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COMPLEMENTS IN THE LATTICE OF ALL TOPOLOGIES OF TOP. GROUPS

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This paper deals with the considerations about the structure of a set of all topologies of topological groups and some properties of the complementary topology of the given topology of a topological group.

In the first part it is proved, that the set $\mathfrak{Y}$ of all topologies of topological groups on the underlyining set G is a complete lattice, and it is proved that if G is an abelian group, then the lattice $\mathfrak{Y}$ is modular. In the second part we shall investigate the complement of the topology $\tau(\Sigma)$ of a top. group (G, Σ) that is compatible with the group operation.

Further we shall deal with the complementarity of factogroups of complementare groups, and in one example we shall show, that lattice $\mathfrak{Y}$ is not distributive.

A topology $\tau(\Sigma^*)$ on a set is always given by a neighbourhood basis Σ^*. A neighbourhood of a set in a topological space is an open set containing that set. A topological group on a group G with a topology $\tau(\Sigma)$, that is defined by a neighbourhood basis of zero Σ we shall denote by (G, Σ).

§ 0. PRELIMINARY REMARKS AND DEFINITIONS

0. 1. A topological space is a space in the meaning of Bourbaki (i.e. $\bar{\varphi} = \varphi \quad A \subset \bar{A}$, $\bar{\bar{A}} = \bar{A} \quad \overline{A \cup B} = \bar{A} \cup \bar{B}$. This space is not a T_0-space.

0. 2. Lattice operations are indicated $\vee$, $\wedge$. Partially ordered set A is called complete lattice, if there exists a supremum and an infimum of every its subset.

0. 3. The sets of all topologies on the set N we shall notice $\mathfrak{F}$ or $\mathfrak{F}(N)$. The set of all topologies on a group G that are compatible with the group operation we shall notice $\mathfrak{Y}$ or $\mathfrak{Y}(G)$.

At next considerations the two following sentences are basic.

0. 4. Let (G, Σ) be a topological group. Then Σ fulfils the following conditions:

1. The intersection of two arbitrary sets of Σ contains a set of Σ.
2. For any set $U \in \Sigma$ there exists a set $V \in \Sigma$ such that $V - V \subset U$.

3. For any set $U \in \Sigma$ and any element $u \in U$ there exists a set $V \in \Sigma$ such that $V + u \subset U$.

4. For any set $U \in \Sigma$ and any element $g \in G$ there exists a set $V \in \Sigma$ such that $-g + V + g \subset U$.

Proof: [1], III., § 18., p. 107.

0. 5. Let G be a group. Let Σ be a system of subsets of G fulfilling the conditions 1.–4. of 0. 4. Then (G, Σ) is a top. group. Topology in G defined by means of Σ is determined uniquelly.

Proof: [1], III., § 18., Th. 9., p. 107.

§ 1. STRUCTURE OF THE SET OF TOPOLOGIES TOPOLOGICAL GROUP

This part will be investigated of the structure of the set of topologies, of the topological groups.

1.1 Definition: Let τ_1 and τ_2 be topologies on the set G. We say, that τ_1 *is stronger than* τ_2 (τ_2 *is weaker than* τ_1), when there exists a base Σ_1^* with regard to the topology τ_1 and Σ_2^* regarding to the topology τ_2 and $\Sigma_1^* \supset \Sigma_2^*$. We write $\tau_1 \geqq \tau_2$.

1. 2. Lemma: Let (G, Σ) be a top. group, $M \subset G$ an arbitrary set and Σ' be the set of all open sets in (G, Σ) containing zero in G. Then (G, Σ') is a top. group and it holds $\overline{M}_{\tau(\Sigma)} = \overline{M}_{\tau(\Sigma')}$.

1. 3. Lemma: Let (G, Σ_1) and (G, Σ_2) be top. grops. Then the following statement are equivalent.

1. $\tau(\Sigma_1) \geqq \tau(\Sigma_2)$
2. For arbitrary set $M \subset G$ it is $\overline{M}_{\tau(\Sigma_1)} \subset \overline{M}_{\tau(\Sigma_2)}$.
3. For arbitrary neighbourhood $U \in \Sigma_2$ there exists a neighbourhood $V \in \Sigma_1$ such that $U \supset V$.
4. For the systems of Σ^1 and Σ^2 of all open sets containing zero in G with regard to topologies $\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ it holds $\Sigma^1 \supset \Sigma^2$.

Proof: See [3].

1. 4. Remark: The relation $\leqq$ introduced in def. 1.1. is partially order on the set $\mathfrak{Y}$ of all topologies of top. groups with the same underlyining set G.

1.5. Theorem: The set $\mathfrak{Y}$ of all topologies of top. groups on the same underlyining set G is a complete lattice with the greatest element $\tau(\mathring{\Sigma})$, $\mathring{\Sigma} = \{X \subseteqq G : 0 \in X\}$ and the smallest element $\tau(\underset{\circ}{\Sigma})$, $\underset{\circ}{\Sigma} = \{G\}$.

Proof: Let $\tau(\Sigma_i) \in \mathfrak{Y}(G)$, $i \in I$. Evidently it is $\tau(\mathring{\Sigma}) \geqq \tau(\Sigma_i) \geqq \tau(\underset{\circ}{\Sigma})$.

Let $\{\tau(\Sigma_i) : i \in I\}$ be an arbitrary set of topologies on the group G Σ_i be a system of all neighbourhoods of zero in the topology $\tau(\Sigma_i)$, $i \in I$. Let $Q = \{\bigcap_{i \in I} U_i : U_i \in \Sigma_i, i \in I$, card $\{i \in I: U_i \neq G\} < \aleph_0\}$ and we prove that $\bigvee_{\mathfrak{Y}(G)} \{\tau(\Sigma_i) : i \in I\} = \tau\{Q\}$.

