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# ASYMPTOTIC BEHAVIOUR OF THE SYSTEM OF TWO DIFFERENTIAL EQUATIONS

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This paper deals with asymptotic properties of solutions of a system

$$\dot{X} = P(X, Y, t),\qquad (1)$$
$$\dot{Y} = Q(X, Y, t),$$
$$\cdot = \frac{\mathrm{d}}{\mathrm{d}t}.$$

The special case of this system was studied in [2]. There C. Kulig generalized Butlewski's theorem [3], [4] about trajectories of a system

$$\ddot{R} = R(\dot{\varphi})^2,$$
$$(R^2\dot{\varphi})^{\cdot} = A(t)\, R^2,$$

which can be transformed into a system of the form

$$\dot{X} = \alpha(t) - X^2 + Y^2,$$
$$\dot{Y} = \beta(t) - 2XY$$

by the substitution

$$X = \dot{R}R^{-1}, \qquad Y = \dot{\varphi}.$$

In the paper [1], there is shown the advantage of transferring the considered system into the equation with complex-valued coefficients

$$\dot{z} = u(t) - z^2,$$

where $z = X + iY$, $u(t) = \alpha(t) + i\beta(t)$. A similar method is used in the present paper.

If we define

$$h(z, t) = P(X, Y, t) + iQ(X, Y, t),$$

where $z = X + iY$ we can replace (1) by

$$\dot{z} = h(z, t).\qquad (2)$$

We can easily see that $z = z(t)$ is a solution of (2) if and only if $z(t) = X(t) + iY(t)$ where $X = X(t)$, $Y = Y(t)$ is a solution of (1). The most part of considerations and formulations of results becomes essentially simpler if we investigate (2) instead of (1). Then it is suitable to give results in the complex form.

Let $I$ denote an open interval $(t_0, \infty)$, $-\infty \leqq t_0 < \infty$ and let $\Omega$ signify a domain in the complex plane. It will be always assumed that $h(z, t)$ is a continuous on $\Omega \times I$ complex-valued function. The description of the asymptotic behaviour of the trajectories of (2) in a neighbourhood of $b \in \Omega$ is the main object of this paper. The equation (2) can be written in a form

$$\dot{z} = G(z, t)\, g(z, t), \tag{3}$$

where $G(z, t) \geqq 0$ is a continuous on $\Omega \times I$ real-valued function and $g(z, t)$ is a continuous on $\Omega \times I$ complex-valued function. (We may take, e.g., $G(z, t) \equiv 1$ and $g(z, t) = h(z, t)$). If the function $g(z, t)$ is in a certain meaning "close" to the (continuous) function of a form $(z - b) f(z)$ we shall show that on suitable assumptions the asymptotic behaviour of the trajectories of (3) depends substantially only on the real part of $f(z)$. The main means of investigation are Lyapunov functions.

Complex numbers and functions are denoted by small Roman characters (except $t, s$), real numbers and functions by capital Roman and small Creek characters (except $I$, $K$). $\bar{d}$ signifies the complex conjugate number of $d$, $\operatorname{Re} d$ and $\operatorname{Im} d$ the real and imaginary parts of $d$, $|d|$ the absolute value of $d$.

We shall put

$$K_C = \{z\colon |z - b| < C\},$$
$$K_{C,D} = \{z\colon C < |z - b| < D\}.$$

If $\Gamma$ is a set, $\bar{\Gamma}$ will denote the closure of $\Gamma$, $\hat{\Gamma}$ the boundary of $\Gamma$.

Let $\Omega^\circ$ be an open subset of $\Omega \times I$. A point $(z_1, t_1) \in (\Omega \times I) \cap \hat{\Omega}^\circ$ is called an egress point of $\Omega^\circ$, with respect to the equation (2), if for every solution $z = z(t)$ of (2), satisfying the initial condition $z(t_1) = z_1$, there is an $\varepsilon > 0$ such that $(z(t), t) \in \Omega^\circ$ for $t_1 - \varepsilon \leqq t < t_1$. The set of all egress points of $\Omega^\circ$ will be denoted by $\Omega^\circ_e$.

