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THE APPROXIMATION OF FUNCTIONS IN THE SENSE OF TCHEBYCHEV I

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INTRODUCTION

This paper gives a sufficiently general and complete approach to the theory of the linear approximation of functions in the sense of Tchebychev. The theory will also serve as a basis for next papers which, perhaps, will follow.

The concepts $\dim_M V$, $\mu(M)$ and the concept of a minimal set are generalizations of the analogous concepts of [4]. The concept of a representative subset is original.

In the paper the following notations are used:

R is the space of all real numbers, C is the space of all complex numbers, N is the system of all natural numbers, $N_0 = N \cup \{0\}$.

For $x \in C$, $x \neq 0$ we define $\operatorname{sign} x = \frac{x}{|x|}$ and $\operatorname{sign} 0 = 0$. For all $x \in C$ we have $x \,.\, \operatorname{sign} \bar{x} = |x|$.

$\emptyset$ is the notation for the empty set.

If X, B are sets, then the system of all mappings of the set B into the set X will be denoted by X^B. We have $X^\emptyset = \{\emptyset\}$.

Let X, B be sets, $f \in X^B$, $M \subset B$. Then the restriction of the function f to the set M will be denoted by f_M. (Exactly written $f_M = f \cap (M \times X)$.) We have $f_M \in X^M$, $f_\emptyset = \emptyset$.

If M is a set, then card M means the cardinal number of M (the number of the elements of M).

If $P \in X^B$, then $P \equiv o$ means: $P(x) = o$ for all $x \in B$.

We denote $\det Q_k(x_j) = \begin{vmatrix} Q_1(x_1) \dots Q_1(x_n) \\ \vdots \\ Q_n(x_1) \dots Q_n(x_n) \end{vmatrix}$ if the order of the determinant is evident from the context.

T-space means a topological space, L-space means a linear space and NL-space means a normed linear space.

Remark. X will denote the space in which the considered functions have their values. We shall mostly assume that X is an NL-space over a field S, where $S = R$

or $S = C$. The zero vector of X will be denoted by o. In some theorems we shall assume that X is a strictly normed NL-space; that means that $| x + y | = | x | + | y |$ and $| x | = | y |$ implies $x = y$.

The most important results (Chapter 2) are derived by the assumption $X = S = R$ (real functions) or $X = S = C$ (complex functions).

Theorem (Helly). Let $n \in N$. Let $\{A_i / i \in I\}$ be a system of convex and closed subsets of R^n containing at least $n + 1$ sets. Let every $n + 1$ distinct sets A_i have a common point and suppose that there exists a finite subsystem, the intersection of which is bounded. Then $\bigcap_{i \in I} A_i \neq \emptyset$.

Proof is given e.g. in [2].

Definition. The approximation problem may be formulated in general in the following way:

Let Y be a set, ϱ be a mapping of $Y \times Y$ into $\langle 0, +\infty \rangle$. Let $V \subset Y$, $V \neq \emptyset, f \in Y$. Let us denote $\mu = \inf_{Q \in V} \varrho(Q, f)$.

An element $P \in V$ is called the element of the best approximation for f in V iff $\varrho(P, f) = \mu$.

Remark. ϱ has mostly the properties of a metric or of a norm. However, we also admit the cases $\varrho(Q, f) = +\infty$ and $\mu = +\infty$. This approach enables us to deal with functions which may be unbounded.

Remark. If Y is a space of functions, then the functions $Q \in V$ are called "polynomials".

1. FUNCTIONS WITH THE VALUES IN AN NL-SPACE

1.1. The Independence and the Dimension in a Subset

Assumption (for § 1.1.). Let B be a set, let X be an L-space over a field S.

Definition 1. If $f, g \in X^B$, we define a function $f + g \in X^B$ by the relation $(f + g)(x) = f(x) + g(x)$. If $f \in X^B$ and $c \in S$, we define a function $c \,.\, f \in X^B$ by the relation $(c \,.\, f)(x) = c \,.\, f(x)$.

Remark. X^B is then an L-space over S.

Definition 2. Let $M \subset B$. Functions $Q_1, \ldots, Q_n \in X^B$ will be called **independent in the set M** iff the restrictions of them to the set M are independent as functions of X^M, i.e. iff there do not exist numbers $a_1, \ldots, a_n \in S$ not all zero such that $\sum_{k=1}^{n} a_k \cdot$ $\cdot\, Q_k(x) = o$ for all $x \in M$.

Theorem 1. Let $M \subset D \subset B$. If functions $Q_1, \ldots, Q_n \in X^B$ are independent in M, then they are independent in D, too.

Theorem 2. Let V be a subspace of X^B, let $M \subset B$. Let us denote $W = \{Q_M / Q \in V\}$.

(1) W is a subspace of X^M.

(2) If V is of a finite dimension, then W is of a finite dimension, too and $\dim W \leqq \leqq \dim V$. (See Theorem 1 for $D = B$.)

Definition 3. Let V be a subspace of X^B, let $M \subset B$. Let us denote $W = \{Q_M / Q \in V\}$.

(1) Let W be of a finite dimension. Then we define $\dim_M V = \dim W$. This number will be called **the dimension** of V **in the set** M.

(2) We shall say that functions $Q_1, \ldots, Q_n \in V$ are **generating for** V (form **a basis of** V) **in the set** M iff the restrictions of them to M are generating for W (form a basis of W).

Theorem 3. Let V be a subspace of X^B, let $M \subset D \subset B$. If functions $Q_1, \ldots, Q_n \in V$ are generating in D, then they are generating in M, too.

Theorem 4. Let V be a subspace of X^B.

(1) We have $\dim_{\emptyset} V = 0$. If V is of a finite dimension, then $\dim_B V = \dim V$.

(2) Let $M \subset D \subset B$ and let $\dim_D V$ exist. Then $\dim_M V$ exists, too and we have $\dim_M V \leqq \dim_D V$.

(3) Let V be of a finite dimension, let $M \subset B$. Then $\dim_M V = \dim V$ iff this condition holds: If $P \in V$ is such that $P(x) = o$ for all $x \in M$, then $P \equiv o$ (i.e. $P(x) = o$ for all $x \in B$).

Proof. (1) is evident, (2) follows from Theorem 1.

(3) Let $Q_1, \ldots, Q_n$ form a basis of V.

a) Let $\dim_M V = \dim V$. Let $P \in V$ be such that $P(x) = o$ for all $x \in M$. We can express P in the form $P = \sum_{k=1}^{n} a_k Q_k$; hence $\sum_{k=1}^{n} a_k Q_k(x) = o$ for all $x \in M$. As $Q_1, \ldots, Q_n$ are generating in M, they form a basis of V in M and therefore they are independent in M. Hence $a_1 = \ldots = a_n = 0$ and $P \equiv o$.

b) Let $\dim_M V < \dim V$. Then $Q_1, \ldots, Q_n$ are dependent in M and there exist $a_1, \ldots, a_n \in S$ not all zero such that $\sum_{k=1}^{n} a_k Q_k(x) = o$ for all $x \in M$. Let us put $P = \sum_{k=1}^{n} a_k Q_k$. Then $P \not\equiv o$ and $P(x) = o$ for all $x \in M$.

Theorem 5. Let V be an n-dimensional subspace of X^B.

(1) Let $M \subset B$ be such that $\dim_M V = m < n$. Then there exists $z \in B$ such that $\dim_{M \cup \{z\}} V \geqq m + 1$.

(2) There exist points $x_1, \ldots, x_m \in B$ such that $m \leqq n$ and $\dim_{\{x_1, \ldots, x_m\}} V = n$.

Proof. (1) Let us admit that such z does not exist. Let $Q_1, \ldots, Q_m \in V$ form a basis of V in M. We can choose $Q \in V$ such that $Q_1, \ldots, Q_m, Q$ are independent (in B). For each $z \in B$ the functions $Q_1, \ldots, Q_m, Q$ are dependent in $M \cup \{z\}$ and there exist numbers $a_1(z), \ldots, a_m(z), a(z) \in S$ not all zero such that $\sum_{i=1}^{m} a_i(z) \, . \, Q_i(x) + a(z) \, . \, Q(x) = o$ for all $x \in M \cup \{z\}$. The functions $Q_1, \ldots, Q_m$ are independent in M, hence necessarily $a(z) \neq 0$. We may assume $a(z) = 1$ (otherwise we can divide all $a_i(z)$ by $a(z)$). Specially $\sum_{i=1}^{m} a_i(z) \, . \, Q_i(z) + Q(z) = o$ for all $z \in B$. Let us choose arbitrary $z, y \in B$. Then for each $x \in M$ we have $\sum_{i=1}^{m} a_i(z) \, . \, Q_i(x) + Q(x) = o = \sum_{i=1}^{m} a_i(y) \, . \, Q_i(x) + Q(x)$, hence $\sum_{i=1}^{m} [a_i(z) - a_i(y)] \, . \, Q_i(x) = o$. Since $Q_1, \ldots, Q_m$ are independent in M, we have $a_i(z) = a_i(y)$ for $i = 1, \ldots, m$. Hence the numbers $a_i(z)$ are not dependent on the point z and we may write only a_i.

For each $z \in B$ we have $\sum_{i=1}^{m} a_i \, . \, Q_i(z) + Q(z) = o$ which is a contradiction with the independence of $Q_1, \ldots, Q_m, Q$ in B.

(2) follows directly from (1).