First we prove, that the system G fulvils conditions 0.5.

1. Let $W_1, W_2 \in Q$. Then $W_1 = \bigcap_{i \in I_0} U_i$, $W_2 = \bigcap_{j \in J} U_j$, where $U_i \in \Sigma_i$, $i \in I_0$, $U_j \in \Sigma_j$, $j \in J_0$, I_0 and J_0 are subsets in I. It is $W_1 \cap W_2 = \bigcap_{k \in K_0} U_k$, where $K_0 = I_0 \cup J_0$, $U_k \in \Sigma_{k_0}$, $K_0 \subset I$, card $K_0 < \aleph_0$.

2.–4. Let $W = \bigcap_{i \in I} U_i \in Q$ is an arbitrary element. Then there exists a set $I_0 \subset I$, card $I_0 < \aleph_0$ such that $U_i \neq G$ for $i \in I_0$ and $U_i = G$ for $i \in I \setminus I_0$. Then for arbitrary elements $w \in W$, $g \in G$, U_i, $i \in I_0$ according to 0. 4. there exists neighbourhoods $V_i \in \Sigma_i$ $i \in I_0$ such that it is $V_i - V_i \subset U_i$ $[V_i + w \subset U_i$, $-g + V_i + g \subset U_i]$. For $i \in I \setminus I_0$ it is $U_i = G$ and then it holds this relation for $V_i = G$. It means, that $\bigcap_{i \in I} V_i - \bigcap_{i \in I} V_i \subset \bigcap_{i \in I} (V_i - V_i) \subset \bigcap_{i \in I} U_i$, $[\bigcap_{i \in I} V_i + w = \bigcap_{i \in I} (V_i + w) \subset \bigcap_{i \in I} U_i$, $-g + \bigcap_{i \in I} V_i + g = \bigcap_{i \in I} (-g + V_i + g) \subset \bigcap_{i \in I} U_i$, where only for $i \in I_0$, card $I_0 < \aleph_0$ is $V_i \neq G$ and then $\bigcap_{i \in I} V_i \in Q$. According to 0. 5. (G, Q) is a top. group. The system Q is by the system Σ_i, $i \in I$ uniquely defined.

Therefore $G \in \bigcap_{i \in I} \Sigma_i$, $\bigcap_{i \in I} \Sigma_i \subset Q$ and according to L. 1. 3. it is $\tau(\Sigma_i) \leqq \tau(Q)$. If $\varphi \in \mathfrak{Y}$ is an arbitrary topology with the property $\varphi \geqq \tau_i$, $i \in I$, and Σ' a system of all neighbourhoods of zero in topology φ, then it is $\Sigma' \supset \bigcap_{i \in I} \Sigma_i$ (see L. 1. 3).

For arbitrary $W \in Q$ it is $W = \bigcap_{i \in I} U_i$, where $U_i \in \Sigma_i$, $i \in I$. From the condition 1. 0. 4. there exists a neighbourhood $W' \in \Sigma'$ such that $W' \subset W$. By the implication 3. $\Rightarrow$ 1. of the lemma 1. 3. it is $\tau(\Sigma') \geqq \tau(Q)$. So the topology $\tau(Q)$ is the supremum of topologies $\tau(\Sigma_i)$, $i \in I$.

We proved, that the set $\mathfrak{Y}$ has the smallest element that is the trivial topology and that every subset of set $\mathfrak{Y}$ has a supremum. So the set $\mathfrak{Y}$ is a complete lattice.

1.6. Corollary: If $\tau(\Sigma_i) \in \mathfrak{Y}(G)$, $i \in I$ then it holds $\bigvee_{\mathfrak{Z}(G), i \in I} \tau(\Sigma_i) = \bigvee_{\mathfrak{Y}(G), i \in I} \tau(\Sigma_i) = \tau(Q)$, where $Q = \{\bigcap_{i \in I} U_i : U_i \in \Sigma_i, i \in I$, card $\{i \in I, U_i \neq G\} < \aleph_0\}$.

Proof: It follows from the proof of the theorem 1.5.

1.7. Definition: Let $\tau(\Sigma_1), \tau(\Sigma_2) \in \mathfrak{Y}(G)$. We say that $\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ are commutative topologies if for arbitrary $U \in \Sigma_1$, $V \in \Sigma_2$ there exist $U_1, U_2 \in \Sigma_1$, $V_1, V_2 \in \Sigma_2$ so that $U + V \supset V_1 + U_1$, $V + U \supset U_2 + V_2$.

1.8. Theorem: If $\tau(\Sigma_1), \tau(\Sigma_2) \in \mathfrak{Y}(G)$, $\Sigma = \{U + V : U \in \Sigma_1, V \in \Sigma_2\}$, $\Sigma' = \{V + U : U \in \Sigma_1, V \in \Sigma_2\}$, then the following statements are equivalent:

1. $\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ are commutative.

2. $\tau(\Sigma) = \tau(\Sigma')$

3. $\tau(\Sigma_1) \wedge_{\mathfrak{Y}} \tau(\Sigma_2) = \tau(\Sigma)$

Proof: 1. $\Rightarrow$ 3.: First we prove that the system Σ fulfils the assumplitions of complete system of neighbourhoods of zero of a topology from $\mathfrak{Y}(G)$.

1. For arbitrary $U + V$, $U_1 + V_1 \in \Sigma$ it is $(U + V) \cap (U_1 + V_1) \supset (U \cap U_1) + (V \cap V_1) = U_2 + V_2$ where $U_2 \in \Sigma_1$, $U_2 \cap U_1$, $V_2 \in \Sigma_2$, $V_2 = V \cap V_1$.