Let $\Gamma_1$ be a topological space and $\Gamma_2 \subset \Gamma_1$. A continuous mapping of $\Gamma_1$ onto $\Gamma_2$ is called a retraction of $\Gamma_1$ onto $\Gamma_2$ if the restriction of this mapping to $\Gamma_2$ is the identity. If there exists a retraction of $\Gamma_1$ onto $\Gamma_2$, $\Gamma_2$ is called a retract of $\Gamma_1$.

The real-valued function $U(z, t)$ defined on $\Omega \times I$ is said to be of class $C^1$ if it has continuous partial derivatives with respect to $t$ and the real and imaginary parts of $z$.

The real-valued function $U(z, t)$ defined on an open subset of $\Omega \times I$ is said to possess a trajectory derivative $\dot{U}(z, t)$ at the point $(z_1, t_1)$ along the solution $z = z(t)$ of (2), $z(t_1) = z_1$ if $U(z(t), t)$ has a derivative at $t = t_1$; in this case

$$\dot{U}(z_1, t_1) = [U(z(t), t)]^{\cdot}_{t=t_1}.$$

This trajectory derivative exists if $U(z, t)$ is of class $C^1$; there holds

$$\dot{U}(z, t) = \frac{\partial U(z, t)}{\partial t} + \frac{\partial U(z, t)}{\partial \operatorname{Re} z} \operatorname{Re} h(z, t) + \frac{\partial U(z, t)}{\partial \operatorname{Im} z} \operatorname{Im} h(z, t).$$

An open subset $\Omega^\circ$ of $\Omega \times I$ will be called a $(U, V)$-subset of $\Omega \times I$ with respect to (2) if there exists a number of real-valued continuous functions

$$U_1(z, t), \ldots, U_L(z, t), V_1(z, t), \ldots, V_M(z, t),$$

on $\Omega \times I$ such that

$$\Omega^\circ = \{(z, t): U_N(z, t) < 0 \quad \text{for} \quad N = 1, 2, \ldots, L \quad \text{and} \quad V_N(z, t) < 0 \quad \text{for} \quad N = 1, 2, \ldots, M\}$$

and if $\mathscr{U}_\alpha$, $\mathscr{V}_\beta$ are the sets

$$\mathscr{U}_\alpha = \{(z, t): U_\alpha(z, t) = 0, U_N(z, t) \leqq 0 \quad \text{for} \quad N = 1, 2, \ldots, L \quad \text{and} \quad V_N(z, t) \leqq 0 \quad \text{for} \quad N = 1, 2, \ldots, M\},$$

$$\mathscr{V}_\beta = \{(z, t): V_\beta(z, t) = 0, U_N(z, t) \leqq 0 \quad \text{for} \quad N = 1, 2, \ldots, L \quad \text{and} \quad V_N(z, t) \leqq 0 \quad \text{for} \quad N = 1, 2, \ldots, M\},$$

then the trajectory derivatives $\dot{U}_\alpha$, $\dot{V}_\beta$ exist on $\mathscr{U}_\alpha$, $\mathscr{V}_\beta$ and satisfy

$$\dot{U}_\alpha(z, t) > 0 \quad \text{for} \quad (z, t) \in \mathscr{U}_\alpha,$$
$$\dot{V}_\beta(z, t) < 0 \quad \text{for} \quad (z, t) \in \mathscr{V}_\beta,$$

respectively, along all solutions through $(z, t)$. In this definition, either $L$ or $M$ can be zero.

**Lemma 1 (Ważewski).** (i) *Let $\Omega^\circ$ be a $(U, V)$-subset of $\Omega \times I$ with respect to* (2). *Then*

$$\Omega_e^0 = \bigcup_{\alpha=1}^{L} \mathscr{U}_\alpha - \bigcup_{\beta=1}^{M} \mathscr{M}_\beta;$$

(ii) *Let $\Omega^\circ$ be a $(U, V)$-subset of $\Omega \times I$ with respect to* (2). *Let $\Xi$ be a nonempty subset of $\Omega^\circ \cup \Omega_e^0$ satisfying the condition that $\Xi \cap \Omega_e^0$ is not a retract of $\Xi$ but is a retract of $\Omega_e^0$. Let us assume that $\Xi$ is compact and let $U_\alpha(z, t)$, $V_\beta(z, t)$ be of class $C^1$. Then there exists a point $(z_1, t_1) \in \Xi \cap \Omega^\circ$ and a solution $z = z(t)$ of* (2), *$z(t_1) = z_1$ such that $(z(t), t) \in \Omega^\circ$ for all $t$ from the right maximal interval of existence.*

For a proof, the reader is referred to [5].

