Archivum Mathematicum

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Partitions and congruences in algebras. II. Modular and distributive equalities, complements

Archivum Mathematicum, Vol. 10 (1974), No. 3, 159--172

Persistent URL: http://dml.cz/dmlcz/104828

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS,
X: 159—172, 1974

PARTITIONS AND CONGRUENCES IN ALGEBRAS II. MODULAR AND DISTRIBUTIVE EQUALITIES, COMPLEMENTS

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(Received August 15, 1973)

The basic facts used in this paper are contained in [10]. There are: a partition in a set, the connection of partitions with symmetrical and transitive relations (ST-relations) and the notion of congruence in algebra. A partition in a set G is a system A (possibly empty) of nonempty pairwise disjoint subsets in G [1, 2, 3, 4, 5, 8]. The elements of a partition A in G are called blocks of the partition A, the union of all blocks belonging to A is denoted by $\bigcup A$ and is called the domain of the partition A. There exists a 1-1 correspondence between the set of all partitions in G and the set of all ST-relations in G. For this reason there is often made no difference between both the notions. A congruence in a universal algebra (G, Ω) is a stable ST-relation A in the algebra (G, Ω). By $A(0)$, the null-set of A, is meant the block of the congruence A in the Ω-group G containing the zero element 0 of the group G. The same notation will be used for the other partitions in Ω-groups. By $P(G)$ or $\Pi(G)$, we mean the system of all partitions (ST-relations) in the set G or all partitions (equivalence relations) on the set G, respectively; by the symbol $\mathscr{K}(G)$ or $\mathscr{C}(G)$, there is denoted the system of all congruences in the algebra G or on the algebra G, respectively. We denote by $\vee_P$ and $\wedge_P$ or simply $\vee$ and $\wedge$ the lattice operations in $P(G)$ and by $\vee_{\mathscr{K}}$ and $\wedge_{\mathscr{K}}$ $(= \wedge_P)$ those in $\mathscr{K}(G)$.

In this paper there are investigated conditions guaranteeing for the given pair of elements in $\mathscr{K}(G)$ (G an Ω-group) one of the following properties: the (dual) modularity in $P(G)$ (2.2 and 2.3.1), the modular equality in $P(G)$ (2.3.2) and the (dual) distributive equality in $P(G)$ (2.4.3 and 2.4.1). The distributivity in $\mathscr{K}(G)$ of a pair of congruences in an abelian or hamiltonian group G is characterized in 2.5.2, and that of the lattice $\mathscr{K}(G)$ in 2.5.3 by the following condition: the finite subsets in G generate cyclic subgroups (in other words, G is generalized cyclic). In below mentioned theorems, the relative complement is related to the interval $[A, B]$ of the lattice $P(G)$, where $A \leqq B$ are congruences in an Ω-group G. In Theorems 2.8 or 2.8.3 there is given a criterion of the existence of a partition or a congruence, respectively, which is a relative P-complement (Definition 2.6.1), in 2.9.1 and 2.9.2 the existence of a congruence which is a Dedekind P-complement (Definition 2.9). Unique congruences of this kind are A, B and $A/B(0) \cap \bigcup A$ (2.9.2). In the paper there are substantially employed results of [5]. We assume the reader to notice the fact that some theorems are mentioned in this paper only for the sake of completeness. There are those following trivially from analogous results of the paper [5].

2.0 An ordered pair of elements c, b of a lattice L is called a *modular pair*, and we write $(c, b)\ M$, if

$$(2,1) \qquad (a \vee c) \wedge b = a \vee (c \wedge b) \qquad \text{for all } a \leqq b.$$

Dually we define a *dual-modular pair* c, b, $(c, b)\ M^*$:

$$(2,1^*) \qquad (a \wedge c) \vee b = a \wedge (c \vee b) \qquad \text{for all } a \geqq b.$$

([6] Def. 1.1)

In the requirement (2,1) or (2,1*) it is possible to take only the elements a fulfilling $b \wedge c \leqq a \leqq b$ or $b \leqq a \leqq b \vee c$, respectively, [6] Lemma 1.4.

In [5], Th. 2.1, the following assertion is proved.

Theorem 2A. *Let B, C be partitions in a set G. A pair C, B is dually modular in the lattice $P(G)$ iff there holds*

(2,2) *B is strongly connected with C in $P(G)$*

(2,3) $C^1, C^2 \in C,\ B^1 \in B,\ C^1 \neq C^2,\ C^1 \between B^1 \between C^2,\ C^1 \setminus B^1 \neq \emptyset,\ C^2 \setminus B^1 \neq \emptyset \Rightarrow$
$\Rightarrow C^1 \setminus B^1,\ C^2 \setminus B^1$ *are covered by some blocks of B.*

Remark. The symbol $C^1 \between B^1$ denotes the incidence of the sets C^1, B^1 (i.e. $C^1 \cap B^1 \neq \emptyset$).

By the condition (2,2), is meant: for any two distinct blocks B^1, B^2 in B contained in the same block of the partition $B \vee_P C$, there exists a block C^1 in C such that $B^1 \between C^1 \between B^2$.