1.2. The Approximation

Assumption (for § 1.2.). Let B be a set, let X be an NL-space over a field S, where $S = R$ or $S = C$. The norm of $x \in X$ will be denoted only by $|x|$. Let V be an n-dimensional subspace of X^B.

Theorem 6. Let $x_1, \ldots, x_m \in B$ be such points that $\dim_{\{x_1, \ldots, x_m\}} V = n$. (Se eTheorem 5(2).) Let functions $Q_1, \ldots, Q_n$ form a basis of V. Then for each number $d \geqq 0$ the set $M_d = \{(a_1, \ldots, a_n) \in S^n / \max_{j=1,\ldots,m} |\sum_{k=1}^{n} a_k \, . \, Q_k(x_j)| \leqq d\}$ is bounded.

Proof. We can easily prove that the function $F(a_1, \ldots, a_n) = \max_{j=1,\ldots,m} |\sum_{k=1}^{n} a_k \, . \, Q_k(x_j)|$ is a continuous non-negative function in S^n and is equal to 0 only at $(0, \ldots, 0)$. The minimum of it in the compact set $\{(a_1, \ldots, a_n) \in S^n / \sum_{k=1}^{n} |a_k| = 1\}$ is therefore a positive number $c > 0$. Let $(a_1, \ldots, a_n) \in M_d$, $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$. Let us denote $a = \sum_{k=1}^{n} |a_k|$. Then $d \geqq \max_{j=1,\ldots,m} |\sum_{k=1}^{n} a_k \, . \, Q_k(x_j)| = \sum_{k=1}^{n} |a_k| \, . \, \max_{j=1,\ldots,m} |\sum_{k=1}^{n} \frac{a_k}{a} \, . \, Q_k(x_j)| \geqq \sum_{k=1}^{n} |a_k| \, . \, c$. Hence $\sum_{k=1}^{n} |a_k| \leqq \frac{d}{c}$ and M_d is bounded.

Remark. We can easily prove that if $y_1, \ldots, y_m \in B$ are such points that $\dim_{\{y_1,\ldots,y_m\}} V < n$, then for each $d \geqq 0$ the set $M'_d = \{(a_1, \ldots, a_n) \in S^n / \max_{j=1,\ldots,m} |\sum_{k=1}^{n} a_k \cdot Q_k(y_j)| \leqq d\}$ contains straight lines and is unbounded.

Definition 4. For $g \in X^B$ we shall denote $\| g \| = \sup_{x \in B} | g(x) |$.

Remark. $\| g \|$ is not a norm of X^B because we admit also the case $\| g \| = +\infty$. The other properties of a norm ($\| g \| \geqq 0$, $\| g \| = 0$ iff $g \equiv o$, $\| c \cdot g \| = | c | \cdot \| g \|$, $\| g + h \| \leqq \| g \| + \| h \|$) are preserved. We have $\| g \| < +\infty$ iff g is bounded in B.

Theorem 7. Let $f \in X^B$, let us denote $\mu = \inf_{Q \in V} \| Q - f \|$. There exists $P \in V$ such that $\| P - f \| = \mu$.

Proof. We have $\mu = +\infty$ iff $\| Q - f \| = +\infty$ for all $Q \in V$, hence Theorem 7 holds for $\mu = +\infty$.

Let us assume $\mu < +\infty$. Let $Q_1, \ldots, Q_n$ form a basis of V. Let us denote $A = \{(a_1, \ldots, a_n) \in S^n / \| \sum_{k=1}^{n} a_k Q_k - f \| < \mu + 1\}$. We have $A \neq \emptyset$; let us choose $(b_1, \ldots, b_n) \in A$. Let us denote $d = 2\mu + 2$. By Theorem 5 there exist points $x_1, \ldots, x_m \in B$ such that $\dim_{\{x_1,\ldots,x_m\}} V = n$, by Theorem 6 the set $M_d = \{(a_1, \ldots, a_n) \in S^n / \max_{j=1,\ldots,m} |\sum_{k=1}^{n} a_k \cdot Q_k(x_j)| \leqq d\}$ is bounded. If $(a_1, \ldots, a_n) \in A$, then $\| \sum_{k=1}^{n} (a_k - b_k) \cdot Q_k \| = \| (\sum_{k=1}^{n} a_k Q_k - f) - (\sum_{k=1}^{n} b_k Q_k - f) \| < 2(\mu + 1) = d$, hence $(a_1 - b_1, \ldots, a_n - b_n) \in M_d$ and the set A is bounded.

For each $m \in N$ there exists $P_m = \sum_{k=1}^{n} a_{km} Q_k \in V$ such that $\| P_m - f \| < \mu + \frac{1}{m}$. For each $m \in N$ we have $(a_{1m}, \ldots, a_{nm}) \in A$, therefore the sequence $\{(a_{1m}, \ldots, a_{nm})\}_{m=1}^{\infty}$ is bounded. By the Theorem of Weierstrass this sequence has a convergent subsequence; let us assume for brevity that $\lim_{m \to \infty} a_{km} = a_k$ for $k = 1, \ldots, n$. Let us denote $P = \sum_{k=1}^{n} a_k Q_k$.

For each $x \in B$ we have $\lim_{m \to \infty} | P_m(x) - f(x) | = | P(x) - f(x) |$ and $| P_m(x) - f(x) | < \mu + \frac{1}{m}$ for all $m \in N$. Hence $| P(x) - f(x) | \leqq \mu$, i.e. $\| P - f \| = \mu$.

Corollary. We have $\mu = 0$ iff $f \in V$.

1.3. The Approximation on a Subset

Assumption (for § 1.3.): Let B be a set, let X be an NL-space over a field S, where $S = R$ or $S = C$. Let $V \subset X^B$, $V \neq \emptyset$, $f \in X^B$. Let us denote $\mu = \inf_{Q \in V} \| Q - f \|$.

Definition 5. Let $M \subset B$.

(1) Let $Q \in V$. If $M = \emptyset$, we put $\| Q - f \|_M = 0$. If $M \neq \emptyset$, we put $\| Q - f \|_M = \sup_{x \in M} | Q(x) - f(x) |$.

(2) We put $\mu(M) = \inf_{Q \in V} \| Q - f \|_M$.

(3) We say that $P \in V$ is **a polynomial of the best approximation to f in the set M** iff $\| P - f \|_M = \mu(M)$.

Remark. (1) $\| Q - f \|_B = \| Q - f \|$ for all $Q \in V$, $\mu(\emptyset) = 0$, $\mu(B) = \mu$.

(2) We admit, of course, also the cases $\| Q - f \|_M = +\infty$ and $\mu(M) = +\infty$. We have $\mu(M) < +\infty$ iff there exists $Q \in V$ such that the function $Q - f$ is bounded in M. It holds e.g. if M is finite.

(3) Mostly V will be a subspace of X^B.

Theorem 8. Let V be a subspace of X^B, let $M \subset B$ and let $\dim_M V$ exist. Then there exists $P \in V$ such that $\| P - f \|_M = \mu(M)$.

Proof. The assertion follows from Theorem 7 if we apply it to M, $\{Q_M / Q \in V\}$, f_M, $\mu(M)$ instead of to B, V, f, μ.

Theorem 9. (1) If $M \subset D \subset B$, then $\mu(M) \leqq \mu(D)$.

(2) If $M \subset B$, then $0 \leqq \mu(M) \leqq \mu$.

(3) Let $M \subset D \subset B$ and $\mu(M) = \mu(D)$. If $P \in V$ has the property $\| P - f \|_D = \mu(D)$, then also $\| P - f \|_M = \mu(M) = \mu(D)$.

(4) Let $M \subset B$ and $\mu(M) = \mu$. If $P \in V$ has the property $\| P - f \| = \mu$, then also $\| P - f \|_M = \mu(M) = \mu$.

(5) Let $M \subset D \subset B$ and $\mu(M) = \mu(D)$. Let $P \in V$ have the property $\| P - f \|_M = \mu(M)$ and let no other function of V have this property. If $Q \in V$ is such that $\| Q - f \|_D = \mu(D)$, then $Q = P$ and hence $\| P - f \|_D = \mu(D)$.

Proof. (1) If $M = \emptyset$, the assertion is obvious. Let us assume $M \neq \emptyset$. For all $Q \in V$ we have $\| Q - f \|_M \leqq \| Q - f \|_D$, hence $\mu(M) \leqq \mu(D)$.

(2) follows from (1) for $D = B$.

(3) We have $\mu(D) = \mu(M) \leqq \| P - f \|_M \leqq \| P - f \|_D = \mu(D)$ therefore the equalities hold.

(4) follows from (3) for $D = B$.

(5) By (3), we have $\| Q - f \|_M = \mu(M)$ hence $Q = P$.

1.4. The Passage to a Finite Subset

Assumption (for § 1.4.). Let B be a set, let X be an NL-space over R. Let V be an n-dimensional subspace of X^B ($n \in N_0$), let $f \in X^B$, let us denote $\mu = \min_{Q \in V} \| Q - f \|$.

Theorem 10. $\mu = \sup_{x_1, \ldots, x_{n+1} \in B} \mu(\{x_1, \ldots, x_{n+1}\})$.

Proof. Let us denote $p = \sup_{x_1, \ldots, x_{n+1} \in B} \mu(\{x_1, \ldots, x_{n+1}\})$. By Theorem 9 (2), $p \leqq \mu$.

If $p = +\infty$, then also $\mu = +\infty$. Therefore we may assume $p < +\infty$. Let $Q_1, \ldots, Q_n$ form a basis of V.