Now, we shall bring two assertions the analogies of which can be found, e.g., in [6], [7].

Let functions $W(t)$, $C(t)$ be defined on an interval $[\alpha, \beta]$ and let $C(t)$ be continuous on $[\alpha, \beta]$.

**Lemma 2.** *Let $C(t)$ have left and right derivatives in each point of $(\alpha, \beta)$. Then the set of all points $t^* \in (\alpha, \beta)$ satisfying $\dot{C}_-(t^*) < \dot{C}_+(t^*)$ is at most countable.*

**Lemma 3.** *If* $\dot{C}(t) = W(t)$ *on* $[\alpha, \beta]$ *except on at most countable subset of* $[\alpha, \beta]$ *and fi* $W(t)$ *is (Lebesgue) integrable, then*

$$\int_{\alpha}^{\beta} W(t)\,\mathrm{d}t = C(\beta) - C(\alpha).$$

**Theorem 1.** *Let* $\varepsilon > 0$, $K = K_\varepsilon \subset \Omega$ *and let there exist a* $\delta$, $0 \leqq \delta < \varepsilon$, *a time* $T > t_0$ *and a function* $f(z)$ *such that*

1° $(z - b) f(z)$ *is continuous on* $K$;

2° $\operatorname{Re} f(z) \leqq 0$ *for any* $z \in K_{\delta,\varepsilon}$;

3° *there holds*

$$|z - b|\,|g(z, t) - (z - b) f(z)| \leqq \delta^2\,|\operatorname{Re} f(z)| \tag{4}$$

*for all* $z \in K_{\delta,\varepsilon}$ *and all* $t \geqq T$;

4° *a trajectory* $z = z(t)$ *of* (3) *satisfies at* $t = t_1 \geqq T$ *the inequality*

$$|z(t_1) - b| < \varepsilon. \tag{5}$$

*Then*

$$|z(t) - b| \leqq \max(\delta, |z(t_1) - b|) \tag{6}$$

*for all* $t \geqq t_1$.

*If, moreover,*

5° $\operatorname{Re} f(z) < 0$ *for any* $z \in K_{\delta,\varepsilon}$;

6° *there exists a continuous on K function* $F(z)$ *such that*

$$G(z, t) \geqq F(z) > 0$$

*for all* $z \in K_{\delta,\varepsilon}$ *and all* $t \geqq T$,
*then to each* $\varepsilon_1$, $\delta < \varepsilon_1 < \varepsilon$ *there exists a time* $t_2 \geqq t_1$ *so that*

$$|z(t) - b| < \varepsilon_1 \tag{7}$$

*for all* $t \geqq t_2$.

Proof. Let us define

$$d(z, t) = g(z, t) - (z - b) f(z)$$

and

$$V(z) = |z - b|^2.$$

The equations $V(z) = C$ represent a family of circles the centres of which are in the point $b$. Each point $z(t)$ of the trajectory $z = z(t)$ belongs to a circle with a parameter $C(t) = V(z(t))$.

Differentiation yields

$$\dot{C}(t) = 2 \operatorname{Re}[\overline{(z - b)}\,\dot{z}].$$

By substituting from (3) we get

$$\dot{C}(t) = 2G(z, t) \operatorname{Re} [\overline{(z - b)}\, g(z, t)].$$

Now, we shall prove the first part of the theorem. Let conditions 1°, 2°, 3°, 4° be satisfied. If $t \geqq T$ and $z(t) \in K_{\delta,\varepsilon}$, then

$$\begin{aligned} \dot{C}(t) &= 2G(z, t) \operatorname{Re} [\overline{(z - b)}\, ((z - b) f(z) + d(z, t))] \leqq \\ &\leqq 2G(z, t) (C(t) \operatorname{Re} f(z) + | z - b | \, | d(z, t) |). \end{aligned}$$

Using (4) and $\operatorname{Re} f(z) \leqq 0$ we get

$$\dot{C}(t) \leqq 2G(z, t) \operatorname{Re} f(z) (C(t) - \delta^2). \tag{8}$$

It is seen that $\dot{C}(t) \leqq 0$ if $t \geqq t_1$ and $\delta^2 < C(t) < \varepsilon^2$. The function $C(t)$ is non-increasing in each such a time.

