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ON THE APPROXIMATIVE SOLUTION OF INTEGRAL EQUATIONS

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1. Introduction.

Finite dimensional approximations of linear bounded operators are needed for the numerical solution of linear equations in normed spaces. The general theory of approximative methods is given in [3] pp. 487-562. In this monography the case of integral equations of the second kind with kernels continuous and periodical (period equal 1) in $<0,1>\times<0,1>$ is studied as example. We shall use the Kantorovich's method for the approximative solution of an integral equation with the kernel $K+\Delta$, where K is a continuous but non periodical kernel and $\Delta(s,t)=g(s)\delta(s-t)$, where δ is the well known Dirac δ -function and g is continuous in $<E_0, E_\infty>$. We shall construct the finite dimensional approximations of the mentioned operator in another way than Kantorovich in [3] p. 512 . Our way is suitable for solving some problems described by equations containing variables which can be obtained by averaging measured quantities. This is the case in many physical problems. As examples we can mention the system of multigroup-diffusion equations ([1]), the system of integro-differential equations of multigroup approximation of neutron transport theory ([4] pp. 282-290) and the equation for describing the energy spectrum of neutrons in the medium consisting of U^{238} and U^{235} ([2]) .

We shall consider the integral equation

$$(1.1) \qquad x(E)-\lambda\left[\int_{E_0}^{E_\infty} K(E,E')x(E')\,dE'+g(E)x(E)\right]=y(E),$$

where the kernel K is continuous in the square $\langle E_0, E_\infty\rangle \times \langle E_0, E_\infty\rangle$ and the function g is continuous in $\langle E_0, E_\infty\rangle$, $(E_0 < E_\infty < +\infty)$. The problems which are mentioned above can be reduced to the equation (1.1) or to a system of equations of the type (1.1).

Let X be a Banach space, $\tilde{X}_n$ an n-dimensional subspace of the space X and $\hat{X}_n$ an n-dimensional space isomorphic to $\tilde{X}_n$. We shall denote this isomorphism by the symbol S_n. Let us suppose that a transformation Q_n exists such that it maps X into $\hat{X}_n$ and that

$$Q_n x = S_n x$$

holds for $x \in \tilde{X}_n$. Thus the transformation $P_n = S_n^{-1} Q_n$ is a projection of X onto $\tilde{X}_n$.

Let us consider the equation

$$(1.2) \qquad x - \lambda A x = y, \quad y \in X,$$

where A is a linear bounded operator mapping X into itself. The "approximative" equation

$$(1.3) \qquad \hat{X} - \lambda \hat{A}_n \hat{X} = \hat{y}, \quad \hat{y} = Q_n y,\ y \in X,\ \hat{y} \in \hat{X}_n$$

corresponds in $\hat{X}_n$ to the equation (1.2). The operator $\hat{A}_n$ in (1.3) can be represented by a square matrix of finite order. We shall also denote this matrix by $\hat{A}_n$.

We shall give the conditions guaranting the convergence of "approximative solutions" $S_n^{-1}\hat{x}_n$ of equation (1.3) to the solution x^* of the equation (1.2). The sufficient conditions are given in [3] pp. 488-489 and 500-501 by Kantorovich. We shall show that the Kantorovich's conditions are fulfilled in our case. We shall construct approximative solutions of equation (1.1) in two different spaces, namely in the spaces $C(\Omega)$ and $L_2(\Omega)$, where $C(\Omega)$ is the set of real functions continuous in $\Omega = \langle E_0, E_\infty \rangle$ with the norm $\|x\| = \max_{E \in \Omega} |x(E)|$ and $L_2(\Omega)$ is the set of real functions, the square of which are integrable in Ω. In $L_2(\Omega)$ the scalar product can be defined by the following formula $(x,y) = \int_{E_0}^{\infty} x(E) y(E) dE$ and the norm $\|x\| = (x,x)^{\frac{1}{2}}$.

The example of the equation (1.1) shows that the construction of the approximative solutions is sometimes more convenient in the space $L_2(\Omega)$ although the exact solution lies in $C(\Omega)$. With other words, we shall show that the sequence of the approximative solutions converges in the norm of $L_2(\Omega)$ to the solution of the equation (1.1) lying in $C(\Omega)$, while the convergence of the mentioned sequence in the norm of $C(\Omega)$ cannot in general be guaranted.

We shall consider a problem which is usual in the neutron transport theory. We shall suppose that

$$K(E,E') > 0 \quad \text{for} \quad E, E' \in (E_0, E_\infty),$$
$$q(E) > 0 \quad \text{for} \quad E \in (E_0, E_\infty)$$

and that the operator A defined by $A = C + D$, where

$$Cx \equiv \int_{E_0}^{E_\infty} K(E,E') x(E') dE'$$

$$Dx \equiv q(E) x(E)$$

is a Radon-Nicolski operator ([5]).