2.1 *If B, C are congruences in an Ω-group, then the condition* (2,2) *is always satisfied and the condition* (2,3) *is equivalent to the following one; $C(0) \subseteq \bigcup B$ or $B(0) \subseteq C(0)$.*

Proof. Condition (2,2). Let $B, C \in \mathscr{K}(G)$, $B^0, B^* \in B$, $B^0 \neq B^*$, B^0, B^* be contained in the same block of the partition $B \vee_P C$. Then there exist blocks $B^1, \ldots, B^{n-1} \in B$, $C^1, \ldots, C^n \in C$ such that

$$B^0 \between C^1 \between B^1 \between \ldots \between B^{n-1} \between C^n \between B^*.$$

Let us denote

$$x_1 \in B^0 \cap C^1, x_2 \in C^1 \cap B^1, \ldots, x_{2n-1} \in B^{n-1} \cap C^n,\ x_{2n} \in C^n \cap B^*,$$
$$\mathfrak{A} = \bigcup B \cap \bigcup C,\ \bar{B} = B \sqcap \mathfrak{A} = \{\mathfrak{A} \cap B^1 : B^1 \in B, \mathfrak{A} \cap B^1 \neq \emptyset\},\ \bar{C} = C \sqcap \mathfrak{A}.$$

B, C are congruences on the Ω-group $\mathfrak{A}$, $x_1, \ldots, x_{2n} \in \mathfrak{A}$. So there hold the relations

$$x_1 \bar{C} x_2 \bar{B} x_3 \ldots x_{2n-1} \bar{C} x_{2n} \qquad \text{and} \qquad x_1 \bar{C}\bar{B}\bar{C} \ldots \bar{C} x_{2n}.$$

As congruences on an Ω-group commute, we have

$$x_1\bar{C}^n\bar{B}^{n-1}x_{2n},$$

(n and $n-1$ are exponents), i.e., there exists $y \in \mathfrak{A}$ such that

$$x_1\bar{C}^n y\bar{B}^{n-1}x_{2n}.$$

From the transitivity of the partitions $\bar{B}$, $\bar{C}$ we get

$$x_1\bar{C}y\bar{B}x_{2n}.$$

From the relation $x_1\bar{C}y$ or $y\bar{B}x_{2n}$ there follows the existence of $C^+ \in C$ or $B^+ \in B$ with the property $x_1, y \in C^+$ or $y, x_{2n} \in B^+$, respectively. Because of $x_{2n} \in B^+ \cap B^*$, we have $B^+ = B^*$. Thus $B^0 \between C^+ \between B^*$ is proved and the condition (2,2) is satisfied.

Condition (2,3). 1. Let $C^1, C^2 \in C$, $B^1 \in B$, $C^1 \neq C^2$, $C^1 \between B^1 \between C^2$, $C^1 \setminus B^1 \neq \emptyset$, $C^2 \setminus B^1 \neq \emptyset$.

a) If there holds $C(0) \subseteq \bigcup B$, then from the relation $C^1 \between B^1$ or $B^1 \between C^2$ we get $C^1 \subseteq \bigcup B$ or $C^2 \subseteq \bigcup B$, respectively, since e.g. for any element $c \in C^1 \cap B^1$ ($\subseteq \bigcup B$) there holds $C^1 = c + C(0) \subseteq \bigcup B$. Thus blocks of the partition B cover both $C^1 \setminus B^1$ and $C^2 \setminus B^1$. The condition (2,3) is verified.

b) If there holds $B(0) \subseteq C(0)$, then the relations $C^1 \between B^1 \between C^2$ imply $B^1 \subseteq C^1 \cap C^2$, thus $C^1 = C^2$, contrary to hypothesis.

2. Let the condition (2,3) be satisfied. Let us suppose $C(0) \nsubseteq \bigcup B$, $B(0) \nsubseteq C(0)$. For $B^1 = B(0)$ there holds $C(0) \cap B^1 \neq \emptyset$. By hypothesis there exists $x \in B^1 \setminus C(0) = B(0) \setminus C(0)$. If we denote $C(0) = C^1$, $x + C(0) = C^2$, then $C^1 \between B^1 \between C^2$, $C^1 \neq C^2$, $C^1 \setminus B^1 = C(0) \setminus B(0) \neq \emptyset$ (since in the opposite case there would be $\bigcup B \supseteq B(0) \supseteq C(0)$, a contradiction), $C^2 \setminus B^1 = [x + C(0)] \setminus B(0) \neq \emptyset$ (since in the opposite case there would be $\bigcup B \supseteq B(0) \supseteq x + C(0)$ and hence $\bigcup B = -x + \bigcup B \supseteq C(0)$, a contradiction), and in spite of it $C^2 \subseteq \bigcup B$ (since in the opposite case there would be $\bigcup B \supseteq C^2 = x + C(0)$ thus $\bigcup B = -x + \bigcup B \supseteq C(0)$, a contradiction). Consequently, contrary to hypothesis, the set $C^2 \setminus B^1$ is not covered by blocks of the partition B.

2.2 *Let B, C be congruences in an Ω-group G. Then C, B is a dually modular pair in the lattice $P(G)$ if and only if $C(0) \subseteq \bigcup B$ or $B(0) \subseteq C(0)$.*

Proof follows directly from 2.1 and from Theorem 2A.

2.2.1 Corollary. *Any two congruences on an Ω-group G form a dually modular pair in $P(G)$.*

2.3 *Two congruences on an Ω-group G form a modular pair in $P(G)$ if and only if they are comparable.*

Remark. Th. 2.2 [5] states the following necessary and sufficient condition for two partitions C, A in a set G to form a modular pair in $P(G)$:

(0) *There does not exist a chain of blocks of the partitions A, C such that*

$$A^1 \between C^1 \between A^2 \between \ldots \between A^k \between C^k \between A^1, k > 1,$$
$$C^1 \neq C^k, A^1 \neq A^i, i = 2, \ldots, k.$$

Proof of Theorem 2.3.1. Comparable congruences on G (even comparable partitions in the set G) form evidently a modular pair in $P(G)$.

2. Let A, C be incomparable congruences on the Ω-group G. Then each of them contains at least two blocks. Choose $A^1 \in A$ and $C^1 \in C$ for which $A^1 \between C^1$. $C^1 \nsubseteq A^1$ holds (since otherwise, for $x \in A^1 \cap C^1$ we would get $A(0) = A^1 - x \supseteq C^1 - x = C(0)$ thus $A \geqq C$) hence there exists $A^2 \in A$, $A^2 \neq A^1$ such that $C^1 \between A^2$. Analogously it may be proved $C^1 \nsupseteq A^2$. Thus there exists $C^2 \in C$, $C^2 \neq C^1$, such that $C^2 \between A^2$. The set $A^1 \cup A^2 \cup C^1 \cup C^2$ is contained in some block of the partition $A \vee_P C$. Since the congruences on Ω-group commute, there will be $C^2 \between A^1$. Hence there is constructed a chain

$$A^1 \between C^1 \between A^2 \between C^2 \between A^1, \quad C^1 \neq C^2, \quad A^1 \neq A^2.$$

By Th. 2.2 [5], $(C, A)\, M$ does not hold in $P(G)$.