If $| z(t_1) - b | > \delta$, then $C(t) \leqq C(t_1)$ for $t \geqq t_1$. Thus the trajectory $z = z(t)$ satisfies the inequality

$$| z(t) - b | \leqq | z(t_1) - b |$$

for all $t \geqq t_1$.

If $| z(t_1) - b | \leqq \delta$, then

$$| z(t) - b | \leqq \delta$$

for all $t \geqq t_1$ because contrariwise we get a contradiction to the previous assertion. This completes the proof of the first part of the theorem.

Now, we shall prove the second part. Let conditions 1° up to 6° be satisfied. Let $\varepsilon_1$, $\delta < \varepsilon_1 < \varepsilon$ be given arbitrary.

If $| z(t_1) - b | < \varepsilon_1$, then we have

$$| z(t) - b | \leqq \max (\delta, | z(t_1) - b |) < \varepsilon_1$$

for all $t \geqq t_1$ and we may take $t_2 = t_1$.

Let $| z(t_1) - b | \geqq \varepsilon_1$. It holds:

$$| z(t) - b | \leqq | z(t_1) - b |$$

for all $t \geqq t_1$. There exist constants $L > 0$, $M < 0$ so that $F(z) \geqq L$ and $\operatorname{Re} f(z) \leqq M$ for all $z \in \bar{K}_{\varepsilon_1, |z(t_1) - b|}$.

Using (8) we get

$$\dot{C}(t) \leqq 2ML(\varepsilon_1^2 - \delta^2) < 0$$

as long as $t \geqq t_1$ and $C(t) \geqq \varepsilon_1^2$. Thus there exists a time $t_2 \geqq t_1$ so that $C(t_2) < \varepsilon_1^2$. Clearly,

$$| z(t) - b | \leqq \max (\delta, | z(t_2) - b |) < \varepsilon_1$$

for all $t \geqq t_2$.

**Theorem 2.** *Let $\varepsilon > 0$, $K = K_\varepsilon \subset \Omega$ and let there exist a function $f(z)$ such that*

1° $(z - b) f(z)$ *is continuous on* $K$;

2° $\operatorname{Re} f(z) < 0$ *for all* $z \in K$;

3° $(z - b)\, g(z, t) \to (z - b)^2 f(z)$ *uniformly on* $K$ *as* $t \to \infty$;

4° *there exists a continuous on* $K_\varepsilon$ *function* $F(z)$ *so that*

$$G(z, t) \geqq F(z) \geqq 0$$

*on* $K_\varepsilon \times I$ *and* $F(z) \neq 0$ *on the set* $K_{0,\varepsilon}$.

*If there exists a sequence of real numbers* $\{t_N\}$, $N = 1, 2, \ldots$, *where* $t_N \to \infty$, *as* $N \to \infty$, *such that a trajectory* $z = z(t)$ *of* (3) *satisfies*

$$|z(t_N) - b| < \varepsilon \tag{9}$$

*for* $N = 1, 2, \ldots$,
*then*

$$\lim_{t \to \infty} z(t) = b.$$

Proof. Let the assumptions of the theorem be satisfied. Let $\varepsilon_1$, $0 < \varepsilon_1 < \varepsilon$ be an arbitrarily given number. Putting $\delta = \varepsilon_1/2$ we can observe that there exists a time $T > t_0$ such that

$$|(z - b)\, g(z, t) - (z - b)^2 f(z)| < \delta^2 \min_{z \in K} |\operatorname{Re} f(z)|$$

for all $t \geqq T$ and all $z \in K$.

Thus we have

$$|z - b|\, |g(z, t) - (z - b) f(z)| \leqq \delta^2 |\operatorname{Re} f(z)|$$

for all $z \in K$ and all $t \geqq T$.

From Theorem 1 it can be seen that there exists a time $t^* \geqq T$ such that

$$|z(t) - b| < \varepsilon_1$$

for all $t \geqq t^*$.