For each $x \in B$ let us put $W(x) = \{(a_1, \ldots, a_n) \in R^n / | \sum_{k=1}^{n} a_k \cdot Q_k(x) - f(x) | \leqq p\}$. Let $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in W(x)$, $0 < r < 1$. Then $| \sum_{k=1}^{n} [r a_k + (1 - r) b_k] \cdot Q_k(x) - f(x) | = | r \cdot (\sum_{k=1}^{n} a_k \cdot Q_k(x) - f(x)) + (1 - r) \cdot (\sum_{k=1}^{n} b_k \cdot Q_k(x) - f(x)) | \leqq \leqq r \cdot p + (1 - r) \cdot p = p$. Hence $W(x)$ is a convex subset of R^n. We can easily prove that $W(x)$ is also closed.

Let $x_1, \ldots, x_{n+1} \in B$ be arbitrary. By Theorem 8, there exists $Q \in V$ such that $| Q(x_k) - f(x_k) | \leqq \mu(\{x_1, \ldots, x_{n+1}\}) \leqq p$ for $k = 1, \ldots, n + 1$. If $Q = \sum_{k=1}^{n} A_k Q_k$, then $(a_1, \ldots, a_n) \in W(x_1) \cap \ldots \cap W(x_{n+1})$. Hence each $n + 1$ sets $W(x)$ have a common point.

By Theorem 5, there exist points $x_1, \ldots, x_m \in B$ such that $\dim_{\{x_1, \ldots, x_m\}} V = n$. Let us denote $d = p + \max_{j=1,\ldots,m} | f(x_j) |$. The set $M_d = \{(a_1, \ldots, a_n) \in R^n / \max_{j=1,\ldots,m} | \sum_{k=1}^{n} a_k \cdot Q_k(x_j) | \leqq d\}$ is bounded by Theorem 6. If $(a_1, \ldots, a_n) \in W(x_1) \cap \ldots \cap W(x_m)$, then for $j = 1, \ldots, m$ we have $| \sum_{k=1}^{n} a_k Q_k(x_j) | \leqq | \sum_{k=1}^{n} a_k Q_k(x_j) - f(x_j) | + | f(x_j) | \leqq p + \max_{j=1,\ldots,m} | f(x_j) | = d$, hence $(a_1, \ldots, a_n) \in M_d$. Hence the set $W(x_1) \cap \ldots \cap W(x_m)$ is bounded.

If card $B \leqq n$, we can choose such points $x_1, \ldots, x_{n+1} \in B$ that $B = \{x_1 \ldots, x_{n+1}\}$ and the assertion of Theorem 10 is obvious. Let card $B \geqq n + 1$. Then the system $\{W(x) / x \in B\}$ satisfies the conditions of Helly's theorem therefore there exists $(a_1, \ldots, a_n) \in \bigcap_{x \in B} W(x)$. Then $| \sum_{k=1}^{n} a_k Q_k(x) - f(x) | \leqq p$ for all $x \in B$, hence $\mu \leqq \leqq \| \sum_{k=1}^{n} a_k Q_k - f \| \leqq p$, i.e. $\mu = p$.

Theorem 11. Let D be a compact T-space.

(1) Let $\{x_m\}_{m=1}^{\infty}$ be a sequence of points from D. Then there exists $x_0 \in D$ such that for every neighbourhood U of x_0 there are infinitely many $m \in N$ such that $x_m \in U$.

(2) Let $\{(x_1^m, \ldots, x_{n+1}^m)\}_{m=1}^{\infty}$ be a sequence of $(n + 1)$-tuples of points from D. Then there exist points $x_1, \ldots, x_{n+1} \in D$ such that for every neighbourhoods U_1 of x_1, U_2 of $x_2, \ldots, U_{n+1}$ of x_{n+1} there are infinitely many $m \in N$ such that $x_k^m \in U_k$ for $k = 1, \ldots, n + 1$.

Proof. (1) Let us assume that the assertion does not hold. Then for each $x \in D$ there exists a neighbourhood $U(x)$ of x such that there are only finitely many x_m

in $U(x)$. $\{U(x)/x \in D\}$ is an open covering of D therefore there exist $y_1, \ldots, y_p \in D$ such that $D = U(y_1) \cup \ldots \cup U(y_p)$. But there are only finitely many x_m in each $U(y_i)$, which is a contradiction.

(2) Let us denote $G = D^{n+1}$ the Cartesian product of the T-spaces D with the topology defined in the theory of T-spaces (see e.g. [7], p. 31). By Tichonov's theorem (see [7], p. 37), G is a compact T-space. Moreover, if $U_1, \ldots, U_{n+1}$ are open in D, then $U_1 \times \ldots \times U_{n+1}$ is open in G. The assertion may be obtained by applying (1) to G.

Definition 6. Let $D \subset B$. We shall say that D is **a representative subset** (with respect to B, V, f) iff there may be given such a topology on D that:

(1) D is a compact T-space.

(2) For each $Q \in V$ we have: for each $x \in D$ and $h > 0$ there exists a neighbourhood $U \subset D$ of x such that $|Q(y) - f(y)| < |Q(x) - f(x)| + h$ for all $y \in U$.

(3) For each $x \in B$ there exists $y \in D$ such that for each $Q \in V$ we have $|Q(x) - f(x)| \leqq |Q(y) - f(y)|$.

Remark. (1) If for each $Q \in V$ the function $|Q - f|$ is continuous in D, then the condition (2) of the definition is fulfilled.

(2) Let B be a compact T-space and let for each $Q \in V$ the function $|Q - f|$ be continuous in B. Then B is a representative subset. (This situation may be constructed always when B is finite.)

(3) We define a representative subset in the same way also in the case when X is a complex NL-space.

Theorem 12. Let B have a representative subset D. Then there exist points $x_1, \ldots, x_{n+1} \in D$ such that

$$\mu = \mu(\{x_1, \ldots, x_{n+1}\}) < +\infty.$$

Proof. If $z_1, \ldots, z_{n+1} \in B$ are arbitrary, then there exist points $x_1, \ldots, x_{n+1} \in D$ such that $|Q(z_k) - f(z_k)| \leqq |Q(x_k) - f(x_k)|$ for $k = 1, \ldots, n + 1$ and for all $Q \in V$. Then $\mu(\{z_1, \ldots, z_{n+1}\}) \leqq \mu(\{x_1, \ldots, x_{n+1}\})$ and with respect to Theorem 10 we have $\mu = \sup_{x_1, \ldots, x_{n+1} \in D} \mu(\{x_1, \ldots, x_{n+1}\})$.

Hence for each $m \in N$ we can choose points $x_1^m, \ldots, x_{n+1}^m \in D$ such that $\lim_{m \to \infty} \mu(\{x_1^m, \ldots, x_{n+1}^m\}) = \mu$. There exist points $x_1, \ldots, x_{n+1} \in D$ satisfying the assertion of Theorem 11 (2). By Theorem 8 there exists $P \in V$ such that $\max_{k=1,\ldots,n+1} |P(x_k) - f(x_k)| = \mu(\{x_1, \ldots, x_{n+1}\})$. Let us choose $h > 0$ arbitrarily. There exist neighbourhoods U_1 of $x_1, \ldots, U_{n+1}$ of $x_{n+1}(U_1, \ldots, U_{n+1} \subset D)$ such that for $k = 1, \ldots, n + 1$ and for each $x \in U_k$ we have $|P(x) - f(x)| < |P(x_k) - f(x_k)| + h \leqq \mu(\{x_1, \ldots, x_{n+1}\}) + h$. If $y_1 \in U_1, \ldots, y_{n+1} \in U_{n+1}$ are arbitrary, then $\max_{k=1,\ldots,n+1} |P(y_k) - f(y_k)| <$

$< \mu(\{x_1, \ldots, x_{n+1}\}) + h$, hence $\mu(\{y_1, \ldots, y_{n+1}\}) < \mu(\{x_1, \ldots, x_{n+1}\}) + h$. By Theorem 11 (2), there are infinitely many $m \in N$ such that $x_k^m \in U_k$ for $k = 1, \ldots, n + 1$. Then $\mu(\{x_1, \ldots, x_{n+1}\}) + h > \mu(\{x_1^m, \ldots, x_{n+1}^m\})$. By means of the limit passage for $m \to \infty$ we get $\mu(\{x_1, \ldots, x_{n+1}\}) + h \geqq \mu$. As $\mu(\{x_1, \ldots, x_{n+1}\}) < +\infty$, we have $\mu < +\infty$, too. As $h > 0$ has been chosen arbitrarily, we have $\mu(\{x_1, \ldots, x_{n+1}\}) \geqq \mu$ and therefore $\mu(\{x_1, \ldots, x_{n+1}\}) = \mu$.

1.5. The Minimal Set

Assumption (for § 1.5.). Let B be a set, let X be an NL-space over a field S, where $S = R$ or $S = C$. Let V be an n-dimensional subspace of X^B ($n \in N$), let $f \in X^B$, let us denote $\mu = \min_{Q \in V} \| Q - f \|$.

Theorem 13. Let $m = n + 1$ for $S = R$, $m = 2n + 1$ for $S = C$.

(1) We have $\mu = \sup_{x_1, \ldots, x_m \in B} \mu(\{x_1, \ldots, x_m\})$.

(2) Let B have a representative subset D. Then there exist points $x_1, \ldots, x_m \in D$ such that $\mu = \mu(\{x_1, \ldots, x_m\}) < +\infty$.