2. Let $U + V \in \Sigma$ be arbitrary. Then ther exist $U' \in \Sigma_1$, $V' \in \Sigma_2$ such that $U \supset U' + U'$, $V \supset V' + V'$ and therefore $U' + V' \in \Sigma$. Further there exist $U'' \in \Sigma_1$, $V'' \in \Sigma_2$ such that $U' + V' \supset V'' + U''$ and $U'' + V'' \in \Sigma$, $U'' \subset U'$, $V'' \subset V'$ and it holds $(U'' + V'') + (U'' + V'') = U'' + (V'' + U'') + V'' \subset U'' + (U' + V') + V'' \subset (U' + U') + (V' + V') \subset U + V$.

Further there exist $V''' \in \Sigma_2$, $U''' \in \Sigma_1$ such that $-V''' \subset V''$, $-U''' \subset U''$ and it is $-(U''' + V''') = -V''' - U''' \subset V'' + U'' \subset U' + V' \subset U + V$.

3. For arbitrary $U + V \in \Sigma$ and $u + v \in U + V$ there exist $U' \in \Sigma_1$ and $V' \in \Sigma_2$, $V'' \in \Sigma_2$ such that $U' + u \subset U$, $V' + v \subset V$, $-u + V'' + u \subset V'$. Then $U' + V'' \in \Sigma$ and $(U' + V'') + (u + v) = U' + (V'' + u) + v \subset U' + (u + V') + v = (U' + u) + (V' + v) \subset U + V$.

4. For arbitrary $U + V \in \Sigma$, $g \in G$ there exist $U_1 \in \Sigma_1$, $V_1 \in \Sigma_2$ such that $-g + U_1 + g \subset U_1$, $-g + V_1 + g \subset V$ and $-g + (U_1 + V_1) + g = (-g + U_1 + g) + (-g + V_1 + g) \subset U + V$, $U_1 + V_1 \in \Sigma$.

Together vith regard to 0. 5. it is $\tau(\Sigma) \in \mathfrak{Y}(G)$.

Now we prove, that $\tau(\Sigma_1) \wedge_{\mathfrak{Y}(G)} \tau(\Sigma_2) = \tau(\Sigma)$. Evidently $\tau(\Sigma) \leqq \tau(\Sigma_i)$, $i = 1, 2$ and if $\tau(\Sigma_0) \in \mathfrak{Y}(G)$, $\tau(\Sigma_0) \leqq \tau(\Sigma_i)$, $i = 1, 2$, then for arbitrary neighbournhood $U_0 \in \Sigma_0$ there exists $W_0 \in \Sigma_0$ such that $U_0 \supset W_0 + W_0$. Further there exist $U_1 \in \Sigma_1$, $V_1 \in \Sigma_2$, $W_0 \supset U_1 \cup V_1$ and then $U_0 \supset U_1 + V_1$, $U_1 + V_1 \in \Sigma$ and by 1.3. it is $\tau(\Sigma_0) \leqq \tau(\Sigma)$, i.e., $\tau(\Sigma)$ is an infimum of topologies $\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ in the lattice $\mathfrak{Y}$.

3. $\Rightarrow$ 1.: For arbitrary $U \in \Sigma_1$, $V \in \Sigma_2$ there exist $U_1 \in \Sigma_1$, $V_1 \in \Sigma_2$ such that $\pm U_1 \subset U$, $\pm V_1 \subset V$, $U + V \supset (U_1 + V_1) + (U_1 + V_1) \supset V_1 + U_1$. Further, there exist $U_2 \in \Sigma_1$, $V_2 \in \Sigma_2$ such that $U_1 + V_1 \supset -(U_2 + V_2)$ and $V + U \supset -V_1 - U_1 = -(U_1 + V_1) \supset U_2 + V_2$.

Together $\tau(\Sigma)$ and $\tau(\Sigma')$ are commutative.

1. $\Leftrightarrow$ 2. According to the definition 1.7. and lemma 1.3.

1.9. Corollary: Let G be an abelian group, $\tau(\Sigma_1)$, $\tau(\Sigma_2) \in \mathfrak{Y}(G)$. Then $\tau(\Sigma_1) \bigwedge_{\mathfrak{Y}(G)} \tau(\Sigma_2) = \tau(\Sigma)$, where $\Sigma = \{U + V : U \in \Sigma_1, V \in \Sigma_2\}$.

1.10. Theorem. If G is an abelian group, then the lattice $\mathfrak{Y}(G)$ is modular.

Proof: Let topology $\tau(\Sigma_i) \in \mathfrak{Y}(G)$, $i = 1, 2, 3$, $\tau(\Sigma_1) \leqq \tau(\Sigma_3)$. Regarding to Lemma 1.2. we may suppose that Σ_i is formed with all open sets in $\tau(\Sigma_i)$, which contains zero in G ($i = 1, 2, 3$). Let us indicate: $\tau' = \tau(\Sigma_i)\ V_{\mathfrak{Y}}\ (\tau(\Sigma_2) \wedge_{\mathfrak{Y}} \tau(\Sigma_3))$, $\tau'' = (\tau(\Sigma_1)\ V_{\mathfrak{Y}}\ \tau(\Sigma_2)) \wedge_{\mathfrak{Y}} \tau(\Sigma_3)$. According to 1.6. and 1.9. it is $\Sigma' = \{U_1 \cap (U_2 + U_3) : U_i \in \Sigma_i, i = 1, 2, 3\}$, ($\Sigma'' = \{(U_1 \cap U_2) + U_3 : U_i \in \Sigma_i, i = 1, 2, 3\}$, respectivelly) a complete system of neighbourhoods of zero of the topology τ' (resp. τ''). Let $U'' \in \Sigma''$ be an arbitrary neighbourhood. Then $U'' = (U_1 \cap U_2) + U_3$, $U_i \in \Sigma_i$, $i = 1, 2, 3$ and there exist $U_1^\circ \in \Sigma_1$, $U_3^\circ \in \Sigma_3$ such that $-U_1^\circ + U_1^\circ \subset U_1$, $U_3 \subset U_1 \cap U_3$ because $U_1^\circ \in \Sigma_1 \subset \Sigma_3$. From that for arbitrary $u' \in U' = U_1^\circ \cap (U_2 + U_3^\circ)$ it holds $u' = u_2 + u_3 \in U_1^\circ$, where $u_2 \in U_2$, $u_3 \in U_3^\circ$. Then $u_2 = -u_3 + u' \in -U_3 + U_1^\circ \subset -U_1^\circ \subset -U_1^\circ + U_1^\circ \subset U_1$, what it means, that $u' \in (U_1 \cap U_2) + U_3$, i.e., $U' \subset U''$ and by lemma 1.3. it is $\tau' \geqq \tau''$. Relation $\tau' \leqq \tau''$ is obvious, then $\tau' = \tau''$. Together we proved, that $\mathfrak{Y}(G)$ is a modulare lattice.