Hence we get

$$\lim_{t \to \infty} z(t) = b.$$

**Theorem 3.** *Let $\varepsilon > 0$, $K = K_\varepsilon \subset \Omega$ and let there exist a $\delta$, $0 \leqq \delta < \varepsilon$, a time $T > t_0$ and a function $f(z)$ such that*

1° $(z - b) f(z)$ *is continuous on* $K$;

2° $\operatorname{Re} f(z) \geqq 0$ *for any* $z \in K_{\delta,\varepsilon}$;

3° *there holds* (4) *for all* $z \in K_{\delta,\varepsilon}$ *and all* $t \geqq T$;

4° *a trajectory* $z = z(t)$ *of* (3) *satisfies at* $t = t_1 \geqq T$ *the condition*

$$\delta < |z(t_1) - b| < \varepsilon. \tag{10}$$

*Then*

$$| z(t) - b | \geqq | z(t_1) - b |$$

*for all* $t \geqq t_1$ *for which the solution* $z = z(t)$ *exists.*

*If, moreover,*

5° $\operatorname{Re} f(z) > 0$ *for any* $z \in K_{\delta, \varepsilon}$;

6° *there exists a continuous on K function* $F(z)$ *such that*

$$G(z, t) \geqq F(z) > 0$$

*for all* $z \in K_{\delta, \varepsilon}$ *and all* $t \geqq T$,
*then to any given* $\varepsilon_1$, $0 < \varepsilon_1 < \varepsilon$, *there exists a time* $t_2 \geqq t_1$ *such that*

$$| z(t) - b | > \varepsilon_1$$

*for all* $t \geqq t_2$ *for which the solution* $z = z(t)$ *exists.*

Proof. The proof is similar to that of Theorem 1.

**Theorem 4.** *Let* $\delta > 0$, $\bar{K}_\delta \subset \Omega$, $K = \hat{K}_\delta$ *and let there exist a function* $f(z)$ *so that*

1° $(z - b) f(z)$ *is continuous on* $K$;

2° $\operatorname{Re} f(z) > 0$ *on* $K$;

3° *there exists a time* $T > t_0$ *such that*

$$G(z, t) > 0$$

and

$$| g(z, t) - (z - b) f(z) | < \delta \operatorname{Re} f(z)$$

*for all* $z \in K$ *and* $t \geqq T$.

*Then, for every* $t_1 > T$, *there exists a trajectory* $z = z(t)$ *of* (3) *so that*

$$| z(t) - b | < \delta$$

*for all* $t \geqq t_1$.

Proof. Let us denote

$$d(z, t) = g(z, t) - (z - b) f(z),$$
$$U(z, t) = | z - b |^2 - \delta^2,$$
$$V(z, t) = T - t,$$
$$\Omega^\circ = \{(z, t) : | z - b | < \delta, t > T\},$$
$$\mathcal{U} = \{(z, t) : | z - b | = \delta, t \geqq T\}$$

and

$$\mathcal{V} = \{(z, t) : | z - b | \leqq \delta, t = T\}.$$

We can see that

$$\dot{U}(z, t) = 2G(z, t) \operatorname{Re} [\overline{(z - b)} g(z, t)]$$

on the set $\mathcal{U}$.

Similarly as in the proof of Theorem 1 we get

$$\dot{U}(z, t) \geqq 2G(z, t) (| z - b |^2 \operatorname{Re} f(z) - | z - b | \, | d(z, t) |).$$

Using assumptions 2°, 3° we can see that

$$\dot{U}(z, t) \geqq 2G(z, t)\,\delta(\delta \operatorname{Re} f(z) - |d(z, t)|) > 0$$

on the set $\mathscr{U}$.

Clearly

$$\dot{V}(z, t) = -1 < 0$$

on the set $\mathscr{V}$.

Thus $\Omega^\circ$ is a $(U, V)$-subset of $\Omega \times I$ with respect to (3).

Using the first part of Lemma 1 we see that the set of all egress points of $\Omega^\circ$ is

$$\Omega_e^0 = \{(z, t): |z - b| = \delta, t > T\}.$$

Let $t_1 > T$ be arbitrary and let us denote

$$\Xi = \{(z, t): |z - b| \leqq \delta, t = t_1\}.$$

The set

$$\Xi \cap \Omega_e^0 = \{(z, t): |z - b| = \delta, t = t_1\}$$

is a retract of $\Omega_e^0$, as it can be seen by choosing the retraction $(z, t) \mapsto (z, t_1)$.

