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A GENERALIZATION OF A THEOREM OF R. BAER

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The purpose of this note is to present a theorem that generalizes R. Baer's theorem on the complete decomposability of pure subgroups of a completely decomposable torsion free abelian groups (see [1], theorem 46.6).

Theorem. If a completely decomposable torsion free abelian group G is a direct sum of groups of rank 1 whose types are inversely well-ordered in the natural partial order of types, then every pure subgroup H of G is again completely decomposable.

Proof. The following proof runs on the same lines as that of Baer's theorem in [1] .

Let $G = \sum_{\alpha < \tau} J_\alpha$ be a direct decomposition of G where J_α $(\alpha < \tau)$ are torsion free groups of rank 1 , and τ is an ordinal. According to the assumptions in our theorem it can be supposed that

(1) $\quad$ type $J_\alpha \geqq$ type J_β whenever $\alpha \leqq \beta < \tau$.

Put, for each ordinal β $(0 < \beta \leqq \tau)$,

(2) $\quad G_\beta = \sum_{\alpha < \beta} J_\alpha$ and $H_\beta = H \cap G_\beta$

By (2) we have $G_\beta \subseteq G_\gamma$ and also $H_\beta \subseteq H_\gamma$ for all ordinals β , γ such that $\beta \leqq \gamma \leqq \tau$. Hence it follows for $\beta \leqq \gamma \leqq \tau$

(3) $\quad H_\beta = H_\gamma \cap H_\beta = H_\gamma \cap (H \cap G_\beta) = H_\gamma \cap G_\beta$

so that $H_\beta = H_{\beta+1} \cap G_\beta$ for every β preceding τ . Consequently, the following relation holds for $\beta < \tau$

(4) $H_{\beta+1} / H_\beta = H_{\beta+1} / (H_{\beta+1} \cap G_\beta) \cong \{G_\beta, H_{\beta+1}\}/G_\beta \leq$
$\leq G_{\beta+1} / G_\beta$.

By (2) it is $G_{\beta+1} = G_\beta \dot{+} J_\beta$ and therefore $G_{\beta+1} / G_\beta \cong J_\beta$. From this we obtain by (4)

(5) $$H_{\beta+1} / H_\beta \cong \bar{J}_\beta \leq J_\beta$$

where $\bar{J}_\beta$ can also vanish.

Let $H_{\beta+1} / H_\beta$ be a nonzero group and let $x \in H_{\beta+1} - H_\beta$. If type(x) is the type of x in G , then, by the purity of $H_{\beta+1}$ in G , its type taken in $H_{\beta+1}$ is the same as that taken in G . As an immediate consequence of this remark, of the relation $0 \neq x \in G_{\beta+1}$ and of (1), (2) we obtain

(6) $$\text{type}(x) \geq \text{type } J_\beta \quad .$$

If we put $x^* = x + H_\beta \in H_{\beta+1} / H_\beta$, then $\text{type}(x^*) \geq \text{type}(x)$ ($\text{type}(x^*)$ is taken in $H_{\beta+1} / H_\beta$), and we have according to (5)

(7) $$\text{type}(x) \leq \text{type}(x^*) = \text{type } \bar{J}_\beta \leq \text{type } J_\beta \quad .$$

From the inequalities (6) and (7) we conclude

(8) $$\text{type}(x) = \text{type}(x^*) = \text{type } \bar{J}_\beta = \text{type } J_\beta \quad .$$

The relation (8) makes it possible to apply a Baer's lemma (see [1], lemma 46.3) and therefore we can write

(9) $$H_{\beta+1} = H_\beta \dot{+} J^*_\beta \qquad (\beta < \tau)$$

where, according to (5) and (8), $J^*_\beta \cong \bar{J}_\beta \cong J_\beta$. We have thus proved that for each ordinal $\beta < \tau$ a direct decomposition (9) is true where J^*_β either vanishes or $J^*_\beta \cong$ $\cong J_\beta$.

Now we shall prove that $H = \sum_{\alpha < \tau} J^*_\alpha$.

First of all we shall show the equality

(10) $$\{J^*_\alpha \ ; \ \alpha < \tau\} = \sum_{\alpha<\tau} J^*_\alpha$$

holds. Let $\alpha_1 < \alpha_2 < \dots < \alpha_n < \tau$, let $x_i \in J^*_{\alpha_i}$ ($i = 1, 2, \dots, n$), and let $x_1 + x_2 + \dots + x_n = 0$. Since $\alpha_j < \alpha_n$ ($j = 1, \dots, n-1$) we have $\alpha_j + 1 \leqq \alpha_n$ ($j = 1, \dots, n-1$), and, by (9) and (3), $x_j \in J^*_{\alpha_j} \subseteq H_{\alpha_j+1} \subseteq$ $\subseteq H_{\alpha_n}$. According to (9)

$$(x_1 + \dots + x_{n-1}) + x_n \in H_{\alpha_n} \dot{+} J^*_{\alpha_n}$$

where $(x_1 + \dots + x_{n-1}) \in H_{\alpha_n}$, $x_n \in J^*_{\alpha_n}$, and we conclude that $x_n = 0 = x_1 + \dots + x_{n-1}$. By the same method we can show that also $x_1 = x_2 = \dots = x_{n-1} = 0$, and (10) is established. Next we shall show by transfinite induction that, for each β with $0 < \beta \leqq \tau$, the following relation holds

(11) $$H_\beta = \sum_{\alpha<\beta} J^*_\alpha .$$

If $\beta = 1$, then we have by (9) $H_1 = H_0 \dot{+} J^*_0 = J^*_0$, since $H_0 = (0)$.

Let $\beta_0 > 1$ and let us assume that (11) is true whenever $\beta < \beta_0$. If β_0 is of the form $\beta_0 = \beta + 1$, then, by (9) and by the inductive assumption, we have

$$H_{\beta_0} = H_{\beta+1} = H_\beta \dot{+} J^*_\beta = \sum_{\alpha<\beta} J^*_\alpha \dot{+} J^*_\beta = \sum_{\alpha<\beta_0} J^*_\alpha .$$

If β_0 is a limit ordinal, then it follows from the definition of G_α ($\alpha < \tau$) that $G_{\beta_0} = \bigcup_{\alpha<\beta_0} G_\alpha$ and also $H_{\beta_0} =$ $= \bigcup_{\alpha<\beta_0} H_\alpha$. We conclude (according to (10))

$$H_{\beta_0} = \bigcup_{\alpha<\beta_0} \left(\sum_{\gamma<\alpha} J^*_\gamma\right) = \sum_{\gamma<\beta_0} J^*_\gamma ..$$

Thus (11) is proved for each $\beta \leqq \tau$. In the special case $\beta = \tau$ we have by (11)

(12) $$H = H_\tau = \sum_{\alpha<\tau} J^*_\alpha$$

which completes the proof of our theorem.

Remark. It follows immediately from the preceding proof that every pure subgroup H of such a group G is isomorphic to a direct summand of G (see (12), (9), (5) and (8)).

References

[1] L. FUCHS, Abelian groups, Budapest, 1958.