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BOUNDS FOR SOLUTIONS OF THE EQUATION $[p(t)\,x']' + q(t)x = h(t,x,x')$

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The aim of this paper is to find bounds for solutions of the equation

$$[p(t)\,x']' + q(t)\,x = h(t, x, x'). \tag{1}$$

Throughout the paper we shall suppose $p(t)$, $q(t)$ to be continuous functions on the interval $J = [a, \infty)$, $p(t) > 0$ and $h(t, x, x')$ continuous on D,

$$D : t \in J, \quad -\infty < x, \quad x' < \infty.$$

Let us denote $u(t)$, $v(t)$ the solutions of

$$[p(t)\,y']' + q(t)\,y = 0 \tag{2}$$

defined on J and satisfying initial conditions $u(a) = 1$, $u'(a) = 0$; $v(a) = 0$, $v'(a) = 1$. Then $y(t) = c_1u(t) + c_2v(t)$ is a solution of (2) and $y(a) = c_1$, $y'(a) = c_2$. Define

$$c = |\,c_1\,| + |\,c_2\,|, \quad a(t) = \max\,(|\,u(t), |\,v(t)\,|), \quad b(t) = \max\,(|\,u'(t)\,|, |\,v'(t)\,|).$$

Theorem 1. *Suppose that there exists a continuous function $\omega(t, u, v)$ defined for $t \in J$ and $0 \leqq u, v < \infty$ with the following properties*

i) $|\,h(t, u, v)\,| \leqq \omega(t, |\,u\,|, |\,v\,|)$ *on D;* (3)

ii) *$\omega(t, u, v)$ is nonnegative and nondecreasing in u, v for every fixed $t \in J$;* (4)

iii) *there is a constant $d > 0$ such that*

$$\int_a^\infty \omega(t, \mathrm{d}a(t), \mathrm{d}b(t))\,\mathrm{d}t < \infty. \tag{5}$$

Then there exists a $t_1 \geqq t_0$ such that every solution of $y(t)$ of (1) *satisfying initial conditions*

$$y(t_1) = c_1, \qquad y'(t_1) = c_2, \qquad |\,c_1\,| + |\,c_2\,| < d$$

exists for all $t \geqq t_1$ *and*

$$|y(t)| \leqq da(t), \qquad |y'(t)| \leqq db(t). \tag{6}$$

Proof. From (5) it follows that there is a $t_1 \geqq t_0$ such that

$$\int_{t_1}^{\infty} a(s)\,\omega(s, da(s), db(s))\,ds < d - c.$$

Consider the equation

$$x(t) = y(t) + \int_{t_1}^{t} [u(t)v(s) - u(s)\,v(t)]\,h(s, x(s), x'(s))\,ds. \tag{7}$$

Every function $x(t)$ satisfying this equation is a solution of (1) with the same initial conditions in t_1 as $y(t)$. From (7) we receive

$$x'(t) = y'(t) + \int_{t_1}^{t} [u'(t)\,v(s) - u(s)\,v'(t)]\,h(s, x(s), x'(s))\,ds,$$

so that

$$|x(t)| \leqq a(t)\left[c + \int_{t_1}^{t} a(s)\,\omega(s, |x(s)|, |x'(s)|)\,ds\right]$$

and

$$|x'(t)| \leqq b(t)\left[c + \int_{t_1}^{t} a(s)\,\omega(s, |x(s)|, |x'(s)|)\,ds\right].$$

If we put

$$z(t) = c + \int_{t_1}^{t} a(s)\,\omega(s, |x(s)|, |x'(s)|)\,ds$$

than

$$|x(t)| \leqq a(t)\,z(t), \qquad |x'(t)| \leqq b(t)\,z(t) \tag{8}$$

and

$$z(t) \leqq c + \int_{t_1}^{t} a(s)\,\omega(s, a(s)\,z(s), b(s)\,z(s))\,ds.$$

Let us suppose that the solution $x(t)$ exists on the interval $[t_1, T)$. Then the function $z(t)$ exists on this interval too and is $z(t) < d$.

In fact it is

$$z(t_1) \leqq c < d$$

so that the inequality $z(t) \leqq d$ holds in a certain right neighbourhood of t_1. Let t_2 be the smallest value of t such that $z(t_2) = d$. Then

$$d = z(t_2) \leqq c + \int_{t_1}^{t_2} a(s)\,\omega(s, a(s)\,z(s), b(s)\,z(s))\,\mathrm{d}s \leqq$$

$$\leqq c + \int_{t_1}^{t_2} a(s)\,\omega(s, da(s), db(s))\,\mathrm{d}s \leqq$$

$$\leqq c + \int_{t_1}^{\infty} a(s)\,\omega(s, \mathrm{d}a(s), \mathrm{d}b(s))\,\mathrm{d}s < d$$

which is a contradiction. Thus $z(t) < d$ on the whole interval $[t_1, T)$ and with respect to (8) we get (6) on $[t_1, T)$. From these inequalities we conclude that the solution $x(t)$ exists for all $t \geqq t_1$ and satisfies (6).