§ 2. A COMPLEMENT OF A TOPOLOGY OF A TOPOLOGICAL GROUP IN THE LATTICE $\mathfrak{F}(G)$ AND $\mathfrak{Y}(G)$

2.0. Definition: Let G be a set and $\tau(\Sigma)$ be a topology on G a topology $\tau(\Sigma')$ on G is called a complement of the topology $\tau(\Sigma)$, if the supremum of topologies $\tau(\Sigma)$ and $\tau(\Sigma')$ is a discrete topology and the infimum of topologies $\tau(\Sigma)$ an $\tau(\Sigma')$ is a trivial topology.

2.1. Theorem: Let (G, Σ) and (G, Σ') be top. groups. Then topologies $\tau(\Sigma)$ and $\tau(\Sigma')$ are complementary in the lattice $\mathfrak{F}$ if and only if $G = \cap\, \Sigma + \cap\, \Sigma'$ and there exist neighbourhoods $U \in \Sigma$, $V \in \Sigma'$, with the property $U \cap V = \{o\}$.

Proof: $\Rightarrow$: Let $\tau(\Sigma)$ and $\tau(\Sigma')$ be complementary in $\mathfrak{F}$. With the assumption $\tau(\Sigma) \vee_{\mathfrak{F}} \tau(\Sigma') = \tau(\{2^G\})$ it is evident that there exist neighbourhoods $U \in \Sigma$, $V \in \Sigma'$ with the property $U \cap V = \{0\}$. Further for arbitrary neighbourhoods $U \in \Sigma$, $V \in \Sigma'$ it is the set $U + V$ open in $\tau(\Sigma)$, $\tau(\Sigma')$ and with the assumption $\tau(\Sigma) \wedge_{\mathfrak{F}} \tau(\Sigma') = \tau(\{G\})$ it follows $U + V = G$. Let $g \in G$ be an arbitrary element. Then for each $U \in \Sigma$, $V \in \Sigma'$ there exist elements $u \in U$, $v \in V$, that $g = u + v$. If $U, U_1 \in \Sigma$, $V, V_1 \in \Sigma'$ are such neighbourhoods, that $U \cap V \subset \{0\}$, $-U_1 + U_1 \subset U$, $V_1 - V_1 \subset V$, then there exist $u_1 \in U_1$, $v_1 \in V_1$ such that $g = u_1 + v_1$. Let $V_2 \in \Sigma'$, $v_1 \notin V_2 \subset V_1$ exist. Then there exist $V_3 = V_1 \cap V_2$, $V_3 \in \Sigma'$ and it holds $g = u_2 + v_2$, where $u_2 \in U_1$, $v_2 \in V_3$. From there $-u_2 + u_1 = v_2 - v_1 \in (-U_1 + U_1) \cap (V_1 - V_1) \subset U \cap V = \{0\}$. Then $v_1 = v_2 \in V_3 \subset V_2$, what is a contradiction. It means, that $v_1 \in \cap\, \Sigma'$and similarly we prove, that $u_1 \in \cap\, \Sigma$ and $G = \cap\, \Sigma + \cap\, \Sigma'$.

$\Leftarrow$: If there exist arbitrary neighbourhoods $U \in \Sigma$, $V \in \Sigma'$, $U \cap V = \{0\}$, then evidently $\tau(\Sigma) \vee_{\mathfrak{F}} \tau(\Sigma') = \tau(\{2^G\})$. Let $G = \cap\, \Sigma + \cap\, \Sigma'$ be and $\emptyset \neq A \subset G$, A be an open set in $\tau(\Sigma)$ and $\tau(\Sigma')$, $a \in A$ arbitrary element. Then $O \in A - a$ is an open set in $\tau(\Sigma)$ and $\tau(\Sigma')$, then $\cap\, \Sigma \mathrm{U} \cap\, \Sigma' \subset A - a$. Further, for an arbitrary element $z \in \bigcap \Sigma$ there exists neighbourhood $U_z \in \Sigma'$ such that $z + U_z \subset A - a$, what means, that

$G = \cap \Sigma + \cap \Sigma' \subset \bigcup\{z + U_z : z \in \cap \Sigma\} \subset A - a \subset G$. Together $A = G + a = G$ and $\tau(\Sigma) \wedge_{\mathfrak{F}} \tau(\Sigma') = \tau(\{G\})$.

2.2. Theorem: Let (G, Σ) and (G, Σ') be top. groups. Then the topologies $\tau(\Sigma)$ and $\tau(\Sigma')$ are complementary in the lattice $\mathfrak{F}$ if and only if $G = U + V$, for any $U \in \Sigma$, $V \in \Sigma'$ and there exist neighbourhoods $U_0 \in \Sigma$, $V_0 \in \Sigma'$ with the property $U_0 \cap V_0 = \{0\}$.