Proof. For $S = R$ the assertions follow from Theorems 10 and 12. Let $S = C$ and let $Q_1, \ldots, Q_n$ form a basis of V. We may consider X as an NL-space over R; we keep the sum and the norm, only the multiple must be restricted to the multiple only by real numbers. Then X^B is an L-space over R, V remains a subspace of X^B. If $Q \in V$, then there exist numbers $a_1, \ldots, a_n \in C$ such that $Q = \sum_{k=1}^{n} a_k Q_k$. Let $a_k = b_k + ic_k$ (where $b_k, c_k \in R$) for $k = 1, \ldots, n$; $Q = \sum_{k=1}^{n} b_k Q_k + \sum_{k=1}^{n} c_k \cdot iQ_k$. On the other hand, if $b_1, \ldots, b_n, c_1, \ldots, c_n \in R$ are such numbers that $\sum_{k=1}^{n} b_k Q_k + \sum_{k=1}^{n} c_k \cdot iQ_k \equiv o$, then $\sum_{k=1}^{n} (b_k + ic_k) Q_k \equiv o$, hence $b_k + ic_k = 0$ and $b_k = c_k = 0$ for $k = 1, \ldots, n$. Therefore the functions $Q_1, \ldots, iQ_1, \ldots, Q_n, iQ_n$ form a basis of V if we take V for a subspace of the L-space X^B over R. Hence V is a $(2n)$-dimensional subspace of the L-space X^B over R. Both assertions follow again from Theorems 10 and 12.

Definition 7. A subset $M \subset B$ is called **a minimal set** iff $\mu(M) = \mu$ and $\mu(G) < \mu$ for every $G \subset M$ such that $G \neq M$.

Remark. (1) Let $f \in V$, i.e. $\mu = 0$. Then there exists exactly one minimal set, namely $\emptyset$.

(2) Let M be a minimal set. Then $M \neq \emptyset$ holds iff $f \notin V$, i.e. iff $\mu > 0$.

(3) Let $M \subset B$, $\mu > 0$. Then M is a minimal set iff $\mu(M) = \mu$ and $\mu(M - \{x\}) < \mu$ for all $x \in M$.

Proof. (1) Obviously $\emptyset$ is minimal. Let $M \neq \emptyset$; then $\emptyset \subset M, \emptyset \neq M, \mu(\emptyset) = 0 = \mu$, hence M is not minimal.

(2) If $f \notin V$, then $\mu > 0$, therefore $\mu(M) > 0$ and $M \neq \emptyset$.

(3) If M is minimal and $x \in M$, then $\mu(M - \{x\}) < \mu$. On the other hand, let M satisfy the latter condition. If $G \subset M$, $G \neq M$, then let us choose $x \in M - G$. Then $G \subset M - \{x\}$, hence $\mu(G) \leqq \mu(M - \{x\}) < \mu$ and M is minimal.

Theorem 14. Let $M \neq \emptyset$ be a minimal set. Then for $S = R$ we have card $M \leqq \dim_M V + 1 \leqq n + 1$, for $S = C$ we have card $M \leqq 2 . \dim_M V + 1 \leqq 2n + 1$.

Proof. Let us denote $W = \{Q_M / Q \in V\}$; then W is a subspace of X^M, $\dim W = \dim_M V$. Further $f_M \in X^M$, $\mu = \mu(M) = \min_{Q \in W} \sup_{x \in M} | Q(x) - f_M(x) |$. Let us denote $m = \dim_M V + 1$ for $S = R$, $m = 2 . \dim_M V + 1$ for $S = C$. By Theorem 13 (1) applied to M, W, f_M we have $\mu = \sup_{x_1, \ldots, x \in M} \mu(\{x_1, \ldots, x_m\})$.

Let us assume that there exist distinct points $z_1, \ldots, z_{m+1} \in M$. For $k = 1, \ldots, m + 1$, we have $\mu(M - \{z_k\}) < \mu$ and therefore there exist points $x_1, \ldots, x_m \in M$ such that $\mu(\{x_1, \ldots, x_m\}) > \max_{k=1,\ldots,m+1} \mu(M - \{z_k\})$. If $x \in M - \{x_1, \ldots, x_m\}$, then $\{x_1, \ldots, x_m\} \subset M - \{x\}$, therefore $\mu(M - \{x\}) \geqq \mu(\{x_1, \ldots, x_m\}) > \max_{k=1,\ldots,m+1} \mu(M - \{z_k\})$, hence $x \notin \{z_1, \ldots, z_{m+1}\}$. Hence $\{z_1, \ldots, z_{m+1}\} \subset \{x_1, \ldots, x_m\}$, which is a contradiction. Necessarily card $M \leqq m$.

Corollary. If $S = R$ and card $M = n + 1$ or $S = C$ and $2n \leqq$ card $M \leqq 2n + 1$, then $\dim_M V = n$.

Remark. If there exists at least one minimal set M, then M is finite and necessarily $\mu = \mu(M) < +\infty$.

Theorem 15. Let B have a representative subset D. Then there exists a minimal set which is a subset of D.

Proof. By Theorem 13 (2), there exist points $x_1, \ldots, x_m \in D$ such that $\mu(\{x_1, \ldots, x_m\}) = \mu$. We can create a minimal set $M \subset \{x_1, \ldots, x_m\}$ by eventual removing several points x_i.

Theorem 16. (1) Let $M \subset B$. Let points $x, y \in M$ be distinct and let $| Q(x) - f(x) | \leqq | Q(y) - f(y) |$ for all $Q \in V$. Then $\mu(M - \{x\}) = \mu(M)$, i.e. M is not minimal.

(2) Let $M = \{x_1, \ldots, x_m\}$ be a minimal set. Let $y_1, \ldots, y_m \in B$ be such points that $| Q(x_k) - f(x_k) | = | Q(y_k) - f(y_k) |$ for $k = 1, \ldots, m$ and for all $Q \in V$. Then $D = \{y_1, \ldots, y_m\}$ is a minimal set, too.

Proof. (1) There exists $P \in V$ such that $\| P - f \|_{M-\{x\}} = \mu(M - \{x\})$. As $y \in M - \{x\}$, we have $| P(x) - f(x) | \leqq | P(y) - f(y) | \leqq \mu(M - \{x\})$. Hence $\mu(M) \leqq \| P - f \|_M = \mu(M - \{x\}) \leqq \mu(M)$ and the equalities hold.

(2) Evidently $\mu(D) = \mu(M) = \mu$, $\mu(D - \{y_k\}) = \mu(M - \{x_k\}) < \mu$.

Theorem 17. Let $M \neq \emptyset$ be a minimal set, let $P \in V$ be such that $\| P - f \|_M = \mu$. Then $| P(x) - f(x) | = \mu$ for all $x \in M$.

Proof. Conversely, let us admit that there exists $z \in M$ such that $| P(z) - f(z) | < \mu$. The inequality $\mu(M - \{z\}) < \mu$ holds and there exists $Q \in V$ such that $\| Q - f \|_{M-\{z\}} = \mu(M - \{z\}) < \mu$. Then necessarily $| Q(z) - f(z) | \geqq \mu$. Let $0 < a < \dfrac{\mu - | P(z) - f(z) |}{| Q(z) - f(z) | - | P(z) - f(z) |}$, we have $a < 1$. Let us put $T = aQ + (1 - a) P$; then $T \in V$. For all $x \in M$ we have $| T(x) - f(x) | \leqq a \,.\, | Q(x) - f(x) | + (1 - a) \,.\, | P(x) - f(x) |$. If $x \in M - \{z\}$, then $| T(x) - f(x) | \leqq a \,.\, \mu(M - \{z\}) + (1 - a) \,.\, \mu < \mu$. Moreover $| T(z) - f(z) | \leqq | P(z) - f(z) | + a \,.\, (| Q(z) - f(z) | - | P(z) - f(z) |) < \mu$. Hence $\mu(M) \leqq \max_{x \in M} | T(x) - f(x) | < \mu$, which is a contradiction.

Theorem 18. Let X be strictly normed. Let $M \neq \emptyset$ be a minimal set. Let $P, Q \in V$ be such that $\| P - f \|_M = \| Q - f \|_M = \mu$. Then $P(x) = Q(x)$ for all $x \in M$.

Proof. Let us denote $T = \frac{1}{2}(P + Q)$. For all $x \in M$ we have $| T(x) - f(x) | = \frac{1}{2} | [P(x) - f(x)] + [Q(x) - f(x)] | \leqq \frac{1}{2}(| P(x) - f(x) | + | Q(x) - f(x) |) \leqq \mu$, i.e. $\| T - f \|_M = \mu$. By Theorem 17, we have $| T(x) - f(x) | = \mu$ for all $x \in M$, therefore $| [P(x) - f(x)] + [Q(x) - f(x)] | = | P(x) - f(x) | + | Q(x) - f(x) |$, $| P(x) - f(x) | = | Q(x) - f(x) | = \mu$. From the basic property of the strictly norm ed spaces we have $P(x) - f(x) = Q(x) - f(x)$, i.e. $P(x) = Q(x)$ for all $x \in M$.

Remark. Let X be strictly normed and let $M \neq \emptyset$ be a minimal set. By Theorem 18, at each point $x \in M$ all the polynomials of the best approximation to f in M have the same value which is therefore determined unambiguously by V, f, M. Moreover, with respect to Theorem 17, we see that there exists a function $r(x) \in X^M$ such that $| r(x) | = 1$ for all $x \in M$ and if $P \in V$ and $\| P - f \|_M = \mu$, then $P(x) = f(x) + \mu \,.\, r(x)$ for all $x \in M$.