Lemma. *Let $\varphi(t)$, $p(t)$, $g(t)$ are continuous functions defined on $[a, b)$, $\varphi(t) \geqq 0$, $p(t) > 0$, $g(t) \geqq 0$. Let $\omega(y)$ be continuous, positive and nondecreasing for $y \geqq 0$. Let $y(t)$ be a nonnegative, continuous function defined on $[a, b)$ and satisfying*

$$y(t) \leqq \varphi(t) + \int_a^t \frac{1}{p(s)} \int_a^s g(\tau)\,\omega[y(\tau)]\,\mathrm{d}\tau\,\mathrm{d}s. \tag{9}$$

Then it is

$$y(t) \leqq \Omega^{-1}\left[\Omega[\Phi(t)] + \int_a^t \frac{1}{p(s)} \int_a^s g(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right], \tag{10}$$

where $\Omega(t) = \int_a^t \frac{\mathrm{d}s}{\omega(s)}$, Ω^{-1} *is its inverse function and* $\Phi(t) = \max_{s \in [a,t]} \varphi(s)$. *The inequality* (10) *remains valid for all $t \geqq a$ for which the right hand side is defined.*

Proof. Let us define on $[a, b)$ the function $Y(t)$ by means of the relation $Y(t) = \max_{a \leqq s \leqq t} y(s)$. Then $Y(t)$ is positive nondecreasing and since the function $\omega(y)$ has the same property for $y \geqq 0$ we get from (9)

$$y(t) \leqq \Phi(t) + \int_a^t \frac{1}{p(s)} \int_a^s g(\tau)\,\omega[Y(\tau)]\,\mathrm{d}\tau\,\mathrm{d}s$$

Let $\xi \in [a, t]$ be a number in which the function $y(t)$ reaches its greatest value. Then

$$Y(t) = y(\xi) \leqq \Phi(\xi) + \int_a^\xi \frac{1}{p(s)} \int_a^s g(\tau)\,\omega[Y(\tau)]\,d\tau\,ds \leqq$$

$$\leqq \Phi(t) + \int_a^t \omega[Y(s)] \frac{1}{p(s)} \int_a^s g(\tau)\,d\tau\,ds$$

Using Bihari's lemma [1] we get

$$Y(t) \leqq \Omega^{-1}\left[\Omega[\Phi(t)] + \int_a^t \frac{1}{p(s)} \int_a^s g(\tau)\,d\tau\,ds\right]$$

and since $y(t) \leqq Y(t)$ the proof is complete.

For $\omega(y) = y$ we receive the following consequence.

Consequence: Let $\varphi(t)$, $p(t)$, $g(t)$ be continuous functions defined on $[a, b)$, $\varphi(t) \geqq 0$, $p(t) > 0$, $g(t) \geqq 0$. Let $y(t)$ be a nonnegative, continuous function defined on $[a, b)$ and satisfying the inequality

$$y(t) \leqq \varphi(t) + \int_a^t \frac{1}{p(s)} \int_a^s g(\tau)\,y(\tau)\,d\tau\,ds.$$

Then it is

$$y(t) \leqq \Phi(t) \exp\left\{\int_a^t \frac{1}{p(s)} \int_a^s g(\tau)\,d\tau\,ds\right\}$$

Theorem 2. *Let $p(t) > 0$ be continuous on $J = [a, \infty)$. Let $h(t, u, v)$ be continuous in $D : t \in J$, $-\infty < u$, $v < \infty$. Let there exist functions $g(t)$, $\omega(y)$ with the following properties*

$g(t)$ is nonnegative and continuous in J,

$\omega(y)$ is continuous for $y \geqq 0$, positive and

$$\int_a^\infty \frac{dt}{\omega(t)} = \infty; \tag{11}$$

$$|\,h(t, u, v)\,| \leqq g(t)\,\omega(|\,u\,|) \text{ in } D.$$

Then every solution of

$$(p(t)\,x')' = h(t, x, x'), \qquad x(t_0) = x_0, \qquad x'(t_0) = x'_0 \tag{12}$$

is defined for $t \geqq a$ and it holds

$$|x(t)| \leqq \Phi(t)\,\Omega^{-1}\left[\Omega(\Phi(t)) + \int_a^t \frac{1}{p(s)} \int_a^s g(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right] \tag{13}$$

where

$$\Phi(t) = \max_{a \leqq s \leqq t} \left| x_0 + x_0' \int_a^s \frac{\mathrm{d}\tau}{p(\tau)} \right|$$

and Ω has the same meaning as in preceding lemma.

Proof. Integrating twice the equation (12) from a to t we get

$$x(t) = x_0 + x_0' \int_a^t \frac{\mathrm{d}s}{p(s)} + \int_a^t \frac{\mathrm{d}s}{p(s)} \int_a^s h(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

If we denote

$$\Phi(t) = \max_{a \leqq s \leqq t} \left| x_0 + x_0' \int_a^s \frac{\mathrm{d}\tau}{p(\tau)} \right|$$

we receive

$$|x(t)| \leqq \Phi(t) + \int_a^t \frac{\mathrm{d}s}{p(s)} \int_a^s g(\tau)\,\omega(|x(\tau)|)\,\mathrm{d}\tau\,\mathrm{d}s.$$

Using the preceding lemma we get (13) on a certain interval $[a, T)$ on which the right hand side is defined. With respect to (11) the right hand side is meaningful on the whole interval on which $x(t)$ exists. From the boundedness of $x(t)$ and $x'(t)$ we conclude the existence of solutions of (1) for all $t \geqq a$. This completes the proof.

The following corollary is a consequence of the preceding theorem for $\omega(u) = u$.

Corollary. *Let $p(t) > 0$ be a continuous function defined on J. Let $h(t, u, v)$ be continuous on $D: t \in J,\ -\infty < u,\ v < \infty$. Let there exist a continuous nonnegative function $g(t)$ such that*

$$|h(t, u, u')| \leqq g(t)\,|u|$$

Then every solution $x(t)$ of (12) is defined for $t \geqq a$ and

$$|x(t)| \leqq \Phi(t) \exp\left\{\int_a^t \frac{1}{p(s)} \int_a^s g(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right\}$$

REFERENCES

[1] Bihari, I., *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations.* Acta Math. Acad. Sci. Hungar 7 (1956), 71—94.

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