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REMARK ON THE TENSOR ALGEBRAS

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In this note we define a local convex topology on the tensor algebra constructed on the sequence of DF -spaces (see [3]), which makes from this topological linear space [1] a locally m-convex algebra. All spaces are automatically supposed to be convex and Hausdorff.

Let A be an algebra over the complex numbers C. The pair $(A, \mathcal{C})$, where A is algebra and $\mathcal{C}$ topology on A, is called the topological algebra (further t. algebra), if the following conditions are satisfied:

1^o $(A, \mathcal{C})$ is a t. l. space,

2^o The multiplication in A is continuous in every component separately.

Under the proper t. algebra we understand every t. algebra in which the multiplication is continuous mapping from $A \times A$ into A. Locally m-convex algebra (briefly l. m-c. algebra) is the t. algebra in which a fundamental system of idempotent neighborhoods of zero (see [4]) exists.

We shall use the following assertion: Every proper t. algebra can be completed and its completion becomes again the proper t. algebra. (The proof of this theorem, which must be certainly known, is of the technical character and will be omitted.) It is an easy consequence of the last proposition that the completion of the l. m.-c. algebra is again l. m.-c. algebra.

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1) further only t. l. space

(In this connection remember the following fact, due to Wael - brock (see [5]): If the completion of a t. algebra is again t. algebra, then the product $M \cdot N = \{z = x \cdot y \in A : x \in M, y \in N\}$ of any bounded subsets M, N of A is also bounded.)

Let us give a sequence $E_1, E_2, \ldots$ of t. l. spaces, put $M = \sum_{n=1}^{\infty} E_n$, where $\sum$ means the topological direct sum and further $\mathop{X}_{n=1}^{\infty} \Sigma_n = \sum_{p=0}^{\infty} \otimes^p M$, where $\otimes^p M = M \otimes \ldots \otimes M$ (p-times) are the projective tensor products (see [3]). It is clear that $\mathop{X}_{n=1}^{\infty} E_n$ is also t. l. space. Call this space the projective tensor product of the sequence $(E_n)_{n \geq 1}$. From an algebraic point of view it is an algebra with the obviously defined multiplication (see [2]).

<u>Theorem</u>. Let $E_n \, (n \geq 1)$ be the sequence of DF-spaces. Denote by $[p]$ the set of all ordered sequences of p natural numbers. Then the projective tensor product of $(E_n)_{n \geq 1}$ is a DF-space and at the same time a l. m.-c. algebra, the completion of which is

$$\widehat{\mathop{X}_{n=1}^{\infty} E_n} = \sum_{p=0}^{\infty} \hat{\otimes}^p M = \sum_{p=0}^{\infty} \sum_{l \in [p]} \hat{\bigotimes_{i \in l}} E_i \, .$$

Proof: It follows from [3], p. 46, proposition 6.2, that we have $\otimes^p M = \sum_{l \in [p]} \bigotimes_{i \in l} E_i$; therefore $\mathop{X}_{n=1}^{\infty} E_n = \sum_{p=0}^{\infty} \sum_{l \in [p]} \bigotimes_{i \in l} E_i$. Now we shall show that $\mathop{X}_{n=1}^{\infty} E_n$ is l. m.-c. algebra. Let $(V_n^{\iota})_{\iota \in I_n}$ be a fundamental system of barrelled neighborhoods of zero in E_n. From the usual propositions about the t. l. spaces and their projective tensor products (see [1], [3]) follows the family of all the sets of the form

$$\sum_{p=0}^{\infty} \sum_{l \in [p]} \Gamma \left(\bigotimes_{i \in l} V_i^{\iota_i} \right) \quad \text{(where } \iota_i \in I_i \text{ for all } p = 0, 1, \ldots \text{ and } l \in [p] \text{)}$$

forms the fundamental system of neighborhoods of zero in $\overset{\infty}{\underset{n=1}{X}} E_n$

Take U as one of these neighborhoods (we omit the indexes ι_1).

If $x \in \mathcal{U}$, then $x = \sum_{j=1}^{k} x_{k_j}$ and for every $j = 1, 2, \dots k$ is $x_{k_j} \in \sum_{I \in [k_j]} \Gamma(\bigotimes_{i \in I} V_i)$ i.e. $x_{k_j} = \sum_{s=1}^{m_j} w_{j,s}$, where $w_{j,s} \in \Gamma(\bigotimes_{i \in I_{j,s}} V_i)$ and $I_{j,s} \in [k_j]$ for all $s = 1, 2, \dots, m$, so that $w_{j,s} = \sum_{\ell \in M_{j,s}} \lambda_\ell^{j,s} \bigotimes_{i \in I_{j,s}} t_i^\ell$ ($M_{j,s}$ being finite sets, we shall omit them for shortness), $t_i^\ell \in V_i = V_i^{\iota_{j,s}}$ and for all possible j, s holds

(1)
$$\sum_\ell |\lambda_\ell^{j,s}| \leq 1 .$$

We have altogether

(2)
$$x = \sum_{j=1}^{k} \sum_{s=1}^{m_j} \sum_{\ell} \lambda_\ell^{j,s} \bigotimes_{i \in I_{j,s}} t_i^\ell$$

If y is another arbitrary element of U we can put down

(3)
$$y = \sum_{J=1}^{R} \sum_{S=1}^{M_J} \sum_{L} \Lambda_L^{J,S} \bigotimes_{k \in I_{J,S}} T_k^L \quad ;$$

here all capitals have the same significance and properties as the corresponding small letters in (2), especially $T_k^L \in V_k = V_k^{\iota_{J,S}}$ and for all J , S we have