Proof: $\Rightarrow$: It follows from the proof the Theorem 2.1.

$\Leftarrow$: Let $G = U + V$, be for arbitrary $U \in \Sigma$, $V \in \Sigma'$ and let neighbourhoods $U_0 \in \Sigma$, $V_0 \in \Sigma'$ exist such that $U_0 \cap V_0 = \{0\}$. Evidently $\{0\} \in \tau(\Sigma) \vee_{\mathfrak{F}} \tau(\Sigma')$ and $\tau(\Sigma) \vee_{\mathfrak{F}} \vee_{\mathfrak{F}} \tau(\Sigma')$ is a discrete topology. Further, if $A \subset G$ is a not empty open set in $\tau(\Sigma)$ and $\tau(\Sigma')$ and $a \in A$ an arbitrary element. Then $A - a$ is open in $\tau(\Sigma)$ and so in $\tau(\Sigma')$ and $O \in A - a$. There exists a neighbourhood $U \in \Sigma$, $U \subset A - a$. Further, for any $z \in U$ there exists a neighbourhood $U_z \in \Sigma'$ such that $z + U_z \subset A - a$. One of these neighbourhoods $U_z \in \Sigma'$ we sign V. It holds $G = U + V \subset U\{z + U_z : z \in U\} \subset A - a \subset \subset G$. Together $A = G + a$ and then $\tau(\Sigma) \wedge_{\mathfrak{F}} \tau(\Sigma') = \tau(\{G\})$.

2.3. Definition: Let G be a group, A, B normal subgroups in G. If $G = A + B$, $A \cap B = \{O\}$, then it is called, that G is *a direct product of A, B*.

2.4. Lemma: Let (G, Σ) be a top. group and let $U \in \Sigma$ be a subgroup in G. Then U is a closed set in G (U is a clopen set).

Proof: Let $g \in \overline{U}$. $U + g$ is a neighbourhood of g, $(U + g) \cap U \neq \emptyset$. Then there exist elements $f, h \in U$ such, that $f + g = h$. From there $g = -f + h \in U$, or $\overline{U} \subset U$ an then $\overline{U} = U$.

2.5. Theorem: Let (G, Σ) and (G, Σ') be top. groups. Then topologies $\tau(\Sigma)$ an $\tau(\Sigma')$ are complementary in $\mathfrak{F}$ if and only if, there exist basis Σ_0 and Σ'_0 of topologies $\tau(\Sigma)$ and $\tau(\Sigma')$, $\Sigma_0 \subset \Sigma$, $\Sigma'_0 \subset \Sigma'$ such that G is a direct product of any neighbourhood $U \in \Sigma_0$ and $V \in \Sigma'_0$.

Proof: $\Rightarrow$: Topologies $\tau(\Sigma)$ and $\tau(\Sigma')$ are complementary in $\mathfrak{F}$, then with regard to theorem 2.2. there exists a neighbourhood $U_0 \in \Sigma$ and $V_0 \in \Sigma'$ such that $U_0 \cap V_0 = \{O\}$ and for any $U \in \Sigma$ and any $V \in \Sigma'$ it holds $U + V = G$.

We chose the systems $\Sigma_1 = \{U \subset U_0 : U \in \Sigma\}$, $\Sigma'_1 = \{V \subset V_0 : V \in \Sigma'\}$. For any neighbourhoods $U \in \Sigma_1$, $V \in \Sigma'_1$ it is $U \cap V = \{O\}$, $U + V = G$. For any neighbourhood $U \in \Sigma\ (U \in \Sigma')$ there exists a neighbourhood $U_1 \in \Sigma_1 \subset \Sigma\ (U_1 \in \Sigma'_1 \subset \Sigma')$ such that $U_1 \subset U$. Then Σ_1 (Σ'_1) is a base of zero of topology $\tau(\Sigma)$ $(\tau(\Sigma'))$.

We shall prove that for every neighbourhoods $U \in \Sigma_1$ there exists a neighbourhood $W \in \Sigma_1$, $W \subset U$ such that W is a subgroup. For any $U \in \Sigma_1$ there exists $W_1 \in \Sigma_1$ such that $W \subset U$, $-W - W + W + W \subset U$. Evidently it is $W \cap V = \{O\}$, $W + V = G$ for any $V \in \Sigma'_1$.

Let $w_1, w_2 \in W$ be. Then we can write $w_1 + w_2$ in the form: $w_1 + w_2 = w + v$, where $w \in W$, $v \in V \in \Sigma_1'$. From there $v = -w + w_1 + w_2 \subset U$, i.e., $v \in U$, $v \in V$ but $U \cap V = \{O\}$, i.e., $v = O$, when $w_1 + w_2 = w \in W$ and then $W + W \subset W$. Let $w_1 \in W$. The element $-w_1$ we can write in the form $-w_1 = w + v$, where $w \in W$, $v \in V \Rightarrow v = -w - w_1 \subset U$, i.e., $v \in V$, $v \in U$. But $U \cap V = \{O\}$ follows $v = 0$, $-v = 0$, $-w_1 = w \in W$, $-W \subset W$. Together W is a subgroup. According to 2.4. W is a clopen subgroup in G.