Theorem 19. Let $M \subset B$, $\mu(M) = \mu$. Let $P \in V$, $\| P - f \|_M = \mu$ and let no other polynomial of the best approximation to f in M exist.

(1) If $M \subset D \subset B$, then $\| P - f \|_D = \mu$ and there is no other polynomial of the best approximation to f in D.

(2) $\| P - f \| = \mu$ and there is no other polynomial of the best approximation to f (in B).

Proof. (1) By Theorem 8, there is at least one polynomial $Q \in V$ such that $\| Q - f \|_D = \mu(D) = \mu$. If Q is such a polynomial, we have $\| Q - f \|_M = \mu(M) = \mu$ by Theorem 9 (3), hence $Q = P$. Theorefore $\| P - f \|_D = \mu$ and the assertion holds.

(2) follows from (1) for $D = B$.

Theorem 20. Let X be strictly normed, let M be a minimal set such that $\dim_M V = n$.

(1) There exists exactly one $P \in V$ such that $\| P - f \|_M = \mu$.

(2) If $M \subset D \subset B$, then there exists exactly one $P \in V$ such that $\| P - f \|_D = \mu$.

(3) There exists exactly one $P \in V$ such that $\| P - f \| = \mu$.

Proof. (1) Let $P, Q \in V$ and $\| P - f \|_M = \| Q - f \|_M = \mu$. By Theorem 4 (3), we have $P - Q \equiv o$, i.e. $P = Q$.

(2) and (3) follow from (1) and from Theorem 19.

2. REAL AND COMPLEX FUNCTIONS

2.1. Some Auxiliary Results

Remark. In the following we restrict ourselves to real and complex functions. That means $X = S$ where $S = R$ or $S = C$. In both cases S is a strictly normed NL-space over S and all the previous results hold.

Theorem 21. Let B be a set, let $S = R$ or $S = C$. Functions $Q_1, \ldots, Q_n \in S^B$ are independent iff there exist points $x_1, \ldots, x_n \in B$ such that $\det Q_k(x_j) \neq 0$.

Theorem 22. If B is a finite set, then $\dim S^B = \operatorname{card} B$. If B is infinite, then S^B is not of a finite dimension.

Theorem 23. Let B be a set, let $S = R$ or $S = C$. Let V be a subspace of S^B, let $n \in N$.

(1) Let $M \subset B$, let M be finite. Then $\dim_M V \leqq \operatorname{card} M$.

(2) Let $M \subset B$, $r \in N$ and let $\dim_M V$ exist. Then $\dim_M V \geqq r$ holds iff there exist $P_1, \ldots, P_r \in V$ and $x_1, \ldots, x_r \in M$ such that $\det P_k(x_j) \neq 0$.

I.e. $\dim_M V \geqq r$ iff there exists $D \subset M$ such that $\dim_D V = \operatorname{card} D = r$.

(3) If points $x_1, \ldots, x_n \in B$ are such that $\dim_{\{x_1, \ldots, x_n\}} V = n$, then for arbitrary $y_1, \ldots, y_n \in S$ there exists $P \in V$ such that $P(x_k) = y_k$ for $k = 1, \ldots, n$. If, moreover, $\dim V = n$, then there exists exactly one such P.

(4) Let $M \subset B$, $\dim_M V = t \in N_0$ and $z \in B$. Let us denote $D = M \cup \{z\}$. Then $t \leqq \dim_D V \leqq t + 1$.

(5) Let $\dim V = n$, let $Q_1, \ldots, Q_n$ form a basis of V. Let $M = \{x_1, \ldots, x_m\} \subset B$. Then $\dim_M V$ is equal to the rank of the matrix

$$A = \begin{pmatrix} Q_1(x_1) & \ldots & Q_1(x_m) \\ \vdots & & \\ Q_n(x_1) & \ldots & Q_n(x_m) \end{pmatrix}.$$

Proof. (1) Let us denote $W = \{Q_M / Q \in V\}$. W is a subspace of S^M, hence $\dim_M V = \dim W \leqq \operatorname{card} M$ by Theorem 22.

(2) If the latter condition is fulfilled, then by Theorem 21 the functions $P_1, \ldots, P_r$ are independent in M and hence $\dim_M V \geqq r$. On the other hand, if $\dim_M V \geqq r$, then there exist $P_1, \ldots, P_r \in V$ independent in M and by Theorem 21, there exist points $x_1, \ldots, x_r \in M$ such that $\det P_k(x_j) \neq 0$. By putting $D = \{x_1, \ldots, x_r\}$ we can prove the assertion concerning D.

(3) By (2) there exist $P_1, \ldots, P_n \in V$ such that $\det P_k(x_j) \neq 0$. Then there exist $a_1, \ldots, a_n \notin \in S$ such that $\sum_{k=1}^{n} a_k P_k(x_j) = y_j$ for $j = 1, \ldots, n$. We may put $P = \sum_{k=1}^{n} a_k P_k$.

Let $\dim V = n$. Then the functions $P_1, \ldots, P_n$ form a basis of V. If $Q = \sum_{k=1}^{n} b_k P_k \in V$ is such that $Q(x_j) = y_j$ for $j = 1, \ldots, n$, then $\sum_{k=1}^{n} b_k P_k(x_j) = y_j$ for $j = 1, \ldots, n$ and hence $a_k = b_k$ for $k = 1, \ldots, n$. Hence $Q = P$.

(4) Let us admit that there exist functions $P_1, \ldots, P_{t+2} \in V$ which are independent in D. By Theorem 21, there exist distinct points $x_1, \ldots, x_{t+2} \in D$ such that $\det P_k(x_j) \neq \neq 0$. If $z \notin \{x_1, \ldots, x_{t+2}\}$, then by (2) we have $\dim_M V \geqq t + 2$, which is a contradiction. Let then e.g. $x_{t+2} = z$. Then at least one subdeterminant of the order $t + 1$, determined by the first $t + 1$ columns, is non-zero; then by (2), we have $\dim_M V \geqq t + 1$ which is a contradiction again. Hence necessarily $\dim_D V \leqq t + 1$. On the other hand, $\dim_D V \geqq \dim_M V = t$.

(5) Let t be the rank of the matrix A, $s = \dim_M V$. By (2), we have $s \geqq t$. As $Q_1, \ldots, Q_n$ are generating in M, there exist $Q_{i_1}, \ldots, Q_{i_s}$ among them which form a basis in M. Then the rows with the indices $i_1, \ldots, i_s$ are independent and hence $s \leqq t$; together $s = t$.

Theorem 24. Let B a set, let $S = R$ or $S = C$, let V be a subspace of S^B and $f \in S^B$. Let $M \subset B$ be finite and $\dim_M V = \operatorname{card} M$. Then $\mu(M) = 0$.

Proof. By Theorem 23 (3), there exists $P \in V$ such that $P(x) = f(x)$ for all $x \in M$. Hence $\mu(M) = 0$.

Theorem 25. Let B be a set, let $S = R$ or $S = C$ and $n \in N$. Let V be an n-dimensional subspace of S^B, let $f \in S^B$. Let $M \neq \emptyset$ be a minimal set. If $S = R$, then we have $\operatorname{card} M = \dim_M V + 1$. If $S = C$, then we have $\dim_M V + 1 \leqq \operatorname{card} M \leqq \leqq 2 . \dim_M V + 1$.

Proof. By Theorem 23 (1), $\dim_M V \leqq \operatorname{card} M$. If $\dim_M V = \operatorname{card} M$, then by Theorem 24 we have $\mu = \mu(M) = 0$, which is in a contradiction with $M \neq \emptyset$. Hence $\dim_M V + 1 \leqq \operatorname{card} M$. The assertions follow now from Theorem 14.

Theorem 26. Let B be a set, let $S = R$ or $S = C$ and $n \in N$. Let V be an n-dimensional subspace of S^B, let $f \in S^B$.

(1) Let $M \subset B$ and $z \in M$ be such that $\mu(M - \{z\}) < \mu(M)$. Then $\dim_{M-\{z\}} V = \dim_M V$.

(2) Let M be a minimal set and let $z \in M$. Then $\dim_{M-\{z\}} V = \dim_M V$. If, moreover, $S = R$, then we have $\dim_{M-\{z\}} V = \text{card } M - 1 = \text{card } (M - \{z\})$ and hence $\mu(M - \{z\}) = 0$.

Proof. (1) Let us denote $D = M - \{z\}$, $r = \dim_D V + 1$ and let us assume that $\dim_M V \geqq r$. Then we can choose $P_1, \ldots, P_r \in V$ independent in M. They must be dependent in D and there exist numbers $a_1, \ldots, a_r \in S$ not all zero such that $\sum_{k=1}^{r} a_k P_k(x) = 0$ for all $x \in D$. Then necessarily $\sum_{k=1}^{r} a_k P_k(z) \neq 0$. There exists $Q \in V$ such that $\| Q - f \|_D = \mu(D)$. Let us denote $b = \dfrac{Q(z) - f(z)}{\Sigma a_k P_k(z)}$, $T = Q - b \,.\, \sum_{k=1}^{r} a_k P_k$. We have $T(x) = Q(x)$ for all $x \in D$, i.e. $\| T - f \|_D = \mu(D)$, moreover $T(z) = Q(z) - b \,.\, \sum_{k=1}^{r} a_k P_k(z) = f(z)$. Hence $\mu(M) \leqq \| T - f \|_M = \| T - f \|_D = \mu(D) < \mu(M)$, which is a contradiction. Hence $\dim_M V = r - 1 = \dim_D V$.

