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REMARK ON PARTIAL MAPPINGS

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The aim of this remark is to prove three theorems on certain homomorphisms of an arbitrary semigroup of mappings, composition of mappings being taken as semigroup multiplication.

If f is a mapping, Df denotes the domain of f, and Rf the range of f. We accept the empty set as a mapping, with both domain and range void.

If f and g are any mappings – we put no restrictive assumptions on Df, Dg and Rg – a map $f \circ g$ is defined as follows:

$$D(f \circ g) = \{x : x \in Dg \text{ and } g(x) \in Df\} ;$$

$$(f \circ g)(x) = f(g(x)) , \text{ for any } x \in D(f \circ g) .$$

Evidently

$$(f \circ g) \circ h = f \circ (g \circ h)$$

if f, g, h are arbitrary mappings.

If X is a set, then all mappings f such that $Df \subset X$ and $Rf \subset X$, form a semigroup under composition. This semigroup will be referred to as the semigroup of partial mappings of X; it always contains the empty mapping.

Let f be a mapping, and let Y be an arbitrary set. By $f|Y$ we denote the following map:

$$D(f|Y) = Df \cap Y ;$$

$$(f|Y)(x) = f(x) , \text{ for any } x \in D(f|Y) .$$

By $f\|Y$ we denote the mapping such that

$$D(f \| Y) = \{x : x \in Df \cap Y \text{ and } f(x) \in Rf \cap Y\};$$

$$(f \| Y)(x) = f(x) \text{ for any } x \in D(f \| Y) .$$

If F is any semigroup of mappings, and if Y is an arbitrary set, then both

$$F|Y = \{f|Y : f \in F\}$$

and

$$F \| Y = \{f \| Y : f \in F\}$$

generate semigroups under composition.

Let F be a semigroup of mappings, and let Y be a set. We put

$$F(Y) = \{f(y) : f \in F \text{ and } y \in Y \cap Df\} ;$$

we write $F(y)$ and $f(Y)$ instead of $F(\{y\})$ and $\{f\}(Y)$, respectively. The set Y is called invariant under F if $F(Y) \subset Y$.

If F is a collection of maps, we put $DF = \bigcup_{f \in F} Df$ and $RF = \bigcup_{f \in F} Rf$.

<u>In this note we only consider semigroups F such that $DF = RF$.</u>

Let F be a semigroup such that $DF = RF$. Let Y be a subset of DF. We discuss the following problem. Let φ_1 and φ_2 be the mappings of the semigroup F into the semigroup generated by $F|Y$ and $F \| Y$, respectively, defined by

$$\varphi_1(f) = f|Y ,$$

$$\varphi_2(f) = f \| Y .$$

Under what condition on Y is $\varphi_1 (\varphi_2)$ a homomorphism. This problem is solved completely by the following theorems:

<u>Theorem 1.</u> φ_1 is a homomorphism if and only if Y is invariant under F.

<u>Theorem 2.</u> φ_2 is a homomorphism if and only if there are

sets Y_1, Y_2, such that Y_1 and $DF \setminus Y_2$ are invariant under F, and $Y = Y_1 \cap Y_2$.

Proof of theorem 1. If Y is invariant under F, φ_1 is clearly a homomorphism. If Y is not invariant under F, then there exists a $g \in F$ and an $x_0 \in Y \cap Dg$ such that $g(x_0) \notin Y$. As $DF = RF$, there is an $f \in F$ such that $g(x_0) \in Df$. Then $x_0 \in D(f \circ g|Y) \setminus D(f|Y \circ g|Y)$. Hence $f \circ g|Y \neq f|Y \circ g|Y$, which shows that φ_1 is not a homomorphism.

Proof of theorem 2. The following three assertions are evident:

(a) if Y is invariant, then φ_2 is a homomorphism;

(b) if $Y = DF \setminus Y_1$, Y_1 invariant under F, then φ_2 is a homomorphism;

(c) if for $i = 1, 2$, $f \| Y_i \circ g \| Y_i = f \circ g \| Y_i$, for all $f, g \in F$, then also $f \| (Y_1 \cap Y_2) \circ g \| (Y_1 \cap Y_2) = f \circ g \| \| (Y_1 \cap Y_2)$, for all $f, g \in F$.

It follows that if $Y = Y_1 \cap Y_2$, Y_1 and $Df \setminus Y_2$ invariant, then φ_2 is a homomorphism.