We notice $\Sigma_2 = \{W : W \in \Sigma_1, W \text{ is a clopen subgroup in } G\}$. In the same way we can construct the system

$$\Sigma_2' = \{W' : W' \in \Sigma_1', W' \text{ is a clopen subgroup in } G\}.$$

For any neighbourhood $U \in \Sigma (U \in \Sigma')$ we ca find a neighbourhood $W \in \Sigma_2 \subset \subset \Sigma (W \in \Sigma_2' \subset \Sigma')$ such that $W \subset U$. Then $\Sigma_2 (\Sigma_2')$ is a base of zero of topology $\tau(\Sigma)$, $(\tau(\Sigma'))$. For any $W \in \Sigma_2$ there exist $W_0 \in \Sigma_2$, $W_0 \subset W$ such, that W_0 is a norm. subgroup in G. Let $W \in \Sigma_2$, $W' \in \Sigma_2'$. Then W and W' are clopen subgroups in G and it is $W + W' = G$, $W \cap W' = \{0\}$. If $g \in G$ is an arbitrary element, then there exist $W_0 \subset W$, $W_0' \subset W'$, $W_0 \in \Sigma_2$, $W_0' \in \Sigma_2'$, such that $-g + W_0 + g \subset W$, $-g + W_0' + + g \subset W'$. We choose any $w_0 \in W_0$. Then $-g + w_0 + g = \tilde{w}_0 + \tilde{w}_0'$, where $\tilde{w}_0 \in W_0$, $\tilde{w}_0' \in W_0'$ and $-\tilde{w}_0 - g + w_0 + g = \tilde{w}_0' \in W_0'$. It is $w_0' \in W$ because $-\tilde{w}_0 \in W_0 \subset W$, $-g + w_0 + g \in W$ and it follows $-\tilde{w}_0 - g + w_0 + g = \tilde{w}_0' \subset W$. Therefore $W \cap W_0' = \{0\}$, $\tilde{w}_0' = 0$, $-g + w_0 + g = \tilde{w}_0 \in W_0$, $-g + W_0 + g \subset W_0$, and W_0 is a normal subgroup in G. We sign

$$\Sigma_0 = \{W_0 : W_0 \in \Sigma_2, W_0 \text{ is a normal subgroup in } G\}$$

and with the same method, we construct $\Sigma_0' \subset \Sigma_2'$,

$$\Sigma_0' = \{W_0' : W' \in \Sigma_2', W_0' \text{ is a norm. subgroup in } G\}.$$

Finally we have that in every neighbourhood $U \in \Sigma$ (resp. Σ') if can be found a neighbourhood, $W_0 \in \Sigma_0 \subset \Sigma (W_0 \in \Sigma_0' \subset \Sigma')$ such that $W_0 \subset U$. Then Σ_0, Σ_0' resp., is the base of zero of topology $\tau(\Sigma)$, $\tau(\Sigma')$ resp. For each neighbourhood $W_0 \in \Sigma_0$ and $W_0' \in \in \Sigma_0$ if is $W_0 \cap W_0' = \{0\}$, $W_0 + W_0' = G$. Then Σ_0 and Σ_0' are such bases that G is a direct product of an arbitrary neighbourhood $U \in \Sigma_0$ and $V \in \Sigma_0'$, $\Sigma_0 \subset \Sigma$, $\Sigma_0' \subset \Sigma'$. Evidently (G, Σ_0) and (G, Σ_0') are top. groups.

$\Leftarrow$: Systems Σ_0 and Σ_0' are bases of topologies $\tau(\Sigma)$ and $\tau(\Sigma')$, and G is a direct product of arbitrary neighbourhoods $U \in \Sigma_0$ and $U' \in \Sigma_0'$. For every neighbourhood $W_0 \in \Sigma_0$ and $W_0' \in \Sigma_0'$ it is $W_0 \cap W_0' = \{0\}$, $W_0 + W_0' = G$ and therefore according to the theorem 2.2. topologies $\tau(\Sigma)$ and $\tau(\Sigma')$ are complementary in $\mathfrak{F}$.

2.6. Corollary: Let (G, Σ) and (G, Σ') be top. groups $\tau(\Sigma)$, $\tau(\Sigma')$ be commutative topologies. Then there are the following conditions equivalent:

1. $\tau(\Sigma)$, $\tau(\Sigma')$ are complementary in lattice $\mathfrak{F}$.

2. $G = \cap \Sigma + \cap \Sigma'$ and there exist a neighbourhoods $U \in \Sigma$, $V \in \Sigma'$, such that $U \cap V = \{0\}$.

3. $U + V = G$ for any $U \in \Sigma$, $V \in \Sigma'$ and there exist neighbourhoods $U_0 \in \Sigma$, $V_0 \in \Sigma'$, such that $U_0 \cap V_0 = \{0\}$.

4. There exist bases of zero $\Sigma_0 \subset \Sigma$, $\Sigma_0' \subset \Sigma'$ of topologies $\tau(\Sigma)$, $\tau(\Sigma')$ such, that G is a direct product of any neighbourhood $U \in \Sigma_0$ and neighbourhood $V \in \Sigma_0'$.

5. $\tau(\Sigma)$, $\tau(\Sigma')$ are complementary in the lattice $\mathfrak{Y}$.

Proof: 1. $\Leftrightarrow$ 2.: see Th. 2. 1

1. $\Leftrightarrow$ 3.: see Th. 2. 3

1. $\Leftrightarrow$ 4.: see Th. 2. 5

1. $\Rightarrow$ 5.: If $\tau(\Sigma)$ and $\tau(\Sigma')$ are complementary in lattice $\mathfrak{F}$, then according to the corollary 1. 6. it is $\tau(\Sigma) \vee_{\mathfrak{F}} \tau(\Sigma') = \tau(\Sigma) \vee_{\mathfrak{Y}} \tau(\Sigma')$. By Th. 2. 2. for all $U \in \Sigma$, $V \in \Sigma'$ it holds $U + V = G$. According to the corollary 1. 9. if is $\tau(\Sigma) \wedge_{\mathfrak{Y}} \tau(\Sigma') = \tau(\{U + V : U \in \Sigma, V \in \Sigma'\}) = \tau(\{G\})$ and then $\tau(\Sigma)$ and $\tau(\Sigma')$ are complementary in the lattice $\mathfrak{Y}$.