$\Xi \cap \Omega_e^0$ is not a retract of $\Xi$. For if there exists a retraction $w: \Xi \to \Xi \cap \Omega_e^0$, then there exists a continuous map of $\Xi$ into itself, e.g. $-w$, without fixed points, which is, by the fixed point theorem of Brouwer, impossible.

Using Lemma 1 we can see that there exists a trajectory $z = z(t)$ of (3) such that

$$|z(t) - b| < \delta$$

for all $t \geqq t_1$.

**Theorem 5.** *Let* $\varepsilon > 0$, $K = \bar{K}_\varepsilon \subset \Omega$. *Let there exist a function* $f(z)$ *so that conditions* 1°, 3°, 4° *of Theorem* 2 *are fulfilled and* $\operatorname{Re} f(z) > 0$ *for all* $z \in K$. *Then there exists a trajectory* $z = z(t)$ *of* (3) *such that*

$$\lim_{t \to \infty} z(t) = b.$$

Proof. There exists a time $t_1 > t_0$ such that

$$|z - b|\,|g(z, t) - (z - b) f(z)| < \frac{\varepsilon^2}{4} \min_{z \in K} \operatorname{Re} f(z)$$

for all $t \geqq t_1$ and $z \in K$.

Hence

$$|g(z, t) - (z - b) f(z)| < \frac{\varepsilon}{2} \operatorname{Re} f(z)$$

for all $z \in \hat{K}_{\varepsilon/2}$ and all $t \geqq t_1$.

Let $t_2 > t_1$. By Theorem 4 there exists a trajectory $z = z(t)$ of (3) so that

$$|z(t) - b| < \frac{\varepsilon}{2} \tag{11}$$

for all $t \geqq t_2$.

Let $\varepsilon_1$, $0 < \varepsilon_1 < \varepsilon$ be arbitrary. Putting $\delta = \varepsilon_1/2$ we can see that there exists a time $t_3 \geqq t_2$ such that

$$|z - b|\,|g(z, t) - (z - b)f(z))| < \delta^2 \min_{z \in K} \operatorname{Re} f(z)$$

for all $t \geqq t_3$ and $z \in K$.

Suppose that.

$$\delta < |z(t^*) - b| < \varepsilon$$

for a time $t^* \geqq t_3$.

By Theorem 3 there exists a time $t_4 \geqq t^*$ so that

$$|z(t_4) - b| > \frac{\varepsilon}{2}.$$

This gives a contradiction. Thus

$$|z(t) - b| \leqq \delta < \varepsilon_1$$

for all $t \geqq t_3$ and the theorem is proved.

**Theorem 6.** *Let $\varepsilon > 0$, $K = K_\varepsilon \subset \Omega$ and let there exist a function $f(z)$ so that*

1° *$(z - b)f(z)$ is continuous on $K$;*

2° *there exist numbers $L$, $M$ such that*

$$0 < M \leqq |\operatorname{Re} f(z)| \leqq L$$

*for any $z \in K$;*

3° *there exist constants $M_1$, $M_2$ so that*

$$0 < M_1 \leqq G(z, t) \leqq M_2$$

*on the set $K \times I$;*

4° *there exists a time $t_1 > t_0$ and a continuous function $D(t)$ such that*

$$\int_{t_1}^{\infty} D(t)\,\mathrm{d}t < \infty$$

*and*

$$|g(z, t) - (z - b)f(z)| \leqq D(t)$$

*for all $z \in K$ and all $t \geqq t_1$.*

*If $z = z(t)$ is a trajectory of (3) such that*

$$|z(t) - b| < \varepsilon$$

*for all $t \geqq t_1$, then*

$$\lim_{t \to \infty} z(t) = b$$

*and*

$$\int_{t_1}^{\infty} |z(t) - b|\,\mathrm{d}t < \infty.$$

Proof. Let us define

$$d(z, t) = g(z, t) - (z - b) f(z)$$

and

$$V(z) = |z - b|.$$

The equations $V(z) = C$ represent a family of circles the centres of which are in the point $b$. Each point $z(t)$ of the trajectory $z = z(t)$ belongs to the circle with a parameter (radius) $C(t) = V[z(t)] < \varepsilon$.

There holds

$$\dot{C}(t) = \frac{(C^2(t))^{\cdot}}{2C(t)}$$

as far as $t \geqq t_1$ and $z(t) \neq b$.