(2) follows from (1) and from the definition of a minimal set.

2.2. The Approximation on r Points

Lemma. Let X be a strictly normed NL-space over S where $S = R$ or $S = C$. Let $x_1, \ldots, x_r \in X$ and $| x_1 + \ldots + x_r | = | x_1 | + \ldots + | x_r |$. Then there exist $b \in X$ (we can take it among $x_1, \ldots, x_r$) and real non-negative numbers $a_1, \ldots, a_r \in S$ such that $x_k = a_k \,.\, b$ for $k = 1, \ldots, r$.

Proof. Let $x, y \in X$ and $| x + y | = | x | + | y |$; we may assume $| x | \geqq | y | > 0$. Then $| x | + | y | = | x + y | \leqq \left| \left(1 - \dfrac{| y |}{| x |}\right) . x \right| + \left| \dfrac{| y |}{| x |} . x + y \right| \leqq \left(1 - \dfrac{| y |}{| x |}\right) . | x | + \dfrac{| y |}{| x |} . | x | + | y | = | x | + | y |$. All the terms are equal, especially $\left| \dfrac{| y |}{| x |} . x + y \right| = \left| \dfrac{| y |}{| x |} . x \right| + | y |$. Since $\left| \dfrac{| y |}{| x |} . x \right| = | y |$, we have $y = \dfrac{| y |}{| x |} . x$ and we may take $b = x$. The proof of Lemma can be completed by the induction

Theorem 27. Let $S = R$ or $S = C$, let $r \in N$. Let the numbers $C_1, \ldots, C_r \in S$ be not all zero and let $C \in S$ be arbitrary. Let us denote $A = \{(u_1, \ldots, u_r) \in S^r / \sum_{k=1}^{r} C_k u_k = C\}$, $d = \dfrac{| C |}{\sum_{k=1}^{r} | C_k |}$.

(1) We have $\min\limits_{(u_1, \ldots, u_r) \in A} \max\limits_{k = 1, \ldots, r} | u_k | = d$.

(2) Let $(u_1, \ldots, u_r) \in S^r$. Then we have $\sum_{k=1}^{r} C_k u_k = C$ and $\max\limits_{k=1,\ldots,r} | u_k | = d$ iff we have $u_k = d \,.\, \text{sign} (C\bar{C}_k)$ for $C_k \neq 0$ and $| u_k | \leqq d$ for $C_k = 0$ $(k = 1, \ldots, r)$.

Proof. Then assertion is obvious for $C = 0$. Let us assume $C \neq 0$; then $d > 0$.

If $(u_1, \dots, u_r) \in A$, then $|C| = |\Sigma C_k u_k| \leqq \Sigma |C_k| . |u_k| \leqq (\max |u_k|) . \Sigma |C_k|$, hence $\max |u_k| \geqq d$.

Let $(u_1, \dots, u_r) \in A$, $\max |u_k| = d$. Then we have the equalities in the previous calculation. Therefore we have $|u_k| = d$ for $C_k \neq 0$ and we conclude by Lemma that there exist $b \in S$ and numbers $a_1 \geqq 0, \dots, a_r \geqq 0$ such that $C_k u_k = a_k b$ for $k = 1, \dots, r$. Let us denote $a = \Sigma a_k$. We have $C = \Sigma C_k u_k = \Sigma a_k b = ab$. Let $C_k \neq 0$. Then $|u_k| = d > 0$, $a_k > 0$; necessarily $b \neq 0, a > 0$. Further $|C_k| . d = a_k . |b|$, hence $a_k = \frac{|C_k|}{|b|} . d, u_k = \frac{b . a_k}{C_k} = \frac{b . |C_k| . d}{C_k . |b|} = \frac{a . b . |C_k| . \bar{C}_k}{|ab| . C_k . \bar{C}_k} . d = \frac{C . \bar{C}_k}{|C| . |C_k|} . d = d . \operatorname{sign}(C\bar{C}_k)$. If $C_k = 0$, necessarily $|u_k| \leqq d$.

Let $(u_1, \dots, u_r) \in S^r$ and $u_k = d . \operatorname{sign}(C\bar{C}_k)$ for $C_k \neq 0$. Then $\sum_{k=1}^{r} C_k u_k = \sum_{k=1}^{r} C_k . d . (\operatorname{sign} C) . (\operatorname{sign} \bar{C}_k) = d . (\operatorname{sign} C) . \sum_{k=1}^{r} |C_k| = |C| . \frac{C}{|C|} = C$, hence $(u_1, \dots, u_r) \in A$. If moreover $|u_k| \leqq d$ for $C_k = 0$, then $\max |u_k| = d$. The proof is completed.

Remark. If $C = 0$ or if $C_k \neq 0$ for $k = 1, \dots, r$, then there exists exactly one $(u_1, \dots, u_r) \in A$ such that $\max |u_k| = d$, namely $u_k = d . \operatorname{sign}(C\bar{C}_k)$ for $k = 1, \dots, r$. Otherwise there are infinitely many such $(u_1, \dots, u_r) \in A$.

Theorem 28. Let B be a set, let $S = R$ or $S = C$ and $r \geqq 2$. Let $x_1, \dots, x_r \in B$. Let V be a subspace of S^B such that $\dim_{\{x_1, \dots, x_r\}} V = r - 1$, let $P_1, \dots, P_{r-1} \in V$ form a basis in $\{x_1, \dots, x_r\}$. Let $f \in S^B$. For $k = 1, \dots, r$ let us denote

$$C_k = (-1)^{k-1} \begin{vmatrix} P_1(x_1) & \dots P_1(x_{k-1}) & P_1(x_{k+1}) & \dots P_1(x_r) \\ \vdots & & & \\ P_{r-1}(x_1) & \dots P_{r-1}(x_{k-1}) & P_{r-1}(x_{k+1}) & \dots P_{r-1}(x_r) \end{vmatrix}.$$

Then $\sum_{k=1}^{r} |C_k| > 0$ by Theorem 23 (2). Further let us denote $C = -\sum_{k=1}^{r} C_k . f(x_k)$,

$$d = \frac{C}{\sum_{k=1}^{r} |C_k|}.$$

(1) For each $P \in V$ we have $\sum_{k=1}^{r} C_k . P(x_k) = 0$, hence $\sum_{k=1}^{r} C_k . [P(x_k) - f(x_k)] = C$.

(2) $\mu(\{x_1, \dots, x_r\}) = \min_{Q \in V} \max_{k=1,\dots,r} |Q(x_k) - f(x_k)| = d$.

(3) If $P \in V$ is such that $\max_{k=1,\dots,r} |P(x_k) - f(x_k)| = d$, then $P(x_k) - f(x_k) = d . \operatorname{sign}(C\bar{C}_k)$ for $C_k \neq 0$.

(4) On the other hand, if $v_1, \dots, v_r \in S$ are such that $v_k = f(x_k) + d . \operatorname{sign}(C\bar{C}_k)$ for $C_k \neq 0$, then there exists $P \in V$ such that $P(x_k) = v_k$ for $k = 1, \dots, r$.

(5) If $P \in V$ is arbitrary, then $\mu(\{x_1, \dots, x_r\}) = \dfrac{|\Sigma C_k . [P(x_k) - f(x_k)]|}{\Sigma |C_k|}$.

(6) Let $P \in V$ have the property that there exists $h \in S$, $h \neq 0$ such that $h . C_k . [P(x_k) - f(x_k)] \geqq 0$ for $k = 1, \dots, r$. Then $\mu(\{x_1, \dots, x_r\}) = \dfrac{\Sigma |C_k| . |P(x_k) - f(x_k)|}{\Sigma |C_k|}$, hence $\min_{C \neq_k 0} |P(x_k) - f(x_k)| \leqq \mu(\{x_1, \dots, x_r\}) \leqq \max_{C_k \neq 0} |P(x_k) - f(x_k)|$.

(7) Let $P \in V$ be such that there exist $h \in S$, $h \neq 0$ and $p \geqq 0$ such that $h . C_k . [P(x_k) - f(x_k)] \geqq 0$ for $k = 1, \dots, r$ and $|P(x_k) - f(x_k)| = p$ for $C_k \neq 0$. Then $\mu(\{x_1, \dots, x_r\}) = p$.

Moreover, if $D \subset B$ is such that $\{x_1, \dots, x_r\} \subset D$ and $|P(x) - f(x)| \leqq p$ for all $x \in D$, then $\mu(D) = p$.

Hence, if moreover $\| P - f \| = p$, then $\mu = p$.

Proof. Let $P \in V$. Then $0 = \begin{vmatrix} P(x_1) & P_1(x_1) \dots P_{r-1}(x_1) \\ \vdots & \\ P(x_r) & P_1(x_r) \dots P_{r-1}(x_r) \end{vmatrix} = \sum_{k=1}^{r} C_k . P(x_k)$, hence (1) holds.

If $u_1, \dots, u_r \in S$, $\sum_{k=1}^{r} C_k u_k = C$, then

$$0 = \sum_{k=1}^{r} C_k . [u_k + f(x_k)] = \begin{vmatrix} u_1 + f(x_1) & P_1(x_1) \dots P_{r-1}(x_1) \\ \vdots & \\ u_r + f(x_r) & P_1(x_r) \dots P_{r-1}(x_r) \end{vmatrix}.$$

As the columns of the determinant from the second to the r-th are independent, there exist $a_1, \dots, a_r \in S$ such that $\sum_{k=1}^{r-1} a_k P_k(x_j) = u_j + f(x_j)$ for $j = 1, \dots, r$. Let $Q = \sum_{k=1}^{r-1} a_k P_k$, then $Q(x_j) - f(x_j) = u_j$ for $j = 1, \dots, r$.