2.3.1 *Let A, C be congruences in an Ω-group G. Then the following conditions are equivalent:*

(1) *$(C, A)\, M$ in $P(G)$*
(2) *$(A, C)\, M$ in $P(G)$*
(3) *$A(0) \cap \bigcup C$, $C(0) \cap \bigcup A$ are comparable sets*
(4) *$A \sqcap \bigcup C$, $C \sqcap \bigcup A$ are comparable partitions.*

Proof. First we shall prove $3 \Rightarrow 1 \Rightarrow 4 \Rightarrow 3$. Then from the relation $1 \Leftrightarrow 3$ and from the symmetry of the condition 3 we get $2 \Leftrightarrow 3$.

1. Do not let $(C, A)\, M$ hold in $P(G)$. By Th. 2.2 [5] there exists a chain

$$A^1 \between C^1 \between A^2 \between \ldots \between A^k \between C^k \between A^1, \quad k > 1, \quad C^1 \neq C^k,$$
$$A^i \neq A^1, \quad 1 < i \leqq k.$$

When denoting $\mathfrak{M} = \bigcup A \cap \bigcup C$, for $i = 1, 2, \ldots, k$, there will hold $\bar{A}^i = A^i \cap \mathfrak{M} \neq \emptyset$, $\bar{C}^i = C^i \cap \mathfrak{M} \neq \emptyset$ thus $\bar{A}^i \in A \sqcap \mathfrak{M}$, $\bar{C}^i \in C \sqcap \mathfrak{M}$.

Further

$$\bar{A}^1 \between \bar{C}^1 \between \ldots \between \bar{A}^k \between \bar{C}^k \between \bar{C}^1 \quad k > 1, \quad \bar{C}^1 \neq \bar{C}^k, \quad \bar{A}^i \neq \bar{A}^1,$$
$$1 < i \leqq k.$$

Thus the congruences $\bar{A} = A \sqcap \mathfrak{M}$, $\bar{C} = C \sqcap \mathfrak{M}$ on the Ω-group do not satisfy the condition (0), Th. 2.2 [5], so that $(\bar{C}, \bar{A})\, M$ does not hold in $P(\mathfrak{M})$. By 2.3 the partitions $\bar{A}$, $\bar{C}$ are not comparable. Then, obviously, there are not comparable the null-sets of the congruences $\bar{A}$, $\bar{C}$, i.e., the sets $A(0) \cap \bigcup C$, $C(0) \cap \bigcup A$.

2. Conversely, if the null-sets of the congruences $\bar{A}$; $\bar{C}$ on the Ω-group $\mathfrak{M}$ are incomparable, then also the partitions $\bar{A}$, $\bar{C}$ are incomparable. By 2.3, $(\bar{C}, \bar{A})\, M$ does not hold in $P(\mathfrak{M})$ thus by Th. 2.2[5], there exists a chain

$$\bar{A}^1 \between \bar{C}^1 \between \dots \between \bar{A}^k \between \bar{C}^k \between \bar{A}^1, k > 1, \bar{C}^1 \neq \bar{C}^k, \bar{A}^i \neq \bar{A}^1, \qquad 1 < i \leqq k.$$

Then

$$A^1 \between C^1 \between \dots \between A^k \between C^k \between A^1$$

is a chain of the condition (0), Th. 2.2[5], such that $(C, A)\, M$ does not hold in $P(G)$. The theorem is proved.

2.3.2 *Let A, B be congruences in an Ω-group G, $B \leqq A$. Then there holds the modular equality in $P(G)$*

(2.3,1) $A \wedge (B \vee C) = B \vee (A \wedge C)$ *for all* $C \in P(G)$

if and only if $B(0) = 0$, *or* $A = B$, *or* $\cup A = A(0)$.

Proof results directly from Th. 2.4 [5].

2.4 *Congruences B, C in an Ω-group G form a distributive pair in $P(G)$*

(2.4,1) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ *for all* $A \in P(G)$

if and only if $B(0)$ and $C(0)$ are comparable sets.

Proof. By Th. 3.1 [5], the distributive equality (2.4.1) is satisfied if.

(2.4,2) each block of the partition $B \vee C$ is a block of the partition B, or that of the partition C.

It will be proved the equivalence of the conditions (2.4,2) and (2.4,1). Let (2.4,2) hold, let $D^0 \in B \vee C$. Then $D^0 \in B$ or $D^0 \in C$ thus for $x \in D^0$ there holds $D^0 - x = (B \vee C)(0)$ and simultaneously $= B(0)$ or $= C(0)$. Hence $C(0) \subseteq B(0)$ or $B(0) \subseteq C(0)$.

Conversely, suppose $C(0) \subseteq B(0)$. Then $C(0) \subseteq \cup B$, thus blocks of the partition C are contained in $\cup B$ or in $\cup C \setminus \cup B$. Consequently, blocks of $B \vee C$ are contained in $\cup B$ or in $\cup C \setminus \cup B$. Those contained in $\cup C \setminus \cup B$ are blocks of the partition C and those contained in $\cup B$ are blocks of the partition B. The condition (2.4,2) is satisfied.

2.4.1 *Congruences A, C in an Ω-group G satisfy the distributive equality in $P(G)$*

(2.4,3) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ *for all* $B \in P(G)$

if and only if

$A(0) = 0$, *or* $C(0) = 0$, *or* $A \leqq C$, *or* $C \leqq A$, $\cup A = A(0)$.