5. $\Rightarrow$ 1.: If $\tau(\Sigma)$ and $\tau(\Sigma')$ are complementary in the lattice $\mathfrak{Y}$, then $\tau(\Sigma) \vee_{\mathfrak{Y}} \tau(\Sigma') = \tau(\Sigma) \vee_{\mathfrak{F}} \tau(\Sigma')$. Further, it is $\tau(\Sigma) \wedge_{\mathfrak{Y}} \tau(\Sigma') = \tau(\{G\})$, but $\tau(\Sigma) \wedge_{\mathfrak{Y}} \tau(\Sigma') = \tau(\{U + V : U \in \Sigma, V \in \Sigma'\})$, $U + V = G$, for any $U \in \Sigma$, $V \in \Sigma'$. Then by 2. 2. $\tau(\Sigma)$ and $\tau(\Sigma')$ are complementary in the lattice $\mathfrak{F}$.

2.7. Remark: Let (G, Σ) and (G, Σ') be top. groups. Then the conditions 1., 2., 2., 4. are equivalent to the result 2. 6.

2.8. Corollary: Let G be a group (an abelian group, resp.), $\tau(\Sigma) \in \mathfrak{Y}_0(G)$. Then $\tau(\Sigma)$ has a complement (from $\mathfrak{Y}(G)$) in lattice $\mathfrak{F}$ ($\mathfrak{Y}$, resp.) if and only if $\tau(\Sigma)$ is discrete.

Proof: In both cases it follows from 2. 1. and 2. 6., that $\tau(\Sigma)$ has a complement $\tau(\Sigma')$ there in $\mathfrak{Y}$ if and only if $G = \cap \Sigma + \cap \Sigma'$ and there exist neighbourhoods $U \in \Sigma$, $V \in \Sigma'$ such that $U \cap V = \{0\}$. But then $\cap \Sigma = \{0\}$ when $G = \cap \Sigma'$, $\tau(\Sigma') = \tau(\{G\})$ or $\tau(\Sigma) = \tau(\{2^G\})$. The rest of the proof is evident.

2.9. Corollary: Let G be a group (an abelian group, resp.), $\tau(\Sigma) \in \mathfrak{Y}(G) \setminus \mathfrak{Y}_0(G)$. Then $\tau(\Sigma')$ (from $\mathfrak{Y}(G)$) is a complement to $\tau(\Sigma)$ in lattice $\mathfrak{F}$ ($\mathfrak{Y}$, resp.) in that case when G is a direct product of $\cap \Sigma$ and $\cap \Sigma'$ and $\cap \Sigma \in \Sigma$, $\cap \Sigma' \in \Sigma'$.

Proof: If it is $\tau(\Sigma')$ a complement of $\tau(\Sigma)$, then $\cap \Sigma$, $\cap \Sigma'$ are normal subgroups in G and according to 2. 1. and 2. 6. it is $G = \cap \Sigma + \cap \Sigma'$, $\cap \Sigma \cap \cap \Sigma' = \{0\}$. Further, $U \in \Sigma$, $V \in \Sigma'$ and by 2. 5. there exist normal subgroups in G, $A \subset U$, $B \subset V$, $A \in \Sigma'$, $B \in \Sigma'$ such that $A + B = G$, $A \cap B = \{0\}$. If there exist $U_1 \in \Sigma$, $U_1 \nsubseteq A$ by the certain arguments there exists a normal subgroup $C \subset U_1$, $C \in \Sigma$, $C \nsubseteq A$ such that $C + B = G$, $C \cap B = \{0\}$. With regard to the modularity of the lattice of normal subgroups in G we receive a contradiction and then $A = \cap \Sigma \in \Sigma$. Similarly we prove, that $\cap \Sigma' \in \Sigma'$. The rest of proof is evident.

2.10. Example: If G is an abelian group, $G = A_1 \oplus A_2 = A_1 \oplus A_3$ are direct products, $A_2 \neq A_3$, then for topologies $\tau(\Sigma_i)$, where $\Sigma_i = \{X \subset G : A_i \subset X\}$, it is $\tau(\Sigma_i) \in \mathfrak{Y}(G)$, $i = 1, 2, 3$. According to 2. 1. topology $\tau(\Sigma_1)$ is complementary to $\tau(\Sigma_2)$ and $\tau(\Sigma_3)$, $\tau(\Sigma_2) \neq \tau(\Sigma_3)$. Obviously the condition $\tau_1 \wedge (\tau_2 \vee \tau_3) = (\tau_1 \wedge \tau_2) \vee \vee (\tau_1 \wedge \tau_3)$ is not fulfilled and the lattice $\mathfrak{Y}(G)$ is not distributive.

2.11. Definition: *A top. space G* is called *connected* when it is not decompose it in two (not empty) disjunctive closed (open) sets A, B. In the contrary case the top. space is called *disconnected.*

2.12. Theorem: Let (G, Σ) and (G, Σ') be top. groups. If topologies $\tau(\Sigma)$ and $\tau(\Sigma')$ are complementary in $\mathfrak{F}$ and $\Sigma \neq \{\emptyset, G\} \neq \Sigma'$ then (G, Σ) and (G, Σ') are disconnected spaces.

Proof: Let us assume, that (G, Σ) is a connected top. space. From the theorem 2. 5. it follows that there exists a neighbourhood $U \in \Sigma$, which is a subgroup in (G, Σ), $\{0\} \neq V \neq G$. According to the lemma 2. 4. V is a closed set, i.e., U is clopen. The set $G \setminus U$ is closed in (G, Σ) and $U \cup (G \setminus U) = G$. It can be possible only if $G \setminus U = \emptyset$ or $G = U$ and it is a contradiction, with the fact that U is a subgroup in G, $\{0\} \neq \neq U \neq G$.