(4)
$$\sum_L |\Lambda_L^{J,S}| \leq 1 .$$

Now $x y = \sum_{j,s} \sum_{J,S} \sum_{\ell,\Lambda} \lambda_\ell^{j,s} \Lambda_L^{J,S} (\bigotimes_{i \in I_{j,s}} t_i^\ell) \otimes (\bigotimes_{k \in I_{J,S}} T_k^L)$. Suppose for instance j, s, J, S to be fixed and let $I_{j,s} = \{i_1, \dots, i_{k_j}\}$ $I_{J,S} = \{k_1, \dots, k_{K_J}\}$. Put $I_{j,s;J,S} = \{i_1, \dots, i_{k_j}, k_1, \dots, k_{K_J}\}$ and $\tau_h^{\ell,L} = t_{i_q}^\ell$ if h is equal to some $i_q \in I_{j,s}$ and $\tau_h^{\ell,L} = T_{k_Q}^L$, when h equals to some $k_Q \in I_{J,S}$. Then

$(\otimes t_i^{\ell}) \otimes (\otimes T_k^{L}) = \underset{h \in I_{j,k_j} J,S}{\otimes} \tau_h^{\ell,L} \in V_{i_1} \otimes \cdots \otimes V_{i_{k_j}} \otimes V_{k_1} \otimes \cdots \otimes V_{k_{K_J}}$.
Further is $\sum_{\ell,L} \lambda_\ell^{j,s} \Lambda_L^{J,S} \otimes \tau_h^{\ell,L} \in \Gamma(V_{i_1} \otimes \cdots \otimes V_{k_{K_J}})$ because from (1), (4) it follows that $\sum_{\ell,L} |\lambda_\ell^{j,S} \Lambda_L^{J,S}| \leq 1$ and so $xy \in \mathcal{U}$, i.e. U is idempotent and $\overset{\infty}{\underset{n=1}{X}} E_n$ so as its completion is l. m.-c. algebra (see the text before the theorem) and both are DF -spaces (see [3]).

Remark. The class of DF -spaces contains all normed spaces and so we obtain the projective tensor product of the sequence of normed spaces. But there is another way, how to construct a "natural" tensor product of the family of normed spaces. Let $(E_\iota)_{\iota \in I}$ be such a family. We can embed isomorfically every E_ι in the B -space $C(X_\iota)$ of continuous functions on the unit sphere X_ι of E_ι', which is compact in the topology $\sigma(E', E)$. Put $X = \prod_{\iota \in I} X_\iota$ and denote by θ_ι (ϑ_ι respectively) the mentioned embedding of E_ι in $C(X_\iota)$ (respectively of $C(X_\iota)$ in $C(X)$). In the Banach algebra $C(X)$ we shall consider the algebra A generated by the set $\bigcup_{\iota \in I} \vartheta_\iota(\theta_\iota(E_\iota))$. It is natural to call A the tensor product of the family $(E_\iota)_{\iota \in I}$. Notice in this connection one fact which is not without interest. Let I be the set of all natural numbers and $\mathcal{J}$ the set of all finite sequences of them. Let X_i $(i \in I)$ be compact spaces. If $J \subset I$ denote $C_J = C(\prod_{i \in J} X_i)$. For arbitrary J, K for which is $J \subset K \subset I$, we have canonical isomorphical embedding $C_J \rightarrow C_K$, so that C_J is closed in C_K. Then $C = \bigcup_{J \in \mathcal{J}} C_J$ is (algebraic) subspace in C_I.

Proposition: Denote by $\mathcal{T}_\omega$ the topology on C of inductive limit of the family C_J $(J \in \mathcal{J})$ (with respect to the mappings $C_J \to C$) and $\mathcal{C}_\sigma$ the topology on C as on the subspace of C_I . Then $\mathcal{C}_\sigma \subsetneq \mathcal{C}_\omega$.

Proof: For $J \in \mathcal{J}$ let $\mathcal{C}_J$ be the topology of the space C_J (it means the topology of uniform convergence on $\prod_{i \in J} X_i$ Evidently $\mathcal{C}_\sigma|_{C_J} = \mathcal{C}_J$ [2], but by the definition of $\mathcal{C}_\omega$, this is the finest local convex topology $\mathcal{C}$ for which every $\mathcal{C}|_{C_J}$ is coarser than the original topology $\mathcal{C}_J$ and so we have $\mathcal{C}_\omega \supset \mathcal{C}_\sigma$ (from this inclusion follows that we have even $\mathcal{C}_\omega|_{C_J} = \mathcal{C}_J$: in fact $\mathcal{C}_J = \mathcal{C}_\sigma|_{C_J} \subset \mathcal{C}_\omega|_{C_J} \subset \mathcal{C}_J$). Put $C_n = C(\prod_{i=1}^{n} X_i)$ and define the topology $\overline{\mathcal{C}}_\omega$ on $C = \bigcup_{n=1}^{\infty} C_n$ of inductive limit of spaces C_n . It is not hard to show that $\mathcal{C}_\omega = \overline{\mathcal{C}}_\omega$. But $(C, \overline{\mathcal{C}}_\omega)$ is complete LF-space (see [1], chap. II.) - so $(C, \mathcal{C}_\omega)$ is complete. Suppose that $\mathcal{C}_\omega = \mathcal{C}_\sigma$; then $(C, \mathcal{C}_\sigma)$ must be closed in C_I , but the closure of $(C, \mathcal{C}_\sigma)$ in C_I is the whole space C_I (see [1]) and so we obtain $C = C_I$ what is not possible.

We shall return to these questions later.

I wish to express my gratitude to Professor M. Katětov for introducing in these topics.

R e f e r e n c e s

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2) If $(Y, \mathcal{C})$ is a topological space, Z subset in Y , then $\mathcal{C}|_Z$ means the relative topology of Z .

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