Proof follows directly from Th. 3.2 [5].

2.4.2 *Congruences B, C in an Ω-group G form a dually distributive pair in $P(G)$, i.e.,*

(2.4,4) $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ *for all* $A \in P(G)$

if and only if

$B(0) = 0$, *or* $C(0) = 0$, *or* B *and* C *are comparable partitions.*

Proof follows directly from Th. 3.3 [5].

2.4.3 *Congruences* A, C *in an* Ω*-group stalisfy the dual distributive equality in* $P(G)$

(2.4,5) $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ *for all* $B \in P(G)$

if and only if

$A(0) = 0$, *or* $C(0) = 0$, *or* $C \leqq A$, *or* $A \leqq C$, $\mathsf{U}C = C(0)$.

Proof follows directly from Th. 3.4 [5].

By *trivial partitions* in a set, we mean the partitions with at most one block and the partitions with only one-element blocks.

2.4.4 Corollary. *Nontrivial congruences* A, C *on an* Ω*-group* G *satisfy the dual distributive equality in* $P(G)$ *if and only if* $C \leqq A$ *holds.*

2.4.5 Corollary. *Let* B, C *be congruences on an* Ω*-group* G. *Then the following conditions are equivalent:*

(a) (B, C) *M in* $P(G)$
(b) (C, B) *M in* $P(G)$
(c) B, C *form a distributive pair in* $P(G)$
(d) B, C *form a dually distributive pair in* $P(G)$
(e) B, C *satisfy the distributive equality in* $P(G)$
(f) B *and* C *are comparable partitions.*

Proof follows from Theorems 2.3, 2.4, 2.4.1, 2.4.2 and 2.4.3.

2.5 Let B, C be congruences in an Ω-group G. Now, in one special case there will be found a condition for the distributivity in $\mathscr{K}(G)$ of the pair B, C. The distributivity in $\mathscr{K}(G)$ of the pair $B, C \in \mathscr{K}(G)$, i.e., the condition

$$A \wedge (B \vee_{\mathscr{K}} C) = (A \wedge B) \vee_{\mathscr{K}} (A \wedge C) \text{ for all } A \in \mathscr{K}(G)$$

can be formulated equivalently in the form

$$(2.5,1)\quad \begin{cases} \text{For any } \Omega\text{-subgroup } \mathsf{U}A \text{ of } G \text{ and its arbitrary ideal } A(0) \text{, there holds} \\ \mathsf{U}A \cap \langle \mathsf{U}B, \mathsf{U}C \rangle = \langle \mathsf{U}A \cap \mathsf{U}B, \mathsf{U}A \cap \mathsf{U}C \rangle \ (= R) \\ A(0) \cap \ll B(0), C(0) \gg_{\langle \mathsf{U}B, \mathsf{U}C \rangle} = \ll A(0) \cap B(0), A(0) \cap C(0) \gg_R . \end{cases}$$

Here, by $\langle \mathsf{U}B, \mathsf{U}C \rangle$, is meant the Ω-subgroup generated in G by $\mathsf{U}B \cup \mathsf{U}C$ and $\ll B(0), C(0) \gg_{\langle \mathsf{U}B, \mathsf{U}C \rangle}$ denotes the ideal generated in $\langle \mathsf{U}B, \mathsf{U}C \rangle$ by $B(0) \cup C(0)$.

If G is a hamiltonian or an abelian group, (2.5,1) may be written in the form

$$(2.5,2)\qquad \begin{cases} \text{For any subgroup } S \text{ of } G \text{ there holds} \\ S \cap (\mathsf{U}B + \mathsf{U}C) = (S \cap \mathsf{U}B) + (S \cap \mathsf{U}C) \\ S \cap (B(0) + C(0)) = (S \cap B(0)) + (S \cap C(0)). \end{cases}$$

The first equality or the second one in (2.5,2) means the distributivity of the pair $\cup B$, $\cup C$ or $B(0)$, $C(0)$, respectively, in the lattice of all subgroups $L(G)$ of the group G.

2.5.1 Definition. Let H be a subgroup of a group G, $a \in G$. An *order* of an element a with respect to H is the smallest positive integer n for which $na \in H$. ([7] II, Chap. 3, [9] sec. I.)

2.5.2 *Let G be a hamiltonian or an abelian group. Congruences B, C in G form a distributive pair in $\mathscr{K}(G)$ if and only if the orders with respect to $\cup B$ and $\cup C$ of any element $a \in (\cup B + \cup C) \setminus (\cup B \cup \cup C)$ are finite and mutually prime, and the orders with respect to $B(0)$ and $C(0)$ of any element $a \in (B(0) + C(0)) \setminus (B(0) \cup C(0))$ are finite and mutually prime as well.*

Proof follows from 2.5 and from [7] II Th. 2, Chap. 3 (eventually from [9] Th. 1, sec. 1).

2.5.3 *Let G be a hamiltonian or an abelian group. The following conditions are equivalent:*

(1) *The lattice $\mathscr{K}(G)$ of all congruences in G is distributive*
(2) *The lattice $\mathscr{C}(G)$ of all congruences on G is distributive*
(3) *The lattice $L(G)$ of all subgroups in G is distributive*
(4) *Each finite subset in G generates a cyclic subgroup (i.e. G is a generalized cyclic group).*

Proof. The equivalences between (1) and (3) and between (2) and (3) as well follow from the formulation (2.5,2) of a distributive pair. By condition (4), there is characterized the distributivity of the lattice $L(G)$ in [7] II. Th. 4, Chap. 3 (see also [9], Th. 2, Chap. 1).