Let us denote $A = \{(u_1, \dots, u_r) \in S^r / \sum_{k=1}^{r} C_k u_k = C\}$. Then $A = \{(Q(x_1) - f(x_1), \dots, \dots, Q(x_r) - f(x_r)) / Q \in V\}$. The assertions (2), (3), (4) follow now immediately from Theorem 27. The assertion (5) follows from (1) and (2).

(6) For $k = 1, \dots, r$ we have $h . C_k . [P(x_k) - f(x_k)] = |h| . |C_k| . |P(x_k) - f(x_k)|$, i.e. $C_k . [P(x_k) - f(x_k)] = (\text{sign } \bar{h}) . |C_k| . |P(x_k) - f(x_k)|$, hence $|\sum C_k . [P(x_k) - f(x_k)]| = \sum |C_k| . |P(x_k) - f(x_k)|$. The inequalities follow from the fact that we may sum $|C_k| . |P(x_k) - f(x_k)|$ and $|C_k|$ only for such k for which $C_k \neq 0$.

(7) The first part follows from (6). If D has the required property, then $\mu(D) \geqq \mu(\{x_1, \dots, x_r\}) = p = \| P - f \|_D \geqq \mu(D)$ and the equalities hold. If we put $D = B$, we have the last assertion.

Remark. (1) We have $C_k \neq 0$ iff $\dim_{\{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_r\}} V = r - 1$. This follows from Theorem 23 (2).

(2) Let $C = 0$ or $C_k \neq 0$ for $k = 1, \dots, r$. If $P \in V$ is such that $\max_{k=1,\dots,r} | P(x_k) - f(x_k) | = d$, we have $P(x_k) = f(x_k) + d \,.\, \text{sign}\,(C\bar{C}_k)$ for $k = 1, \dots, r$. That means: the values of such P in $x_1, \dots, x_r$ are not dependent on the choice of P. If, moreover, $\dim V = r - 1$, then there exists exactly one $P \in V$ with this property; it follows from Theorem 23 (3).

(3) We have $d = 0$ iff $C = 0$, i.e. iff there exists $Q \in V$ such that $Q(x_k) = f(x_k)$ for $k = 1, \dots, r$.

Corollary. Let us formulate Theorem 28 for $r = 1$: Let B be a set, let $S = R$ or $S = C$, let $x_1 \in B$. Let V be a subspace of S^B such that $\dim_{\{x_1\}} V = 0$, let $f \in S^B$. We have $Q(x_1) = 0$ for all $Q \in V$, hence $\mu(\{x_1\}) = \min_{Q \in V} | Q(x_1) - f(x_1) | = | f(x_1) |$.

2.3. The Values at the Points of a Minimal Set

Assumption (for § 2.3.). Let B be a set, let $S = R$ or $S = C$, let $n \in N$. Let V be an n-dimensional subspace of S^B, let $Q_1, \dots, Q_n$ form a basis of V. Let $f \in S^B$, let us denote $\mu = \min_{Q \in V} \| Q - f \|$.

Remark. If $M \neq \emptyset$ is a minimal set, then for $S = R$ we always have $\dim_M V = \text{card}\, M - 1$. (This is not true of $S = C$.)

Theorem 29. Let M be such a minimal set that $\text{card}\, M = r \geqq 2$, $M = \{x_1, \dots, x_r\}$, $\dim_M V = r - 1$.

(1) The rank of the matrix $\begin{pmatrix} Q_1(x_1) \dots Q_1(x_r) \\ \vdots \\ Q_n(x_1) \dots Q_n(x_r) \end{pmatrix}$ is $r - 1$.

(2) Let the rows with the indices $i_1, \dots, i_{r-1}$ be independent. For $k = 1, \dots, r$ let us put

$$C_k = (-1)^{k-1} \begin{vmatrix} Q_{i_1}(x_1) & \dots Q_{i_1}(x_{k-1}) & Q_{i_1}(x_{k+1}) & \dots Q_{i_1}(x_r) \\ \vdots & & & \\ Q_{i_{r-1}}(x_1) & \dots Q_{i_{r-1}}(x_{k-1}) & Q_{i_{r-1}}(x_{k+1}) & \dots Q_{i_{r-1}}(x_r) \end{vmatrix}.$$

Then $C_k \neq 0$ for $k = 1, \dots, r$.

Proof. (1) The assertion follows from Theorem 23 (5).

(2) The polynomials $Q_{i_1}, \dots, Q_{i_{r-1}}$ form a basis in M. Let $k \in \{1 \dots, r\}$. Then these polynomials are generating also in $M - \{x_k\}$, by Theorem 26 we have $\dim_{M-\{x_k\}} V = r - 1$, therefore the polynomials $Q_{i_1}, \dots, Q_{i_{r-1}}$ form a basis in $\{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_r\} = M - \{x_k\}$ and are independent here. By Theorem 21, the determinant is non-zero, hence $C_k \neq 0$.

Theorem 30. Let $M = \{x_1, \dots, x_r\}$ be such a minimal set that $r \geqq 2$ and $\dim_M V =$

$= r - 1$. Let $Q_{i_1}, \ldots, Q_{i_{r-1}} \in V$ be independent in M, let us denote $C_1, \ldots, C_r$ like in Theorem 29 and $C = -\sum_{k=1}^{r} C_k . f(x_k)$.

(1) We have $\mu = \dfrac{|C|}{\sum_{k=1}^{r} |C_k|}$.

(2) Let $P \in V$ be such that $\max_{k=1,\ldots,r} |P(x_k) - f(x_k)| = \mu$. Then we have $P(x_k) - f(x_k) = \mu . \operatorname{sign}(C\bar{C}_k)$ for $k = 1, \ldots, r$.

(3) Let $P \in V$ be such that $P(x_k) - f(x_k) = q . \operatorname{sign} \bar{C}_k$ for $k = 1, \ldots, r$ $(q \in S)$. Then $|q| = \max_{k=1,\ldots,r} |P(x_k) - f(x_k)| = \mu$.

(4) Let $P \in V$. Then we have $\| P - f \| = \mu$ iff we have $P(x_k) - f(x_k) = \| P - f \| . \operatorname{sign}(C\bar{C}_k)$ for $k = 1, \ldots, r$.

Proof. (1) We have $\mu = \mu(\{x_1, \ldots, x_r\})$ and the assertion follows from Theorem 28(2).

(2) By Theorem 29 (2), we have $C_k \neq 0$ for $k = 1, \ldots, r$ and hence by Theorem 28 (3) we have $P(x_k) - f(x_k) = \mu . \operatorname{sign}(C\bar{C}_k)$.

(3) If $q = 0$, we have $\mu = \mu(\{x_1, \ldots, x_r\}) = 0$ which is a contradiction. Hence $q \neq 0$. For $k = 1, \ldots, r$ we have $(\operatorname{sign} \bar{q}) . C_k . [P(x_k) - f(x_k)] = |q| . |C_k| > 0$, hence by Theorem 28 (7) we have $\mu = \mu(\{x_1, \ldots, x_r\}) = |q| = \max_{k=1,\ldots,r} |P(x_k) - f(x_k)|$.

(4) If $\| P - f \| - \mu$, then we have $\max_{k=1,\ldots,r} |P(x_k)| - f(x_k) = \mu$ by Theorem 9(4) and the assertion follows from (2). If the latter condition is fulfilled, then by (3) we have $\mu = \max_{k=1,\ldots,r} |P(x_k) - f(x_k)| = \| P - f \|$.

Theorem 31. Let $M = \{x_1, \ldots, x_{n+1}\}$ be such a minimal set that card $M = n + 1$, $\dim_M V = n$. For $k = 1, \ldots, n + 1$ let us denote

$$C_k = (-1)^{k-1} \begin{vmatrix} Q_1(x_1) \ldots Q_1(x_{k-1}) & Q_1(x_{k+1}) \ldots Q_1(x_{n+1}) \\ \vdots & \\ Q_n(x_1) \ldots Q_n(x_{k-1}) & Q_n(x_{k+1}) \ldots Q_n(x_{n+1}) \end{vmatrix},$$

and $C = -\sum_{k=1}^{n+1} C_k . f(x_k)$.

(1) We have $\mu = \dfrac{|C|}{\sum_{k=1}^{n+1} |C_k|}$.

(2) Let $P \in V$. Then the following assertions are equivalent:

(a) $\max_{k=1,\ldots,n+1} |P(x_k) - f(x_k)| = \mu$.

(b) $\| P - f \| = \mu$.

(c) $P(x_k) - f(x_k) = \| P - f \| . \operatorname{sign}(C\bar{C}_k)$ for $k = 1, \ldots, n + 1$.

(d) There exists $q \in S$ such that $P(x_k) - f(x_k) = q . \operatorname{sign} \bar{C}_k$ for $k = 1, \ldots, n + 1$.

Proof. (1) The assertion follows from Theorem 30 (1).

(2) We have $\dim_M V = n$, hence (b) follows from (a) by Theorems 19 and 20; (c) follows from (b) by Theorem 30 (4); (d) follows obviously from (c) and (a) follows from (d) by Theorem 31 (3).