Let $E_0 < E_1 < \dots < E_n = E_\infty$ and let $X \in L_2(\Omega)$. We put $\hat{x} = (\xi_0, \xi_1, \dots, \xi_n)$, where

$$\xi_j = \frac{1}{E_j - E_{j-1}} \int_{E_{j-1}}^{E_j} x(E) dE, \; j = 1, \dots, n; \; \xi_0 = x(E_0). \tag{1.4}$$

Let

(1.5) $$\hat{E}_j = \frac{1}{2}(E_{j-1}+E_j),\quad j=1,\dots,n,\quad \hat{E}_0 = E_0$$

and let us put

(1.6) $$\hat{A}_n = \hat{C}_n + \hat{D}_n,\quad \hat{C}_n = (\hat{c}_{jk}),\quad \hat{D}_n = (\hat{d}_j\,\delta_{jk}),\ 1\le j,k\le n$$

(δ_{jk} -Kroneker's delta), where

(1.7) $$\begin{cases} \hat{C}_{jk} = \dfrac{E_k - E_{k-1}}{E_j - E_{j-1}} \displaystyle\int_{E_{j-1}}^{E_j} K(E,\hat{E}_j)\,dE, & 1\le j,\ k\le n, \\ \hat{d}_j = g(\hat{E}_j), & j = 0,1,\dots,n. \end{cases}$$

Further let

(1.8) $$\begin{aligned} &\hat{c}_{j0} = 0,\quad j = 0,1,\dots,n \\ &\hat{C}_{0k} = (E_k - E_{k-1})\,K(E_0,\hat{E}_k),\quad k = 1,\dots,n\ . \end{aligned}$$

The matrix with the elements $\hat{c}_{jk} + \hat{d}_j\,\delta_{jk},\ 0\le j,k\le n$ will be denoted by the symbol A_n° .

2. Finite-dimensional approximations in $C(\Omega)$.

The approximative system of equations corresponding to (1.1) has the following norm

(2.1) $$\xi_j - \lambda \sum_{k=0}^{n} [\hat{C}_{jk} + \hat{d}_j\,\delta_{jk}]\,\xi_k = \eta_j,\quad j = 0,1,\dots,n,$$

where $\eta_j,\ j = 0,1,\dots,n$ are components of the vector $\hat{y} = (\eta_0,\eta_1,\dots,\eta_n)$ defined analogously as $(\xi_0,\dots,\xi_n)$ in (1.4) . We assume that $y\in C(\Omega)$.

We suppose that $X = C(\Omega),\ \|X\| = \mathrm{Max}\,|x(E)|,\ \hat{X}_n$ is the set of ordered groups of $n+1$ real numbers $(\xi_0,\xi_1,\dots,\xi_n) = \hat{X}$ with the norm $\|X\| = \{\sum_{k=0}^{n} |\xi_k|^2\}^{\frac{1}{2}}$.
We define the map Q_n as follows

(2.2) $$Q_n x = \hat{X},\quad \hat{X} = (\xi_0,\dots,\xi_n),$$

where ξ_j are defined by (1.4) for $j=0,1,\dots,n$. The space $\tilde{X}_n$ consists of functions partially linear in Ω with the vertices $E_0,\dots,E_n$. Thus we have for $\tilde{x}\in\tilde{X}_n$

$$\tilde{x}(E)=\tilde{x}(E_{j-1})+\frac{\tilde{x}(E_j)-\tilde{x}(E_{j-1})}{E_j-E_{j-1}}(E-E_{j-1}),\ E\in\langle E_{j-1},E_j\rangle. \tag{2.3}$$

It is evident that $S_n x = Q_n x$ if $x\in\tilde{X}_n$. The formulae

$$\begin{cases} \|S_n\|=\sup\limits_{\|\tilde{x}\|=1}\|S_n x\|=1, \\ \|Q_n\|=\sup\limits_{\|x\|=1}\|Q_n x\|=1 \end{cases} \tag{2.4}$$

clearly hold.

It can be shown that if $\hat{X}=S_n\tilde{x}$, $\hat{x}=(\xi_0,\dots,\xi_n)$, then

$$\xi_0=\tilde{x}(E_0),\quad \xi_j=\tilde{x}(\hat{E}_j),\ j=1,\dots,n, \tag{2.5}$$

where $\hat{E}_j$ are defined by (1.5). Thus we can construct the vector $\tilde{x}\in\tilde{X}_n$ to every vector $\hat{x}\in\hat{X}_n$ such that $\hat{X}=S_n\tilde{x}$ i.e. $\tilde{x}=S_n^{-1}\hat{x}$. If we know the values $\xi_j=\tilde{x}(\hat{E}_j)$, we can construct the values $\tilde{x}(E_j),\ j=1,\dots,n$, which together with $\xi_0=\tilde{x}(E_0)$ define the vector $\tilde{x}\in\tilde{X}$ uniquely.

The equalities

$$\|S_n^{-1}\|=\sup_{\|\hat{x}\|=1}\|S_n^{-1}\hat{x}\|=1 \tag{2.6}$$

clearly hold, thus the relations

$$\|P\|=\sup_{\|x\|=1}\|S_n^{-1}Q_n x\|\le 1 \tag{2.7}$$

are valid for the operator $P_n=S_n^{-1}Q_n$.

We prove the following statement.

<u>Theorem 1</u>. If the inequalities

$$|\lambda|\le q<\frac{1}{r(D)} \tag{2.8}$$

($r(D)$ is the spectral radius of D)

hold, the equation (1.1) has one and only one solution $x \in X$ and the sequence $\{S_n^{-1} \hat{x}_n\}$, where $\hat{x}_n$ is the solution of the system (2.1) , converges to x in the norm of $C(\Omega)$.

Proof. The assertion of the theorem will be proved if we show that the Kantorovich's conditions Ia, II and III ([3]) pp. 407,501) are fulfilled.