2.5.4 Corollary. *Let G be a finite hamiltonian or a finite abelian group. The lattice $\mathscr{K}(G)$ is distributive, or equivalently the lattice $\mathscr{C}(G)$ is distributive if and only if G is a cyclic group.*

2.5.5 *The lattices $\mathscr{K}(G)$ and $\mathscr{C}(G)$, where G is a hamiltonian or an abelian group, are modular.*

Proof. The mapping $A \in \mathscr{K}(G) \to (\cup A, A(0))$ is an isomorphism of the lattice $\mathscr{K}(G)$ into the direct sum $L(G) \oplus L(G)$ of two copies of the lattice $L(G)$; the lattice $L(G)$ is modular. Similarly, the mapping $A \in \mathscr{C}(G) \to A(0) \in L(G)$ is an isomorphism of $\mathscr{C}(G)$ onto $L(G)$.

2.6 *Let G be an algebra. The lattice $\mathscr{K}(G)$ is upper-continuous.*

Proof. It is to be proved $C \wedge \bigvee_{\gamma}{}_{\mathscr{K}} D_\gamma = \bigvee_{\gamma}{}_{\mathscr{K}} (C \wedge D_\gamma)$ for any $C \in \mathscr{K}(G)$ and any chain $\{D_\gamma\}$ of elements of $\mathscr{K}(G)$. For the chain $\{D_\gamma\}$, however, there holds $\bigvee_{\gamma}{}_{\mathscr{K}} D_\gamma =$

$= \bigvee_{P\gamma} D_\gamma$ thus by Th. 4.1 [5], we have $C \wedge (\bigvee_{\mathscr{K}\gamma} D_\gamma) = C \wedge (\bigvee_{P\gamma} D_\gamma) = \bigvee_{P\gamma}(C \wedge D_\gamma)$. As the last expression is a chain, it equals $\bigvee_{\mathscr{K}\gamma}(C \wedge D_\gamma)$. Hence the assertion.

2.6.1 Definition. Let G be an algebra $A, B \in P(G)$, $A \leqq B$. We define $[A, B] = \{C \in P(G) : A \leqq C \leqq B\}$.

Let $A \leqq C \leqq B$ be partitions in the set G. $D \in P(G)$ is called a *relative P-complement* of C in $[A, B]$ if $C \wedge D = A$, $C \vee D = B$. Let $A \leqq C \leqq B$ be congruences in the algebra G. $D \in \mathscr{K}(G)$ is called a *relative $\mathscr{K}$-complement* of C in $[A, B]$ if $C \wedge D = A$, $C \vee_{\mathscr{K}} D = B$.

2.6.2 *Let G be an algebra, $A, B, C \in \mathscr{K}(G)$, $A \leqq C \leqq B$, $\{D_\gamma\}$ a chain of relative $\mathscr{K}$-complements of C in $[A, B]$. Then $\bigvee_{\mathscr{K}\gamma} D_\gamma$ is a relative $\mathscr{K}$-complement of C in $[A, B]$.*

Proof. It suffices to prove $C \wedge \bigvee_{\mathscr{K}\gamma} D_\gamma = A$, $C \vee_{\mathscr{K}} \bigvee_{\mathscr{K}\gamma} D_\gamma = B$. The first equality follows from 2.6, the second one from the relations $B = C \vee_{\mathscr{K}} D_\gamma$ for all γ, $B = \bigvee_{\mathscr{K}\gamma}(C \vee_{\mathscr{K}} D_\gamma) \leqq C \vee_{\mathscr{K}} \bigvee_{\mathscr{K}\gamma} D_\gamma \leqq B$.

2.7 *In any lattice, the mappings*

$\varphi : x \in [a \wedge b, a] \to b \vee x, \psi : y \in [b, a \vee b] \to a \wedge y$

define (mutually inverse) isomorphisms between the intervals $[a \wedge b, a]$ and $[b, a \vee b]$ if and only if

(1) *$x = (x \vee b) \wedge a$ for all $a \wedge b \leqq x \leqq a$ and $y = (y \wedge a) \vee b$ for all $b \leqq y \leqq a \vee b$,*

or equivalently

(2) *(b, a) M and (a, b) M^*.*

(See Lemma 1.3[6].)

Proof. The mappings φ and ψ are isotonic and for $x \in [a \wedge b, a]$, $y \in [b, a \vee b]$, there holds

(3) $x\varphi\psi = (x \vee b) \wedge a \geqq x \vee (b \wedge a) = x$, $y\psi\varphi = (y \wedge a) \vee b \leqq y \wedge (a \vee b) = y$.

Hence

$$x\varphi\psi\varphi \geqq x\varphi \geqq (x\varphi)\,\psi\varphi, \qquad \text{i.e.,} \qquad (x\varphi)\,\psi\varphi = x\varphi.$$

Similarly $(y\psi)\,\varphi\psi = y\psi$ thus when denoting $S = [a \wedge b, a]$, $H = [b, a \vee b]$, we have

$$\varphi\psi = 1_{H\psi}, \qquad \psi\varphi = 1_{S\varphi}.$$

a) If the mappings φ and ψ are "onto", then $\varphi\psi = 1_S$, $\psi\varphi = 1_H$ thus, by (3), there holds (1).

b) If (1) holds, then $\varphi\psi = 1_S$, $\psi\varphi = 1_H$, i.e., φ and ψ are mutually inverse one-to-one mappings "onto"; since both the mappings are isotonic, they are the isomorphisms between the intervals S and H. The equivalence between (1) and (2) is evident.

Remark. From the proof of the theorem there is evident that the condition

φ or ψ is a mapping onto $[b, a \vee b]$ or onto $[a \wedge b, a]$, respectively

is equivalent to (1) as well.