Remark. The previous Theorems are of a great importance especially for real functions ($S = R$) because in this case for any minimal set $M \neq \emptyset$ the relation $\dim_M V = \operatorname{card} M - 1$ always holds (which is not true of $S = C$). It is then sufficient to prove the existence of a minimal set (e.g. by Theorem 15). We can seldom find it exactly, but even the knowledge of its existence is of a great importance.

Remark. Let a minimal set M have the single element x. By Theorem 25, we have $\dim_{\{x\}} V = 0$, hence $Q(x) = 0$ for all $Q \in V$. Hence we have $\mu = \mu(\{x\}) = |f(x)|$.

2.4. The Application to the Classical Problem

Remark. Let us denote $R^* = \langle -\infty, +\infty \rangle = R \cup \{-\infty, +\infty\}$. Let us put $T = \{(b, c)/b, c \in R\} \cup \{\langle -\infty, c)/c \in R\} \cup \{(b, +\infty\rangle/b \in R\}$. A set $M \subset R^*$ will be called "open" iff for each $x \in M$ there exists $A \in T$ such that $x \in A$, $A \subset M$. We can easily prove that in this way we get a topology on R^*; R^* is then a compact Hausdorff T-space with respect to it. The relative topology induced from R^* to R coincide with the usual topology on R. If $B \subset R^*$, then $C(B)$ will denote the system of all continuous real functions in B.

Assumption (for § 2.4.). Let $I \subset R^*$ be an interval. Let W be an n-dimensional subspace of $C(I)$, let W satisfy the Haar condition on I (i.e. $Q \in W$ and $Q \not\equiv 0$, then Q has at most $n - 1$ zeros in I). Let $Q_1, \ldots, Q_n$ form a basis of W. Let $B \subset I$ be compact, let $\operatorname{card} B \geqq n + 1$, let $f \in C(B)$.

Lemma. Evidently the following conditions are equivalent:

(1) The Haar condition.

(2) If $x_1, \ldots, x_n \in I$ are distinct, then there do not exist $a_1, \ldots, a_n \in S$ not all zero such that $\sum_{k=1}^{n} a_k Q_k(x_j) = 0$ for $j = 1, \ldots, n$.

(3) If $x_1, \ldots, x_n \in I$ are distinct, then $\det Q_k(x_j) \neq 0$.

(4) If $x_1, \ldots, x_n \in I$ are distinct and $y_1, \ldots, y_n \in S$ are arbitrary, then there exists $P \in W$ such that $P(x_j) = y_j$ for $j = 1, \ldots, n$.

(5) If $M \subset I$, then $\dim_M V = \min(n, \operatorname{card} M)$.

Remark. Let us denote $V = \{Q_B/Q \in W\}$. V is a subspace of $C(B)$, $\dim V = \dim_B W = n$. The restrictions of $Q_1, \ldots, Q_n$ to the set B form a basis of V. We shall approximate the function f by means of the polynomials $Q \in V$ in the set B.

If $Q \in V$, then $Q - f \in C(B)$ and $\| Q - f \| = \max_{x \in B} | Q(x) - f(x) | < +\infty$. Let us denote $\mu = \min_{Q \in W} \| Q - f \|$.

If $Q \in W$, then the symbol $\| Q - f \|$ will denote $\max_{x \in B} | Q(x) - f(x) | = \| Q_B - f \|$. We have $\mu = \min_{Q \in V} \| Q - f \|$.

Theorem 32. Let $x_1 < x_2 < \dots < x_{n+1}$ be points in I. For $k = 1, \dots, n+1$ let us denote

$$C_k = (-1)^{k-1} \begin{vmatrix} Q_1(x_1) \dots Q_1(x_{k-1}) \; Q_1(x_{k+1}) \dots Q_1(x_{n+1}) \\ \vdots \\ Q_n(x_1) \dots Q_n(x_{k-1}) \; Q_n(x_{k+1}) \dots Q_n(x_{n+1}) \end{vmatrix}.$$

Then the numbers $C_1, \dots, C_{n+1}$ are non-zero and alternate in sign.

Proof. Let $k \in \{1, \dots, n\}$. For all $x \in I$ let us put

$$Q(x) = \begin{vmatrix} Q_1(x_1) \dots Q_1(x_{k-1}) \; Q_1(x) \; Q_1(x_{k+2}) \dots Q_1(x_{n+1}) \\ \vdots \\ Q_n(x_1) \dots Q_n(x_{k-1}) \; Q_n(x) \; Q_n(x_{k+2}) \dots Q_n(x_{n+1}) \end{vmatrix}.$$

Then $Q \in W$ and Q is continuous on $\langle x_k, x_{k+1} \rangle$. By Lemma (3), we have $Q(x) \neq 0$ for all $x \in \langle x_k, x_{k+1} \rangle$. Hence $Q(x_k) \,.\, Q(x_{k+1}) > 0$. We have $C_k = (-1)^{k-1} Q(x_{k+1})$, $C_{k+1} = (-1)^k Q(x_k)$, hence $C_k \,.\, C_{k+1} < 0$.

Theorem 33. Let $P \in W$ and let $x_1 < \dots < x_{n+1}$ be points in B. Let us define the numbers $C_1, \dots, C_{n+1}$ like in Theorem 32.

(1) We have $\mu \geqq \mu(\{x_1, \dots, x_{n+1}\}) = \dfrac{| \Sigma C_k \,.\, [P(x_k) - f(x_k)] |}{\Sigma | C_k |}$.

(2) Let us suppose that there exists $h \neq 0$ such that $h \,.\, (-1)^k \,.\, [P(x_k) - f(x_k)] \geqq 0$ for $k = 1, \dots, n+1$. Then $\mu \geqq \mu(\{x_1, \dots, x_{n+1}\}) = \dfrac{\Sigma | C_k | \,.\, | P(x_k) - f(x_k) |}{\Sigma | C_k |} \geqq$
$\geqq \min_{k = 1, \dots, n+1} | P(x_k) - f(x_k) |$.

Remark. The inequality between μ and the last term is the well-known relation of de la Vallée–Poussin.

Proof. We have $\dim_{\{x_1, \dots, x_{n+1}\}} V = \dim_{\{x_1, \dots, x_{n+1}\}} W = n$ by Lemma (5). We apply Theorem 28 to the restrictions of $Q_1, \dots, Q_n$ to B and to the function P_B. Then (1) follows from Theorem 28 (5). As to (2): Let the condition in (2) be fulfilled. Then for $k = 1, \dots, n+1$ we have $C_k \neq 0$ and sign $C_k = (-1)^{k-1} \,.\,$ sign C_1, hence $(-h \,.\, \text{sign } C_1) \,.\, C_k \,.\, [P(x_k) - f(x_k)] = -h \,.\, (\text{sign } C_1) \,.\, | C_k | \,.\, (-1)^{k-1} \,.\, (\text{sign } C_1) \,.\, [P(x_k) - f(x_k)] = | C_k | \,.\, h \,.\, (-1)^k \,.\, [P(x_k) - f(x_k)] \geqq 0$. The assertion (2) follows from Theorem 28(6).

Remark. The condition in Theorem 33 (2) says that $P - f$ alternates in sign at the points $x_1, \dots, x_{n+1}$ (or $P(x_k) - f(x_k) = 0$).

Remark. B is a representative subset, hence there exists a minimal set. Let us suppose $f \notin V$, i.e. $\mu > 0$. Let us consider such a subset $M \subset B$ that card $M \leqq n$. By Lemma (4) there exists $P \in W$ such that $P(x) = f(x)$ for all $x \in M$, hence $\mu(M) = 0$. Therefore, if M is a minimal set, necessarily card $M = n + 1$. Hence we have $\dim_M V = n$ by Lemma (5).

Theorem 34. There exists exactly one $P \in W$ such that $\| P - f \| = \mu$.

Proof. By Theorem 20 (3), there exists exactly one $Q \in V$ such that $\| Q - f \| = \mu$ We have $\dim_B W = n$, therefore by Theorem 4 (3) two distinct polynomials of W cannot coincide in B. If $P \in W$ is the only polynomial for which $P_B = Q$, then P is the only polynomial of W such that $\| P - f \| = \mu$.

Theorem 35 (Tchebychev). Let $P \in W$. Then $\| P - f \| = \mu$ iff there exist points $x_1 < \ldots < x_{n+1}$ in B and a number $h \in \{-1, +1\}$ such that $P(x_k) - f(x_k) = h \, . \, (-1)^k \, . \, \| P - f \|$ for $k = 1, \ldots, n + 1$.

Proof. If the latter condition is fulfilled, then by Theorem 33 (2) we have $\mu \geqq \geqq \min | P(x_k) - f(x_k) | = \| P - f \|$, hence $\| P - f \| = \mu$.

Let $\| P - f \| = \mu$. For $\mu = 0$ we may choose the points $x_1 < \ldots < x_{n+1}$ in B arbitrarily; let then $\mu > 0$. Let the points $x_1 < \ldots < x_{n+1}$ in B form a minimal set. We apply Theorem 31 to the restrictions of $Q_1, \ldots, Q_n$ to B and to the function P_B. Let us denote $C_1, \ldots, C_{n+1}, C$ as in Theorem 31. For $k = 1, \ldots, n + 1$ we have $P(x_k) - f(x_k) = \mu \, . \, \text{sign}\, (CC_k) = \mu \, . \, (\text{sign}\, C) \, . \, (-1)^{k-1} \, . \, (\text{sign}\, C_1) = -\text{sign}\, (CC_1) \, . \, (-1)^k \, . \, \| P - f \|$. As $C \neq 0$, we may put $h = -\text{sign}\, (CC_1)$.

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