Conversely, assume that $Y \subset DF$ is such that $\varphi_2 : F \to \to F \| Y$ is a homomorphism. If $y \in Y$, $g \in F$, $y \in Dg$, $g(y) \notin Y$, then $f \circ g(y) \notin Y$, for all $f \in F$. It follows that the set $F(Y) \setminus Y$ is invariant. As $Y \cup F(Y)$ is also invariant – F being a semigroup – and as $Y = Y_1 \cap Y_2$, where $Y_1 = Y \cup F(Y)$ and $Y_2 = DF \setminus (F(Y) \setminus Y)$, the proof of theorem 2 is concluded.

Both theorems can be applied to algebraic semigroups. If $(X; .)$ is a semigroup with unit, and if $x \in X$, we denote by f_x the mapping of X into itself such that $f_x(y) = x \cdot y$, for all $y \in X$. Let $F = \{f_x : x \in X\}$; then F is a semigroup under composition, isomorphic with $(X; .)$. A subset Y of X is a left ideal of $(X; .)$ if and only if Y is invariant under F.

From theorems 1 and 2 we derive the following:

Corollary. Let $(X; .)$ be a semigroup with unit and let Z_1, Z_2 be subsets of X. The mappings φ_1 and φ_2, such that $\varphi_1(x) = f_x | Z_1$, and $\varphi_2(x) = f_x \| Z_2$, are homomorphisms of $(X; .)$ into semigroups generated by $F | Z_1$ and $F \| Z_2$, respectively, if and only if Z_1 is a left ideal of X, and $Z_2 = Y_1 \setminus Y_2$, where both Y_1 and Y_2 are left ideals of $(X; .)$.

This result is very close to that of Lyapin in [1].

Let f, h be mappings such that $Rf \subset Dh$. The mapping f is called compatible with the equivalence relation defined by h, or shortly, compatible with h, if for all x_1, $x_2 \in Df$:

$$h(x_1) = h(x_2) \Rightarrow h \circ f(x_1) = h \circ f(x_2) .$$

If f is compatible with h, we define a map $h \rightarrowtail f$ in the following way:

$$D(h \rightarrowtail f) = R(h | Df) ;$$

and, if $x \in D(h \rightarrowtail f)$:

$$(h \rightarrowtail f)(x) = h \circ f(y) ,$$

where y is any point of $D(f)$ such that $x = h(y)$. Then $h \rightarrowtail f$ is evidently a mapping.

For instance, if $f = f' | Df$, where f' is a mapping commuting with h, $f' \circ h = h \circ f'$, while moreover $Df' = Dh$, then f is compatible with h, so that $f \rightarrowtail h$ is defined.

Theorem 3. Let F be a semigroup of mappings under composition, and let h be a mapping such that $Rf \subset Dh$ and such that each $f \in F$ is compatible with h . Then the mapping ψ , defined on all F , such that $\psi(f) = h \rightarrowtail f$ for $f \in F$, is a homomorphism of F onto the semigroup of all maps $h \rightarrowtail f$, $f \in F$, with composition as the semigroup operation.

Proof. It suffices to prove that

$$(h \rightarrowtail f) \circ (h \rightarrowtail g) = h \rightarrowtail (f \circ g) ,$$

for all $f, g \in F$. As $D(h \rightarrowtail (f \circ g)) = R(h/D(f \circ g))$, the map $h \rightarrowtail (f \circ g)$ sends $f(y)$ onto $h \circ f \circ g(y)$, whenever this term is defined. On the other hand, $h \rightarrowtail g$ maps $h(y)$ onto $h \circ g(y)$, if this is well defined, and $h \rightarrowtail f$ maps $h(z)$ onto $h \circ f(z)$ whenever this term is defined. If we take $z = g(y)$, we find that $(h \rightarrowtail f) \circ (h \rightarrowtail g)$ maps $h(y)$ onto $h \circ f \circ g(y)$, if this is a well defined term. This finishes the proof.

This theorem too admits of an application to algebraical semigroups.

Corollary. Let $(X; .)$ be a semigroup with unit. Let $h : X \to X$ be such that the preimages of points under h define a congruence relation on X ;i.e. such that for all $x_1, x_2, y \in X$

$$h(x_1) = h(x_2) \Rightarrow h(y \cdot x_1) = h(y \cdot x_2) .$$

Then the map ψ , defined by: $\psi(x) = h \rightarrowtail f_x$, is a homomorphism of $(X; .)$ onto the semigroup of all $h \rightarrowtail f_x$, $x \in X$, with composition as the semigroup operation.

R e f e r e n c e :

[1] Е.С. Ляпин, О представлениях полугрупп частичными преобразованиями, Мат.сб., 52(94)(1960), 589 - 596 .