2.7.1 *Let A, B congruences in an Ω-group G. Then the mappings*

$$D \in [A \wedge B, A] \to B \vee D \qquad \text{and} \qquad C \in [B, A \vee B] \to A \wedge C$$

define (mutually inverse) isomorphisms between the intervals $[A \wedge B, A]$ and $[B, A \vee \vee B]$ if and only if

$$A(0) \subseteq B(0), \text{ or } B(0) \subseteq A(0), \text{ or } \mathsf{U}B \supseteq A(0) \supseteq B(0) \cap \mathsf{U}A.$$

Proof. It suffices to express the condition 2.7(2). The first part of the condition is expressed in 2.3.1 as follows:

(1) $A(0) \cap \mathsf{U}B \subseteq B(0) \cap \mathsf{U}A$, *or* (2) $A(0) \cap \mathsf{U}B \supseteq B(0) \cap \mathsf{U}A$.

the second one in 2.2:

(a) $A(0) \subseteq \mathsf{U}B$, or (b) $B(0) \subseteq A(0)$.

It is easy to verify that there holds

$$1 \wedge \mathrm{a} \Leftrightarrow A(0) \subseteq B(0)$$
$$2 \wedge \mathrm{b} \Leftrightarrow B(0) \subseteq A(0)$$
$$1 \wedge \mathrm{b} \Rightarrow B(0) \subseteq A(0) \Rightarrow 2 \wedge b$$
$$2 \wedge \mathrm{a} \Leftrightarrow \mathsf{U}B \supseteq A(0) \supseteq B(0) \cap \mathsf{U}A.$$

Hence the assertion.

2.7.2 Corollary. *Let A, B be congruences on an Ω-group G. Then the mappings $D \in [A \wedge B, A] \to D \vee B$ and $C \in [B, A \vee B] \to C \wedge A$ define (mutually inverse) isomorphisms between the intervals $[A \wedge B, A]$ and $[B, A \vee B]$ if and only if the partitions A, B are comparable.*

2.7.3 Remark. 1. In Theorem 2.7.2 the operations $\wedge$, $\vee$ and the interval $[,]$ can be referred to $\Pi(G)$ or to $\mathscr{C}(G)$ instead of to $P(G)$. The lattice operations are the same in all three lattices and further it is seen that the mappings $D \to D \vee B$ and $C \to C \wedge A$ transform partitions (congruences) "on" into partitions (congruences) "on".

2. Suppose that the condition in 2.7.2 is fulfilled. If $B \leqq A$ then both the mappings are identical mappings of the interval $[B, A]$. If $A \leqq B$, both the mappings are trivial isomorphisms between the one-element intervals $[A, A]$ and $[B, B]$.

2.8 *Let $A \leqq C \leqq B$ be congruences in an Ω-group G. Then C has a relative P-complement in $[A, B]$ if and only if there holds*

(1) $\mathsf{U}A + C(0) = \mathsf{U}C$, *or* $B(0) = C(0)$.

Proof. An analogous theorem in [5] (Th. 5.1) referring to the partitions in the set G gives the following and sufficient condition:

(2) If a block of the partition C contains no block of the partition A, then it is a block of the partition B.

I. If the first part of the condition (1) is satisfied, then it is satisfied (trivially) the condition (2) thus, by Th. 5.1 [5], C has a relative P-complement in $[A, B]$. If the second part of the condition (1) is satisfied, then there holds (2) since all blocks of the partition C are blocks of the partition B.

II. If C has a relative P-complement in $[A, B]$, then there holds (2). Thus, either any block of the partition C contains some block of the partition A and hence $\mathsf{U}A + C(0) = \mathsf{U}C$, or there exists $x \in \mathsf{U}C$ such that $x + C(0)$ does not meet $\mathsf{U}A$; in this case by (2), $x + C(0) = x + B(0)$ thus $C(0) = B(0)$.

2.8.1 Remark. In case of the congruences on an Ω-group, the condition (1) is always satisfied (namely its first part) thus there always exists a relative P-complement. But this is a well-known fact—see 0.4 [6].

2.8.2 *Let $A \leqq C \leqq B$ be congruences in an Ω-group G, let $D \in \mathscr{K}(G)$ be a relative P-complement of C in $[A, B]$. Then*

$$\mathsf{U}C = \mathsf{U}A,\ \mathsf{U}D = \mathsf{U}B \text{ or } \mathsf{U}C = \mathsf{U}B,\ \mathsf{U}B = \mathsf{U}A.$$

Further there holds

$$[C(0) = A(0) \Leftrightarrow D(0) = B(0)] \text{ and } [C(0) = B(0) \Leftrightarrow D(0) = A(0)].$$

Proof. The relation $\mathsf{U}C \cup \mathsf{U}D = \mathsf{U}B$ implies comparability of the sets $\mathsf{U}C$ and $\mathsf{U}D$. Otherwise there exist elements $x \in \mathsf{U}C \setminus \mathsf{U}D$, $y \in \mathsf{U}D \setminus \mathsf{U}C$ and there holds $x + y \bar{\in} \mathsf{U}C \cup \mathsf{U}D = \mathsf{U}B$, a contradiction. Thus $\mathsf{U}C = \mathsf{U}B$, or $\mathsf{U}D = \mathsf{U}B$. We get the other equalities between domains from the relation $\mathsf{U}C \cap \mathsf{U}D = \mathsf{U}A$.

If $\mathsf{U}C = \mathsf{U}A$, $\mathsf{U}D = \mathsf{U}B$, $C(0) = A(0)$, then $C = A$ thus $D = B$, i.e., $D(0) = B(0)$, If $\mathsf{U}C = \mathsf{U}B$, $\mathsf{U}D = \mathsf{U}A$, $C(0) = A(0)$, then $C = \mathsf{U}B/A(0)$. By 3.5.7 [11], there holds

$$B(0) = (C \vee_P D)(0) = [C(0) + \mathsf{U}C \cap D(0)] \cup [\mathsf{U}D \cap C(0) + D(0)] =$$
$$= [A(0) + D(0)] \cup [A(0) + D(0)] = A(0) + D(0) = D(0) \text{ thus again } D(0) = B(0).$$

The reverse implication follows from the symmetry.

The last assertion will be proved similarly.