The equation (1.1) and the following equation

$$(2.9) \qquad x(E) - \frac{\lambda}{1-\lambda g(E)} \int_{E_0}^{E_\infty} K(E,E')x(E')\,dE' = \frac{y}{1-\lambda g(E)}$$

are equivalent for those λ for which (2.8) hold. We shall prove that the conditions Ia, II and III are satisfied for the case (2.9) . It can be shown that the approximative system corresponding to (2.9) has the following form

$$(2.10) \qquad \xi_j - \frac{\lambda}{1-\lambda g(\hat{E}_j)} \sum_{k=0}^{n} \hat{c}_{jk}\, \xi_k = \frac{\eta_j}{1-\lambda g(\hat{E}_j)} .$$

Thus the system (2.10) is equivalent to the system (2.1) and all the assertions, which are valid for (2.10) , are also valid for the system (2.1) .

The operator $(I-\lambda D)^{-1} C$, where I is the identical operator, will be denoted by the symbol $H = H_\lambda$ i.e.

$$y = Hx = \frac{1}{1-\lambda g(E)} \int_{E_0}^{E_\infty} K(E,E')x(E')\,dE' .$$

The corresponding approximative operators will be denoted by $\hat{H}_n$, H_n^0 i.e.

$$\hat{H}_n = (h_{jk}), \quad h_{jk} = \frac{1}{1-\lambda \hat{d}_j \delta_{jk}} \hat{c}_{jk}$$

for $1 \leq j, k \leq n$ and $H_n^0 = (h_{jk})$ for $0 \leq j, k \leq n$.

Condition Ia . A sequence $\{\eta_{on}\}, \eta_{on} \to 0$ exists such that

$$(2.11) \qquad \| Q_n H\tilde{x} - H_n^0 S_n \tilde{x} \| \leq \eta_{on} \|\tilde{x}\|$$

holds for every $\tilde{x} \in \tilde{X}_n$ and for $n = 1, 2, \dots$.

The symbols ω, ω' will denote the moduli of continuity of the kernel K i.e.

$$\omega(\delta) = \sup_{E, E'} |K(E+F, E') - K(E, E')|, \ E, E', E+F \in \Omega, |F| \leq \delta,$$

$$\omega'(\delta) = \sup_{E, E'} |K(E, E'+F') - K(E, E')|, \ E, E', E'+F' \in \Omega, |F'| \leq \delta.$$

Further let

$$\omega_x(S) = \sup_{E \in \Omega} |x(E+F) - x(E)|, \ E, E+F \in \Omega, \ |F| \leq \delta$$

be the modul of continuity of the function $x \in C(\Omega)$.

We shall estimate the absolute value of the expression

$$\int_{E_0}^{E_\infty} x(E)\tilde{x}(E)dE - \sum_{k=1}^{n} f_k \xi_k (E_k - E_{k-1}) ,$$

where $x \in C(\Omega)$, $\tilde{x} \in \tilde{X}_n$, $\hat{x} = (f_0, f_1, \dots, f_n)$, $\hat{\tilde{x}} = (\xi_0, \xi_1, \dots, \xi_n)$.

Similarly as in [3] p. 513 we obtain

$$(2.12) \quad \begin{aligned} &\left|\int_{E_0}^{E_\infty} x(E)\tilde{x}(E)dE - \sum_{k=1}^{n} \Delta_k f_k \xi_k \right| \leq \\ &\leq \sum_{k=1}^{n} \int_{E_{k-1}}^{E_k} |x(E) - f_k| \, |\tilde{x}(E)| dE \leq \omega_x(\Delta)(E_\infty - E_0)\|\tilde{x}\|, \end{aligned}$$

where $\Delta_k = E_k - E_{k-1}$, $k = 1, \dots, n$; $\Delta = \max_{1 \leq k \leq n} \Delta_k$, $\Delta_0 = 0$.

If we take specifically

$$x(E') = (\Delta_j)^{-1}[1 - \lambda g(\hat{E}_j)]^{-1} \int_{E_{j-1}}^{E_j} K(E, E')dE \ ; \ x(E') = \frac{K(E_0, E')}{1 - \lambda g(E_0)} ,$$

we obtain from (2.12) the inequalities

$$(2.13) \quad \begin{aligned} &\left|\int_{E_0}^{E_\infty} [(\Delta_j)^{-1} \int_{E_{j-1}}^{E_j} dE \frac{K(E, E')}{1 - \lambda g(\hat{E}_j)}] \tilde{x}(E')dE' - \sum_{k=1}^{n} \frac{\Delta_k}{\Delta_j} \int_{E_{j-1}}^{E_j} \frac{K(E, \hat{E}_k)}{1 - \lambda g(\hat{E}_j)} dE \, \xi_k \right| \leq \\ &\leq (E_\infty - E_0) . \max_{|\lambda| \leq q; E \in \Omega} |1 - \lambda g(E)|^{-1} . \omega'(\Delta) \|\tilde{x}\| , \end{aligned}$$

$$\begin{aligned} &\left|\int_{E_0}^{E_\infty} \frac{K(E_0, E')}{1 - \lambda g(E_0)} \tilde{x}(E')dE' - \sum_{k=1}^{n} \Delta_k \frac{K(E_0, \hat{E}_k)}{1 - \lambda g(E_0)} \xi_k \right| \leq \\ &\leq (E_\infty - E_0)\omega'(\Delta) \max_{|\lambda| \leq q; E \in \Omega} |1 - \lambda g(E)|^{-1} . \|\tilde{x}\| . \end{aligned}$$