From the proof of Theorem 1 it follows that

$$(C^2(t))^{\cdot} = 2G(z, t)\left(|z - b|^2 \operatorname{Re} f(z) + \operatorname{Re}\left[\overline{(z - b)}\, d(z, t)\right]\right)$$

for all $t \geqq t_1$.

Thus the function $\dot{C}(t)$ is continuous in any time $t \geqq t_1$, $z(t) \neq b$ and there holds

$$\dot{C}(t) = G(z, t)\left(C(t) \operatorname{Re} f(z) + \frac{\operatorname{Re}\left[\overline{(z - b)}\, d(z, t)\right]}{|z - b|}\right).$$

Now, we get

$$|\dot{C}(t) - G(z, t)\, C(t) \operatorname{Re} f(z)| \leqq M_2 D(t)$$

and

$$|\dot{C}(t)| \leqq M_2(LC(t) + |d(z, t)|)$$

as far as $t \geqq t_1$ and $z(t) \neq b$.

Let $t^* \geqq t_1$ be such a time that $z(t^*) = b$. A simple calculation shows that the right derivative

$$\dot{C}_+(t^*) = G(b, t^*)\, |g(b, t^*)|$$

and the left derivative

$$\dot{C}_-(t^*) = -G(b, t^*)\, |g(b, t^*)|.$$

Therefore the derivative $\dot{C}(t^*)$ exists if and only if $g(b, t^*) = 0$. One can see that in this case $\dot{C}(t^*) = 0$.

Let us define

(12) $$W(t) = \begin{cases} \dot{C}(t) & \text{as far as } t \geqq t_1 \text{ and } z(t) \neq b, \\ 0 & \text{as far as } t \geqq t_1 \text{ and } z(t) = b. \end{cases}$$

The function $W(t)$ satisfies estimates

(13) $$|W(t) - G(z, t)\, C(t) \operatorname{Re} f(z)| \leqq M_2 D(t),$$

(14) $$|W(t)| \leqq M_2(LC(t) + |d(z, t)|).$$

Observe that $W(t)$ is continuous in any time $t \geqq t_1$, $z(t) \neq b$ and in any time $t^* \geqq t_1$, $z(t^*) = b$ in which the derivative $\dot{C}(t^*)$ exists. The second argument follows from (14).

Let $t \in (t_1, \infty)$. From Lemma 2 it follows that the set of all points of the discontinuity of $W(s)$ in $(t_1, t)$ is at most countable. Using (14) we can see that $W(s)$ is bounded on a closed interval $[t_1, t]$. Thus the function $W(s)$ is integrable over $[t_1, t]$.

From Lemma 3 it follows

$$\int_{t_1}^{t} W(s)\,\mathrm{d}s = C(t) - C(t_1). \tag{15}$$

Now, we shall only continue the proof for the case

$$-L \leqq \operatorname{Re} f(z) \leqq -M < 0$$

because the proof of the case

$$0 < M \leqq \operatorname{Re} f(z) \leqq L$$

is very similar.

Using (13) and (14) we obtain

$$-M_2(LC(t) + D(t)) \leqq W(t) \leqq -M_1MC(t) + M_2D(t) \tag{16}$$

and

$$|W(t)| \leqq M_2(LC(t) + D(t)). \tag{17}$$

Let us integrate these inequalities over $[t_1, t]$ and let us suppose that

$$\int_{t_1}^{\infty} C(t)\,\mathrm{d}t = \infty. \tag{18}$$

From the integrated inequalities (16) it can be seen that

$$\lim_{t \to \infty} \int_{t_1}^{t} W(s)\,\mathrm{d}s = -\infty.$$

Hence

$$\lim_{t \to \infty} C(t) = -\infty$$

and we get a contradiction.

Therefore

$$\int_{t_1}^{\infty} C(t)\,\mathrm{d}t < \infty. \tag{19}$$

From the integrated inequality (17) we have

$$\int_{t_1}^{\infty} | W(s) | \, ds < \infty.$$

Thus

$$\int_{t_1}^{\infty} W(t) \, dt$$

converges and there exists a finite limit

$$\lim_{t \to \infty} C(t) = 0.$$

Therefore we have

$$\lim_{t \to \infty} z(t) = b$$

and

$$\int_{t_1}^{\infty} | z(t) - b | \, dt < \infty.$$

The theorem is proved.

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