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ON ITERATION METHODS AND THE FORMULATION OF THE ISBELL'S PROBLEM

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Lemma. If a mapping f from a set X into X has only one fixed point $x_0 \in X$, then x_0 is fixed for every mapping g from X into X commuting with f under composition.

Really,

$$f\,[g(x_0)] = g\,[f(x_0)] = g(x_0),$$

and $g(x_0)$ is a fixed point of f. As f has only one fixed point, $g(x_0) = x_0$.

This simple fact enables us to formulate some well known theorems concerning the existence of only one fixed point, with a stronger assertion. We shall show the application of this idea to iteration methods.

Let f be a continuous mapping from a topological space X into X. We shall denote, as usual, the composition of mappings f and g as $f \circ g$ and put $f^{n+1} = f \circ f^n$. We shall say that $X, Y \subset X$, is an iteration set of f with the fixed point x_0, if $x \in Y$ implies

$$\lim_{n \to \infty} f^n(x) = x_0 .$$

Theorem 1. Let X be a topological space, f and g two commuting continuous mappings of X into X. Let Y be an iteration set of f with the fixed point x_0; let $g(x_0) \in Y$. Then x_0 is a fixed point of g.

Proof. Since $g \circ f^n(x) = f^n \circ g(x)$, $g(x_0) \in Y$, we obtain

$$x_0 = \lim_{n\to\infty} f^n \circ g(x) = \lim_{n\to\infty} g \circ f^n(x) = g\left[\lim_{n\to\infty} f^n(x_0)\right] = g(x_0).$$

Theorem. 2. Let (X, ρ) be a complete metric space. Let f be a Lipschitz mapping from X into X, with a constant $\alpha < 1$ (that is, $\rho(f(x_1), f(x_2)) \leq \alpha\rho(x_1, x_2)$ for any $x_1 \in X$, $x_2 \in X$). Then every mapping g from X into X (continuity is not assumed) commuting with f possesses a fixed point.

Proof. The assertion follows from the above Lemma since f has exactly one fixed point.

The above lemma and the theorem 2 can be used for another formulation of the Isbell's problem, which asks: Is it true that every two continuous mappings from the closed unit interval into itself which are commutative under composition have a common fixed point?

Every theorem which deals with a fixed point can be formulated by means of the notion of commutativity owing to the following simple fact: x_0 is a fixed point of a mapping f if and only if the constant mapping h, $h(x) = x_0$ for each $x \in X$, is commutative with f.

Therefore the Isbell's problem is equivalent to the following question: Is it true that to every two commuting continuous mappings from the closed unit interval into itself a constant mapping exists which commutes with both of them?

According to our remarks the constant mapping in the last formulation can be replaced by every mapping with exactly one fixed point, especially by a Lipschitz mapping with constant α, $\alpha < 1$.