If we put

$$(2.14) \qquad \eta_{on} = \max_{|\lambda| \leq q,\, E \in \Omega} |1 - \lambda g(E)|^{-1} \cdot \omega'(\Delta) \cdot (E_\infty - E_o)$$

we obtain from (2.13) the estimate (2.11) . It follows from the continuity of K and g that $\lim_{n \to \infty} \eta_{on} = 0$ so that according to (2.6)

$$(2.15) \qquad \eta_{on} \, \|S_n^{-1}\| \to 0 \qquad \text{for} \qquad n \to +\infty \, .$$

Condition II . For every vector $x \in X$ there exists a vector $\tilde{y} \in \tilde{X}_n$ such that

$$(2.16) \qquad \| Hx - \tilde{y} \| \leq \eta_{1n} \, \|x\| \, .$$

The validity of the inequality (2.16) follows from the estimate

$$\| x - S_n^{-1} Q_n x \| \leq \omega_x (\Delta)$$

which is valid for any vector $x \in X$ because the values of the functions $\tilde{x} = S_n^{-1} Q_n x$ and x are identical in the points $E_o, E_1, \dots, E_n$ and the function $\tilde{x}$ is linear in (E_{j-1}, E_j), $1 \leq j \leq n$ i.e.

$$(2.17) \qquad \begin{aligned} &|x(E) - \tilde{x}(E)| = |x(E) - [(E_j - E)\, x(E_{j-1}) + \\ &+ (E - E_{j-1})\, x(E_j)] (\Delta_j)^{-1}| \leq \omega_x (\Delta) \end{aligned}$$

Let us estimate the modulus of continuity of the function $y = Hx,\ x \in C(\Omega)$. Let $E, E+F \in \Omega$ and $|F| \leq \Delta$. Then we have

$$|y(E+F) - y(E)| = \left| \int_{E_o}^{E_\infty} \left[\frac{K(E+F, E')}{1 - \lambda g(E+F)} - \frac{K(E, E')}{1 - \lambda g(E)} \right] x(E')\, dE' \right| \leq$$

$$\leq (E_\infty - E_o) \left[\frac{1}{a}\, \omega(\Delta) + \frac{bq}{a^2}\, \omega_g(\Delta) \right] \|x\| \, ,$$

where $a = \inf_{|\lambda| \leq q,\, E \in \Omega} |1 - \lambda g(E)|$, $b = \sup_{E, E' \in \Omega} |K(E, E')|$.

We have proved that the modulus of continuity of the function $y = Hx$ is not larger than $\eta_{1n} \|x\|$, so that we have obtained from (2.8) the estimate (2.16) for $y = Hx$, $\tilde{y} = P_n y$ where η_{1n} are defined by

$$(2.18) \qquad \eta_{1n} = (E_\infty - E_0)\left[\frac{1}{a}\,\omega(\Delta) + \frac{bq}{a^2}\,\omega_q(\Delta)\right] .$$

It follows from (2.17) and (2.7) that

$$(2.19) \qquad \eta_{1n} \|P_n\| \to 0 \quad \text{for} \quad n \to +\infty .$$

Condition III. For every vector $y \in C(\Omega)$ there exists a vector $\tilde{y} \in \tilde{X}_n$ such that

$$\|y - \tilde{y}\| \leqq \eta_{2n} \|y\| .$$

Let $\tilde{y} = P_n y$, then the estimate $\|y - \tilde{y}\| \leqq \omega_y(\Delta)$ is valid according to (2.17) so that condition III holds with

$$\eta_{2n} = \frac{\omega_y(\Delta)}{\|y\|} .$$

From this and from (2.7) it follows that

$$(2.20) \qquad \eta_{2n} \|P_n\| \to 0 \quad \text{for} \quad n \to +\infty .$$

It is clear that the assumption of the existence of a solution of the system (2.1) for every right side implies the unicity of the solution. The assertion of theorem 1 is then a consequence of theorem 3δ (2. XIV) [3] p.507, assumptions of which are fulfilled if the inverse operator $(I - \lambda A)^{-1}$ exists for λ, for which (2.8) hold.

Remark. If we assume that $K(E_0, E') = 0$ for $E' \in \Omega$, $q(E_0) = 0$, $y(E_0) = 0$, then $\hat{C}_{0k} = 0$, $\hat{d}_0 = 0$ in (1.7), (1.8). The system (2.1) and the system

$$(2.21) \qquad \xi_j - \lambda \sum_{k=1}^{n} (\hat{C}_{jk} + \hat{d}_k \delta_{jk})\,\xi_k = \eta_j , \quad j = 1, \dots, n ;$$

where $\hat{y} = (\eta_0, \dots, \eta_n)$ is an arbitrary vector with $\eta_0 = 0$ are equivalent. In other words, the solution of the system (2.1) lies in the subspace $X^0_n \subset \hat{X}_n$ consisting of vectors $\hat{x} = (0, \xi_1, \dots, \xi_n)$.

It follows from theorem 1 that the solution x of the equation (1.1) is the limit of the sequence $\{S^{-1}_n \hat{x}_n\}$ where $\hat{x}_n$ is the solution of the system (2.21).

In the next chapter we shall show that the approximative solutions $\{S^{-1}_n \hat{x}_n\}$ where $\hat{x}_n$ is the solution of (2.21), converge to the exact solution x of equation (1.1) in the norm of the space $L_2(\Omega)$ without the additional assumptions $K(E_0, E') = 0$ for $E' \in \Omega$, $g(E_0) = 0$ and $y(E_0)$.