2.8.3 *Let $A \leqq C \leqq B$ be congruences in an Ω-group G. Then C has a congruence in G as a relative P-complement in $[A, B]$ if and only if there holds one of the following conditions*

(1) $C = B$

(2) $\mathsf{U}C = \mathsf{U}A$

(3) $\mathsf{U}A + C(0) = \mathsf{U}B$

Proof. If C has a congruence in G as its relative P-complement in $[A, B]$, then by 2.8 and 2.8.2 there holds

$$\text{(a)}\ \ \mathsf{U}A + C(0) = \mathsf{U}C, \text{ or (b) } B(0) = C(0)$$

and

$$(\alpha)\ \ \mathsf{U}C = \mathsf{U}A, \text{ or } (\beta)\ \ \mathsf{U}C = \mathsf{U}B.$$

It can be easily proved the validity of the following

$$a \wedge \alpha \Leftrightarrow \mathsf{U}C = \mathsf{U}A,\ a \wedge \beta \Leftrightarrow \mathsf{U}A + C(0) = \mathsf{U}B,\ b \wedge \alpha \Leftrightarrow C = \mathsf{U}A/B(0),$$
$$b \wedge \beta \Leftrightarrow C = B.$$

Further it is evident that $b \wedge \alpha \Rightarrow a \wedge \alpha$. It follows that C satisfies one of the conditions 1, 2 or 3.

Conversely, if C satisfies one of these conditions, then it has a relative P-complement in $[A, B]$ since the condition 2.8 (2) is evidently satisfied.

2.9 Definition. Let $A \leqq C \leqq B$ be elements of an arbitrary lattice S. An element $D \in [A, B]$ is called a *Dedekind complement* of the element C in $[A, B]$ if for $E, F \in [A, B]$ there holds the following:

(1) $C \leqq E \leqq B \Rightarrow E = C \vee (E \wedge D)$, (2) $A \leqq F \leqq D \Rightarrow F = D \wedge (F \vee C)$, ([5] Definition p. 61).

Let us note that Dedekind complements of an element C in $[A, B]$ are relative complements of the element C in $[A, B]$. Namely, putting $E = B$ in (1), we obtain $B = C \vee (B \wedge D) = C \vee D$, and for $F = A$ the condition (2) gives $A = D \wedge (A \vee C) = D \wedge C$. Furthermore $C = A$ or $C = B$ has exactly one (Dedekind) complement in $[A, B]$, namely $D = B$ or $D = A$, respectively.

If A, B, C, D are congruences in an algebra G, we can distinguish two types of Dedekind complements. We say that D is a *Dedekind P-complement* of C in $[A, B]$ or a *Dedekind $\mathscr{K}$-complement* of C in $[A, B]$, respectively, if D is a Dedekind complement of C in $[A, B]$ referred to the lattice $S = P(G)$ or $S = \mathscr{K}(G)$, respectively.

2.9.1 *Let A, B, C, D be congruences in an Ω-group G, $A \leqq B$, $C, D \in [A, B]$, let*

D be a relative P-complement of C in $[A, B]$. *Then D is a Dedekind P-complement of C in* $[A, B]$ *if and only if there holds*

$$D(0) \supseteq C(0), \text{ or } C(0) \supseteq D(0), \text{ or } \mathsf{U}C \supseteq D(0) \supseteq C(0) \cap \mathsf{U}D.$$

Proof. Putting $a = D$, $b = C$, $x = F$, $y = E$ in 2.7, the following statement

(1) the mappings $\psi : E \in [C, B] \to E \wedge D$ and $\varphi : F \in [A, D] \to F \vee C$ are mutually inverse isomorphisms between the intervals $[C, B]$ and $[A, D]$, respectively,

will be equivalent to the statement

(2) $E \in P(G)$, $C \leqq E \leqq B \Rightarrow E = C \vee (E \wedge D)$ and $F \in P(G)$, $A \leqq F \leqq D \Rightarrow F = D \wedge (F \vee C)$.

The condition (2) is equivalent to the statement

(3) D is a Dedekind P-complement of C in $[A, B]$.

Since we suppose $C, D \in \mathscr{K}(G)$, condition (1) is, by 2.7.1, equivalent to the following one

(4) $D(0) \supseteq C(0)$, or $C(0) \supseteq D(0)$, or $\mathsf{U}C \supseteq D(0) \supseteq C(0) \cap \mathsf{U}D$.

In conclusion, we have: (3) $\equiv$ (4).

2.9.2 *Let A, B, C, D be congruences in an Ω-group G, $A \leqq B$, $C, D \in [A, B]$, let D be a relative P-complement of C in* $[A, B]$. *Then D is a Dedekind P-complement of C in* $[A, B]$ *if and only if one of the following conditions holds*

1. $D = A$
2. $D = B$
3. $D = \mathsf{U}A/B(0) \cap \mathsf{U}A$.

Proof. Let D be a Dedekind P-complement of C in $[A, B]$. By 2.9.1, this is equivalent to the following

I. $D(0) \supseteq C(0)$, or II. $C(0) \supseteq D(0)$, or III. $\mathsf{U}C \supseteq D(0) \supseteq C(0) \cap \mathsf{U}D$.

Since D is a relative P-complement of C in $[A, B]$, then by 2.8.2 it will hold

a) $\mathsf{U}C = \mathsf{U}A$, $\mathsf{U}D = \mathsf{U}B$, or b) $\mathsf{U}C = \mathsf{U}B$, $\mathsf{U}D = \mathsf{U}A$

and by 2.8, we have

α) $\mathsf{U}A + C(0) = \mathsf{U}C$, or β) $B(0) = C(0)$.

If we denote in addition

I′. $A(0) = C(0)$, $B(0) = D(0)$, II′. $A(0) = D(0)$, $B(0) = C(0)$,

then evidently (see 2.8.2) I $\Leftrightarrow$ I′ and II $\Leftrightarrow$ II′.