Similarly we can prove that (G, Σ') is a disconnected top. space.

2.13. Theorem: Let (G, Σ) and (G, Σ'), be top. groups, topologies $\tau(\Sigma)$ and $\tau(\Sigma')$, be complementary and $N \subset G$ be a subgroup in G. If $\Sigma_N = \{U \cap N : U \in \Sigma\}$, $\Sigma'_N = = \{V \cap N : V \in \Sigma'\}$, then top. groups (N, Σ_N) and (N, Σ'_N) have again complementary topologies $\tau(\Sigma_N)$ and $\tau(\Sigma'_N)$ if $\mathfrak{F}$.

Proof: We prove, that the systems Σ_N and Σ'_N fulfil the conditions Th. 2.2.

1. Let $U_N \in \Sigma_N$, $V_N \in \Sigma'_N$, $U_N = U \cap N$, $V_N = V \cap N$, where $U \in \Sigma$, $V \in \Sigma'$. It holds $U + V = G$, $N \subset G \Rightarrow N = U \cap N + V \cap N = U_N + V_N$.

2. If $F \in \Sigma$, $H \in \Sigma'$, $F \cap H = \{0\}$, then there exist sets $F_N \in \Sigma_N$ and $H_N \in \Sigma'_N$, $F_N = F \cap N$, $H_N = H \cap N$ and it is $F_N \cap H_N = F \cap N \cap H \cap N = F \cap H \cap N = = \{0\}$.

2.14. Remark: If (G, Σ) and (G, Σ') are complementary top groups, $N \subset G$ is a normal subgroup in G, then the top. factogroups $(G \setminus N, \Sigma^*)$ and $(G \setminus N, \Sigma'^*)$ are not complementary generally. (The system Σ^*-see [1], p. 111 def. 24).

The situation, when complementary factogroups we can derive from complementary groups, it is shown in the following, theorem.

2.15. Theorem: Let (G, τ) and (G, τ') be top. groups, and topologies τ and τ' be complementary, Σ and Σ' be bases of topologies τ and τ' composed with normal subgroups. Let $N = \cap \Sigma$ or $N = \cap \Sigma'$. Then top. factogroups $(G / N, \Sigma^*)$ and $(G / N, \Sigma'^*)$ are complementary (in $\mathfrak{F}$).

Proof: We chose bases Σ and Σ' of topologies τ and τ' in such a way, that they are composed only with the normal subgroups in G. Such bases exist (see 2.5.) First, we prove that $N = \cap \Sigma$ is a normal subgroup. By 0. 5. for any neighbourhoods $U \in \Sigma$ and any el$_1$ment $a \in U$ there exists a neighbourhood $V \in \Sigma$ such that $V + a \subset U$. Specially for any $U \in \Sigma$ and any $n \in N$ it is $N + n \subset U$ and $N + N \subset N$. Further, for any neighbourhood $U \in \Sigma$ there exists a neighbourhood $V \in \Sigma$ such that $-V \subset U$. However $V \supset N$, or $-N \subset -V \subset U$ and from there $-N \subset \cap \Sigma = N$. Together N is a subgroup in G. Further for any neighbourhood $U \in \Sigma$ and any element $g \in G$ there exists a neighbourhood $V \in \Sigma$ such that $-g + V + g \subset U$. Then $-g + N + + g \subset -g + V + g \subset U$ for any neighbourhood $U \in \Sigma$. It means, that $-g + N + + g \subset N$. According to [1], p. 19, def. 4 N is a normal subgroup.

We prove, that systems Σ^* and Σ'^* fulfil the asumptions of the theorem 2.2., i.e., for any $U_1 \in \Sigma^*$, $V_1 \in \Sigma'^*$ it holds $U_1 + V_1 = G$ and exist $F_1 \in \Sigma^*$, $H_1 \in \Sigma'^*$, such that $F_1 \cap H_1 = \{0\}$ in G / N.

1. For any $U \in \Sigma$, $V \in \Sigma'$ it is $U + V = G$. We denote U_1 and V_1 elements of bases Σ^* and Σ'^* inducet with the set U and V. Evidently it holds $U_1 \supset U$, $V_1 \supset V$ and $U_1 + V_1 = G$. Evidently so for any neighbourhood $U^* \in \Sigma^*$ and $V \in \Sigma'^*$ it holds $U^* + V^* = G$.

2. There exist sets $F \in \Sigma$ and $H \in \Sigma'$ such that $F \cap H = \{0\}$. We denote F_1 and H_1, belonging element of bases Σ^* and Σ'^* induced with sets F and H. We prove, that $F_1 \cap H_1 = N$.

We assume, $\bar{x} = N + x \subset F_1 \cap H_1$ and show, that $\bar{x} = N$. Therefore $0 \in N$ and $x \in F_1 \cap H_1 = F_1 \cap H + N$, $F_1 = \cup \{N + x : x \in F\}$ follows $x \in F \cap H + N$. We can x write $x = h + a$, where $h \in H$, $a \in N$ and $h = x - a \in F$. Therefore $N \subset F$ and F is a subgroup. But $F \cap H = \{0\}$, therefore $h = 0$ and $x \in N$. Then $\bar{x} = N + x$ is the zero element of the factogroup G / N. Then $F_1 \cap H_1 = N$.

2.16. Remark: Let (G, Σ) and (G, Σ') be complementary top. groups. If at least one of there systems Σ, Σ' is such, that every its element is unit in a factogroup G / N with regard to a normal subgroup N in G, then the top. factogroups $(G / N, \Sigma^*)$ and $(G / N, \Sigma'^*)$ are complementary. (In $\mathfrak{F}$).

Proof: Is similar as the proof of the Th. 2.15.

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