3. Finite-dimensional approximations in $L_2(\Omega)$.

In this chapter we shall consider the real space $L_2(\Omega)$ with the scalar product $(x, y) = \int_{E_0}^{E_\infty} x(E) y(E) dE$. The symbol $\hat{X}_n$ denotes the set of vectors $\hat{x} = (\xi_1, \dots, \xi_n)$ with the scalar product $(\hat{x}, \hat{y}) = \sum_{k=1}^{n} \xi_k \eta_k$. We define the map Q_n as

$$(3.1) \quad Q_n x = \hat{x} \Longleftrightarrow \hat{x} = (\xi_1, \dots, \xi_n), \; \xi_k = \frac{1}{\Delta_k} \int_{E_{k-1}}^{E_k} x(E) dE, \quad k = 1, \dots, n.$$

The space $\tilde{X}_n$ is the subset of $L_2(\Omega)$ consisting of constant functions in (E_{j-1}, E_j) i.e.

$$(3.2) \quad \tilde{x} \in \tilde{X} \Longleftrightarrow \tilde{x}(E) = \tilde{x}(\tilde{E}_j) \quad \text{for } E \in (E_{j-1}, E_j) \text{ almost everywhere}$$

where $\tilde{E} \in (E_{j-1}, E_j)$, $j = 1, \dots, n$.

If $x \in \tilde{X}_n$, we then put $S_n x = Q_n x = \hat{x} = (\xi_1, \dots, \xi_n)$.

We evidently have

$$(3.3) \quad \begin{cases} \| Q_n \| = \sup\limits_{\|x\| = 1} \| Q_n x \| = 1 \\ \| S_n \| = \sup\limits_{\|\tilde{x}\| = 1} \| S_n \tilde{x} \| = 1 . \end{cases}$$

It can be verified easily that the coordinates of $\hat{x} = S_n \tilde{x},\ \tilde{x} \in \tilde{X}_n$ $\hat{x} = (\xi_1, \dots, \xi_n)$ have the following form

$$\xi_j = \tilde{x}(\tilde{E}_j),\ \tilde{E}_j \in (E_{j-1}, E_j),\ j = 1, \dots, n. \tag{3.4}$$

For every vector $\hat{x} \in \hat{X}_n$ a vector $x \in \tilde{X}_n$ can be constructed such that $\hat{X} = S_n \tilde{x}$ and hence $\tilde{x} = S_n^{-1} \hat{x}$.

However, if $\hat{x} = (\xi_1, \dots, \xi_n)$ then $\tilde{x}(E) = \xi_j$, for $E \in (E_{j-1}, E_j)$ and $\tilde{x} \in \tilde{X}_n$.

Clearly

$$\| S_n^{-1} \| = \sup_{\|\hat{x}\| = 1} \| S_n^{-1} \hat{x} \| = 1, \tag{3.5}$$

so that

$$\| P_n \| = \sup_{\|x\| = 1} \| S_n^{-1} Q_n x \| \leqq 1 \tag{3.6}$$

for $P_n = S_n^{-1} Q_n$.

Let $\hat{y} \in \hat{X}_n$, $\hat{y} = (\eta_1, \dots, \eta_n)$ and $\hat{x} = (\xi_1, \dots, \xi_n)$ be a solution of the system

$$\xi_j - \lambda \sum_{k=1}^{n} (\hat{c}_{jk} + \hat{d}_k \delta_{jk}) \xi_k = \eta_j, \quad j = 1, \dots, n, \tag{3.7}$$

where $\hat{c}_{jk}, \hat{d}_j,\ 1 \leqq j, k \leqq n$ are defined by (1.7), (1.8).

Then we put

$$\tilde{x} = S_n^{-1} \hat{x}_n. \tag{3.8}$$

Theorem 2. Let us suppose that the inequalities (2.8) are valid and that the bounded inverse operator $(I - \lambda A)^{-1}$ exists for the operator $I - \lambda A$. Then the formula

$$\lim_{n \to \infty} S_n^{-1} \hat{x}_n = x \tag{3.9}$$

holds in the norm of the space $L_2(\Omega)$. In (3.9) x denotes the solution of the equation (1.1) for the given element $y \in L_2(\Omega)$.

Proof. We shall show that the Kantorovich's conditions Ia, II, III of [3] pp. 488, 501 hold in this case similarly as in the previous chapter (i.e. for the operator H).

Condition Ia . There exists a sequence of numbers $\{\eta_{on}\}$ such that the relations

$$(3.10) \qquad \| Q_n H\tilde{x} - \hat{H}_n S_n \tilde{x} \| \leqq \eta_{on} \| \tilde{x} \|$$

hold for every vector $\tilde{x} \in \tilde{X}_n$ and for $n = 1, 2, \dots$.

Let us estimate the term $\left| \int_{E_0}^{E_\infty} x(E)\tilde{x}(E)dE - \sum_{k=1}^{n} \Delta_k \xi_k \xi_k \right|$,

where $x \in C(\Omega)$, $\xi_k = (\Delta_k)^{-1} \int_{E_{k-1}}^{E_k} x(E)\,dE$ and $\tilde{x} \in \tilde{X}_n$.

We obtain

$$\left| \int_{E_0}^{E_\infty} x(E)\tilde{x}(E)dE - \sum_{k=1}^{n} \Delta_k \xi_k \xi_k \right| = \left| \sum_{k=1}^{n} \int_{E_{k-1}}^{E_k} [x(E) - \xi_k]\tilde{x}(E)dE \right| \leqq$$

$$\leqq \omega_x(\Delta)\,[E_\infty - E_0]^{\frac{1}{2}} \,\|\tilde{x}\|_L \,,$$

where the symbol $\|x\|_L$ denotes the norm of x in $L_2(\Omega)$. Similarly as in chapter 2 $\Delta = \underset{1 \leqq k \leqq n}{Max}\, \Delta_k$, $\Delta_k = E_k - E_{k-1}$.