It holds

$\mathrm{I} \wedge \mathrm{a} \Rightarrow \mathrm{I}' \wedge \mathrm{a} \Rightarrow D = B$,

$\mathrm{I} \wedge \mathrm{b} \wedge \alpha \Rightarrow \cup C = \cup B,\ C(0) = A(0),\ \cup A + C(0) = \cup C \Rightarrow \cup A = \cup A + A(0) = \cup A + C(0) = \cup C,\ C(0) = A(0) \Rightarrow C = A \Rightarrow D = B$.

$\mathrm{I} \wedge \mathrm{b} \wedge \beta \Rightarrow \cup C = \cup B,\ C(0) = A(0) = B(0) \Rightarrow C = B \Rightarrow D = A$.

$\mathrm{II} \wedge \mathrm{a} \Rightarrow \mathrm{II}' \wedge \mathrm{a} \Rightarrow B(0) \subseteq \cup A,\ C = \cup A/B(0)$, $A(0)$ is an ideal of $\cup B$ and $D = \cup B/A(0)$.

By hypothesis, there is $C \vee_P D = B$ thus for $x \in \cup B \setminus \cup A$ we have (with respect to the relation $B(0) \subseteq \cup A$) $(C \vee_P D)(x) = D(x)$[1] $= A(0) + x$ thus $A(0) + x = B(0) + x$, $A(0) = B(0)$, $C = A$, $D = B$.

$\mathrm{II} \wedge \mathrm{b} \Rightarrow \mathrm{II}' \wedge \mathrm{b} \Rightarrow D = A$.

$\mathrm{III} \wedge \mathrm{a} \Rightarrow \cup D = \cup B,\ D(0) \supseteq C(0) \cap \cup D \Rightarrow \cup D = \cup B,\ D(0) \supseteq C(0) \cap \cup B = C(O) \Rightarrow \mathrm{I} \wedge \mathrm{a} \Rightarrow D = B$.

Now, we shall prove

(*) $\mathrm{III} \wedge \mathrm{b} \Leftrightarrow (A(0) = C(0) \cap \cup A) \wedge \mathrm{b} \Leftrightarrow 3$.

First, $\mathrm{III} \wedge \mathrm{b} \Rightarrow A(0) = C(0) \cap \cup A$ because of $D(0) \cap C(0) = A(0)$, $D(0) \supseteq C(0) \cap \cup D = C(0) \cap \cup A$ implies $A(0) = D(0) \cap C(0) \supseteq C(0) \cap \cup A \supseteq A(0)$.

Next, $(A(0) = C(0) \cap \cup A) \wedge \mathrm{b} \Rightarrow 3$. In proving that we shall prove $B(0) = C(0) + D(0)$. Indeed by 3.5.7[11], $B(0) = (C \vee_P B)(0) = [C(0) + \cup C \cap D(0)] \cup [\cup D \cap C(0) + D(0)] = [C(0) + D(0)] \cup [\cup A \cap C(0) + D(0)] = C(0) + D(0)$.
Finally, we shall prove $D(0) = B(0) \cap \cup A$. Obviously $[C(0) + D(0)] \cap \cup A \supseteq C(0) \cap \cup A + D(0) \cap \cup A$. But there holds also the reverse inclusion $\subseteq$. In fact, for an element a of the set on the left side there will be $a = c + d \in \cup A$ for suitable $c \in C(0)$, $d \in D(0)$. Hence $c \in \cup A - d \subseteq \cup A + D(0) \subseteq \cup A + \cup D = \cup A + \cup A = \cup A$ thus $c \in C(0) \cap \cup A$ ($\subseteq \cup A$) and hence $d \in \cup A$. It follows $d \in D(0) \cap \cup A$. In conclusion $a = c + d \in C(0) \cap \cup A + D(0) \cap \cup A$, thus the inclusion $\subseteq$. From this the demanded equality follows since $B(0) \cap \cup A = [C(0) + D(0)] \cap \cup A = C(0) \cap \cup A + D(0) \cap \cup A = A(0) + D(0) \cap \cup D = A(0) + D(0) = D(0)$.

The implication $3 \Rightarrow \mathrm{III} \wedge \mathrm{b}$ is evident.

The necessity of the condition is proved.

Sufficiency:

$1 \Rightarrow D(0) = A(0) \subseteq C(0) \Rightarrow \mathrm{II}$

$2 \Rightarrow D(0) = B(0) \supseteq C(0) \Rightarrow \mathrm{I}$

Suppose 3. Since by 2.8.2 $\cup C = \cup B$, there will hold $\cup B \supseteq B(0) \cap \cup A \supseteq C(0) \cap \cup A \Rightarrow \cup C \supseteq D(0) \supseteq C(0) \cap \cup B \Rightarrow \mathrm{III}$. In all the cases the condition of Theorem 2.9.1 is satisfied thus D is a Dedekind P-complement of C in $[A, B]$.

[1]) $D(x)$ is defined as $\{y \in G : yDx\}$ (see [11] Def. 3.5).

2.9.3 Remark. 1. In 1. there holds $C = B$, in 2. $C = A$. In 3. $C = \mathsf{U}B/C(0)$, where $C(0)$ are precisely those ideals in $\mathsf{U}B$ which satisfy $A(0) = C(0) \cap \mathsf{U}A$ (since in the preceding theorem (*) was proved).

2. Theorem 2.9.2 may be derived also by using 2.8.2 and Theorem 5.6 [5]. This theorem represents a variant of 2.9.2 for partitions in a set. Its condition is as follows: All the blocks of the partition D contained in the same block of the partition B are blocks of the partition A with the exception of at most one block. This is equivalent to the statement: $D(0) = A(0)$, or $D(0) = B(0) \cap \mathsf{U}D$.
From 2.8.2 there follows $\mathsf{U}D = \mathsf{U}A$, or $\mathsf{U}D = \mathsf{U}B$.
From here results the assertion of Theorem 2.9.2 (with taking account of II $\wedge$ a in the proof of 2,9.2).

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