E_k)}\right) \leqslant \sum_{k=1}^{n} \mu(E_k) \frac{\int_{E_k} \Phi(f) d\mu}{\mu(E_k)} =$$

$$= \int_X \Phi(f) d\mu .$$

On the other hand, let (5) be satisfied. Then ν is absolutely continuous with respect to μ : If we have $\mu(E) = 0$ then, by continuity of μ , $F_n = E_n \cup E \downarrow E$, and hence $\mu(F_n) \downarrow 0$. By (5), we have

$$\mu(F_n) \, \Phi\left(\frac{\nu(F_n)}{\mu(F_n)}\right) < C .$$

If $\nu(E) \neq 0$, then, by (1) we have

$$\lim_{n\to\infty} \mu(F_n)\, \Phi\left(\frac{\nu(F_n)}{\mu(F_n)}\right) = \lim_{n\to\infty} \nu(F_n) \frac{\Phi\left(\frac{\nu(F_n)}{\mu(F_n)}\right)}{\frac{\nu(F_n)}{\mu(F_n)}} = +\infty .$$

By Radon-Nikodym´s theorem there exists a function f such that

$$\nu(E) = \int_E f d\mu$$

Suppose that ν is a measure. If $\{f_n\}$ is a sequence of elementary functions (2), $\mu(E_i) > 0$, $f_n \uparrow f$, then

$$(6)\quad C \geqslant \sum_{k=1}^{n} \mu(E_k)\, \Phi\left(\frac{\alpha_k\, \mu(E_k)}{\mu(E_k)}\right) = \int_X \Phi(f_n) d\mu ,$$

and, by Beppo-Levi´s theorem, we conclude that $f \in L_\Phi$. Generally, let $X = A \cup B$ be a Hahn´s decomposition (cf. 3 , §29), ν^+, ν^- the upper and the lower variations of ν ,

$$\nu^+(E) = \int_E f^+ d\mu , \quad \nu^-(E) = \int_E f^- d\mu .$$

Evidently, ν^+(ν^- resp.) satisfies (5) on A (B resp.). The assertion can be obtained by means of the following equality

$$(7)\qquad \int_X \Phi(f) d\mu = \int_A \Phi(f^+)\, d\mu + \int_B \Phi(f^-) d\mu .$$

The last equality can be got from (6), for $f \geqslant 0$, and from (7), for arbitrary f .

B i b l i o g r a p h y:

[1] F. RIESZ et B. SZ. NAGY, Leçons d´analyse fonctionelle, Budapest 1953

[2] М. А. КРАСНОСЕЛЬСКИЙ, Я.Б. РУТИЦКИЙ, Выпуклые функции и пространства Орлича, Москва 1958.

[3] P. R. HALMOS, Measure Theory, New York 1950.