If we put

$$x(E') = (\Delta_j)^{-1} \int_{E_{j-1}}^{E_j} \frac{K(E,E')}{1-\lambda g(E_j)}\, dE \,,$$

we obtain using (3.11) the inequality

$$\left| \int_{E_0}^{E_\infty} dE [(\Delta_j)^{-1} \int_{E_{j-1}}^{E_j} \frac{K(E,E')}{1-\lambda g(E_j)} dE]\tilde{x}(E') - \sum_{k=1}^{n} \Delta_k \frac{\hat{c}_{jk}}{1-\lambda \hat{d}_k \delta_{jk}} \xi_k \right| \leqq$$

$$\leqq [E_\infty - E_0]^{\frac{1}{2}} \left[\frac{\omega(\Delta)}{a} + \frac{bq}{a^2}\,\omega_g(\Delta) \right] \|\tilde{x}\|_L \,,$$

or

$$(3.12) \quad \| Q_n H\tilde{x} - \hat{H}_n S_n \tilde{x} \|_L \leqq (E_\infty - E_0) \left[\frac{\omega(\Delta)}{a} + \frac{bq}{a^2}\omega_g(\Delta) \right] \|\tilde{x}\|_L .$$

If we put

$$\eta_{on} = (E_\infty - E_0) \left[\frac{\omega(\Delta)}{a} + \frac{bq}{a^2}\,\omega_g(\Delta) \right]$$

we obtain (3.10) from (3.12) . It follows from (3.5) that the relation

$$\eta_{on} \| S_n^{-1} \| \longrightarrow 0 \quad \text{for} \quad n \to +\infty \tag{3.14}$$

is valid.

Condition II . There exists a vector $\tilde{y} \in \tilde{X}_n$ to every vector $x \in L_2(\Omega)$ such that

$$\| Hx - \tilde{y} \|_L \leqq \eta_{1n} \| x \|_L \ . \tag{3.15}$$

It can be easily proved that the relations

$$\begin{aligned} |x(E) - \tilde{x}(E)| &= |x(E) - (S_n^{-1} Q_n x)(E)| = \\ &= |x(E) - (\Delta_j)^{-1} \int_{E_{j-1}}^{E_j} x(E) dE| = |x(E) - x(\tilde{E}_j)| \leqq \omega_x(\Delta) \end{aligned} \tag{3.16}$$

hold for $x \in C(\Omega)$ in the interval $(E_{j-1}, E_j), j = 1, \ldots, n,$ $\tilde{E}_j \in \langle E_{j-1}, E_j \rangle$.

It can be shown that the function $y = Tx$ (where T is an integral operator with a continuous in $\Omega \times \Omega$ kernel $T(E, E')$), is a continuous for $x \in L_2(\Omega)$. We shall estimate the modulus of continuity of the function $y = Hx$. Let $E, E+F \in \Omega$, $|F| \leqq \delta$, then we have

$$\begin{aligned} |y(E+F) - y(E)| &= |\int_{E_0}^{E_\infty} [\frac{K(E+F, E')}{1-\lambda g(E+F)} - \frac{K(E, E')}{1-\lambda g(E)}] x(E') dE'| \leqq \\ &\leqq \omega_H(\delta) \sum_{k=1}^{n} \int_{E_{k-1}}^{E_k} |x(E)| dE \ , \end{aligned} \tag{3.17}$$

where $\omega_H(\delta) = [\frac{\omega(\delta)}{a} + \frac{2b}{a^2} \omega_g(\delta)] (E_\infty - E_0)^{\frac{1}{2}}$.

From (3.16) and (3.17) it follows easily

$$\| Hx - S_n^{-1} Q_n Hx \|_L \leqq [\frac{\omega(\Delta)}{a} + \frac{2b}{a^2} \omega_g(\Delta)] (E_\infty - E_0) \| x \|_L . \tag{3.18}$$

Let us put

$$\eta_{1n} = (E_\infty - E_0) [\frac{\omega(\Delta)}{a} + \frac{2b}{a^2} \omega_g(\Delta)] \tag{3.19}$$

we obtain (3.15) with $\tilde{y} = S_n^{-1} Q_n H x$.

According to (3.6)

(3.20) $$\eta_{1n} \| P_n \| \to 0 \quad \text{for} \quad n \to +\infty .$$

Condition III . A vector $\tilde{y}_n \in \tilde{X}_n$ exists for every vector $y \in X$ so that the inequality

(3.21) $$\| y - \tilde{y}_n \| \leq \eta_{2n} \| y \|$$

holds. In the considered case we can put $\tilde{y}_n = S_n^{-1} Q_n y$, $\eta_{2n} = \| y \|_L^{-1} \cdot \| y - \tilde{y}_n \|_L$ since, as is well known, ([7] p. 16), $\| y - \tilde{y}_n \| \to 0$ for $n \to +\infty$ and for every vector $y \in L_2(\Omega)$.

According to (3.6)

(3.22) $$\eta_{2n} \| P_n \| \to 0 \quad \text{for} \quad n \to +\infty .$$

The assertion of theorem 2 is then a consequence of the theorem 3σ (2.XIV) in [3] p. 507 , the assumptions of which are fulfilled.

4. Approximations of eigenvalues and eigenvectors

We shall consider the conditions from which follows the convergence of eigenvalues and eigenvectors of the operator $\hat{A}_n$ to eigenvalues and eigenvectors of the operator A . We shall consider the case of an approximation in the space $L_2(\Omega)$ only, because the caseof the space $C(\Omega)$ can be studied analogously.

<u>Theorem 3</u> . Let μ, $|\mu| > r(D)$ be an eigenvalue of the operator $A = C + D$, where $r(D)$ is the so-called spectral radius of the operator D . Let A_n, $n = 1, 2, \dots$ be the finite-dimensional approximation of A defined by (1.7), (1.8) . Then the formula

(4.1) $$\lim_{n \to \infty} \hat{\mu}_n = \mu_0$$

holds, where $\hat{\mu}_n$ is an eigenvalue of the operator $\hat{A}_n$, $n = 1, 2, \dots$.

If μ_0 is a simple eigenvalue and x_0 ($\|x_0\| = 1$) the corresponding eigenvector of the operator A, then the sequence $\{S_n^{-1} \hat{x}_n\}$, where $\hat{x}_n$, $\|S_n^{-1} \hat{x}_n\| = 1$ is an eigenvector corresponding to the value $\hat{\mu}_n$ ($\hat{\mu}_n \to \mu_0$) converges in the norm of the space $L_2(\Omega)$ to the vector y_0, where $y = c x_0$, $|c| = 1$:

$$\lim S_n^{-1} \hat{x}_n = y_0 .$$

Proof. We remark that for every $\varepsilon > 0$ the at most finite number of eigenvalues of the operator A lies outside the circle $|\lambda| \leqq \kappa(D) + \varepsilon$. Let us denote these eigenvalues by the symbols $\lambda_1, \dots, \lambda_p$. Let Γ be such domain outside the circle $|\lambda| \leqq \kappa(D)$, having no common points with the circle $|\lambda| = \kappa(D)$. Let us denote the region which is originated by removing suitable δ-neighborhoods of eigenvalues $\lambda_1, \dots, \lambda_p$ by the symbol Γ_0. The resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ exists in the region Γ_0 and $\|R(\lambda, A)\|$ is bounded in Γ_0 as a function of λ. The resolvents $R(\lambda, \hat{A}_n) = (\lambda I_n - \hat{A}_n)^{-1}$ (I_n -identical operator in $\hat{X}_n$) exist for sufficiently large n and $\|R(\lambda, \hat{A}_n)\|$ are bounded independently of n by theorem 3δ(2.XIV) [3] p. 507 . The eigenvalues of the operator $\hat{A}_n$ may lie in the δ-neighborhoods of the points $\lambda_1, \dots, \lambda_p$. Thus, if the approximation eigenvalues have some limit, one of the points $\lambda_1, \dots, \lambda_p$ must be this limit. We shall prove that every eigenvalue λ_j, $|\lambda_j| > \kappa(D)$ of the operator A is the limit of a sequence of eigenvalues of approximation operators $\hat{A}_n$, $n = 1, 2, \dots$. However, if for example the eigenvalue λ_j is not such a limit, then no one eigenvalue of the operators $\hat{A}_n$ lies in some circle Γ_j with the centre λ_j for sufficiently large n . The function $\| R(\lambda, A) \|$ is bounded on the boundary of Γ_j, thus the norms $\| R(\lambda), \hat{A}_n) \|$ *are uniformly bounded on the boundary of* Γ_j *for suffi*ciently large n . From the maximum modulus principle it follows the uniform boundedness of $\| R(\lambda, \hat{A}_n \|$ inside Γ_j follows. According to [3] p. 507 **theorem 6** (2.XIV), the resolvent $R(\lambda, A)$ exists in Γ_j and this is a contradiction to the assumption about λ_j . Thus we have proved that every eigenvalue $\lambda_j, |\lambda_j| > \varkappa(D)$ is the limit of some sequence of eigenvalues of the operators $\hat{A}_n,\ n = 1, 2, \dots$ and thus (4.1) holds for any eigenvalue $\lambda_1, \dots, \lambda_p$.

We shall prove that a sequence of indices $m_1 < m_2 < \dots$ can be chosen so that for $n = m_k$ $\hat{x}_n$, where $\hat{x}_n$ is the eigenvector corresponding to the eigenvalue $\hat{\mu}_n$ $(\hat{\mu}_n \to \mu_0)$, the relation

(4.2) $$y_0 = \lim_{n \to \infty} S_n^{-1} \hat{x}_n$$

holds in the norm of the space $L_2(\Omega)$, where y_0 is an eigenvector corresponding to the eigenvalue μ_0 of the operator A .

If the vector x_0 is an eigenvector of the operator A corresponding to the eigenvalue $\mu_0, |\mu_0| > \varkappa(D)$, the equations

(4.3) $$A x_0 = \mu_0 x_0 , \qquad R(\mu_0, D) C x_0 = x_0$$

are valid simultaneously. This means that the validity of the assertion of theorem 3 can be proved for the compact operator $H = H_\lambda = (\frac{1}{\lambda})^{-1} R(\lambda, D) C$. We shall assume that μ_0 is a characteristic value of the operator $H = H_{\mu_0}$, i.e that there exists a vector x_0 such that $Hx_0 = \mu_0 x_0$. We can suppose that $x_0 = c y_0$ which follows from the simplicity of μ_0. It follows from the compactness of H that a convergent subsequence $\{ H S_{n_k}^{-1} \hat{x}_{n_k} \}$ of the sequence $\{ H S_n^{-1} \hat{x}_n \}$ exists, where $\hat{x}_n = (\hat{\mu}_n)^{-1} \hat{H}_n \hat{x}_n$, $\| S_{n_k}^{-1} \hat{x}_n \| = 1$. Let us call its limit x_0. We shall show that $x_0 = c x_0$. For a moment we shall assume that the whole sequence $\{ H S_n^{-1} \hat{x}_n \}$ converges, i.e.

$$\lim H S_n^{-1} \hat{x}_n = x_0 . \tag{4.4}$$

The sequence $\{ P_n H S_n^{-1} \hat{x}_n \}$ is also convergent, since

$$\| x_0 - P_n H S_n^{-1} \hat{x}_n \| \leq \| x_0 - P_n x_0 \| + \| P_n x_0 - P_n H S_n^{-1} \hat{x}_n \| \leq$$
$$\leq \| x_0 - P x_0 \| + \| P_n \| \, \| x_0 - H S_n^{-1} \hat{x}_n \| \to 0$$

and from here we deduce that

$$\lim P_n H S_n^{-1} \hat{x} = x_0 . \tag{4.5}$$

The sequence $\{ S_n^{-1} \hat{x}_n \}$ is also convergent, since

$$\frac{1}{\mu_n} H S_n^{-1} \hat{x}_n \to \frac{1}{\mu_0} x_0 \tag{4.6}$$

follows from the convergence $\hat{\mu}_n \to \mu_0$ and evidently

$$\frac{1}{\mu_0} H S_n^{-1} \hat{x}_n \to \frac{1}{\mu_0} x_0 . \tag{4.7}$$

Further

$$\| P_n (S_n^{-1} \hat{x}_n - \mu_0^{-1} H S_n^{-1} \hat{x}_n) \| \leq \| S_n^{-1} (\hat{x}_n - \mu_0^{-1} Q_n H S_n^{-1} \hat{x}_n) \| \leq$$
$$\leq \| S_n^{-1} ((\hat{\mu}_n)^{-1} \hat{H}_n \hat{x}_n - \mu_0^{-1} Q_n H S_n^{-1} \hat{x}_n) \| \leq | \mu_0^{-1} - (\hat{\mu}_n)^{-1} | \, \| Q_n H S_n^{-1} \hat{x}_n \| +$$
$$+ \frac{\eta_{0n}}{\mu_0}$$

so that

$$\| S_n^{-1} \hat{x}_n - \mu_0^{-1} x_0 \| \leq \| P_n (S_n^{-1} \hat{x}_n - \mu_0^{-1} H S_n^{-1} \hat{x}_n) \| + \| \mu_0^{-1} P_n H S_n^{-1} \hat{x}_n - \mu_0^{-1} x_0 \| \leq$$

$$\leq \frac{\eta_{0n}}{\varkappa(D)} + | \mu_0^{-1} - (\hat{\mu}_n)^{-1} | \, \| Q_n H S_n^{-1} \hat{x}_n \| \leq \frac{\eta_{0n}}{\varkappa(D)} + | \mu_0^{-1} - (\hat{\mu}_n)^{-1} | \, \| H \| \to 0 .$$

We have proved that $\lim\limits_{n\to\infty} S_n^{-1}\hat{x}_n = \mu_o^{-1}x_o = \lim\limits_{n\to\infty} \mu_o^{-1} H S_n^{-1}\hat{x}_n$,

i.e. $\mu_o^{-1} H x_o = H(\lim\limits_{n\to\infty} S_n^{-1}\hat{x}_n) = \lim\limits_{n\to\infty} H S_n^{-1}\hat{x}_n = x_o$.

The simplicity of μ_o implies that either $x_o = \vartheta$, or $x = c x_o$, $c \neq 0$. The first case is impossible because $\| S_n^{-1}\hat{x}_n \| = 1$, so that $x = c x_o$, $|C| = 1$.

We return to the chosen subsequence $\{S_{n_k}^{-1}\hat{x}_{n_k}\}$. Our purpose is to prove that the whole sequence $\{S_n^{-1}\hat{x}_n\}$ converges to $x_o = c x_o$. In the contrary case there exists an infinite number of indices $\{m'\}$ such that $\{S_{m'}^{-1}, \hat{x}_{m'}\}$ does not converge to x_o . Similarly as was shown above, it can be proved that there exists a subsequence $\{S_{m'_k}^{-1}\hat{x}_{m'_k}\}$ for which $\lim S_{m'_k}^{-1}\hat{x}_{m'_k} = w_o$, where $w_o = c' x_o$, $|C'| = 1$. From this it follows that $w_o = {c'}/{c}\, x_o$ and thus the convergence of the whole sequence $\{S_n^{-1} x_n\}$ can be obtained by help of the suitable choice of norm-factors. Thus the theorem is completely proved.

<u>Corollary</u> . Let μ_o be the dominant eigenvalue of the operator $A = C + D$ and x_o $(\|x_o\| = 1)$ the corresponding positive eigenvector. Let $\hat{\mu}_n, \hat{x}_n$ $(\|S_n^{-1}\hat{x}_n\| = 1)$ be the approximative eigenelements of the operators $\hat{A}_n = \hat{C}_n + \hat{D}_n$. Then $\lim\limits_{n\to\infty} S_n^{-1}\hat{x}_n = x_o$ holds in the norm of the space $L_2(\Omega)$ and $\lim\limits_{n\to\infty} \hat{\mu}_n = \mu_o$.

Proof. The validity of corollary is a consequence of the simplicity of the eigenvalue μ_o [6] and the preceding theorem.

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