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PARTITIONS AND CONGRUENCES IN ALGEBRAS I. BASIC PROPERTIES

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0 A partition in a set G is a system A (possibly empty) of nonempty mutually disjoint subsets in G [4, 2, 7, 10, 11]. The empty system A will be called an *empty partition* and will be denoted by $\emptyset$.

The elements of a partition A in G are called *blocks* of the partition A; they are nonempty subsets in G. Let us denote by $\bigcup A$ the union of all blocks belonging to the partition A. A is, of course, a partition *on* $\bigcup A$. The set $\bigcup A$ will be called a *domain* of the partition A.

0.1 Now, let A be a symmetric and transitive binary relation in the set G. A is an equivalence relation in the set $\bigcup A = \{x \in G : xAx\}$. To find it out, it suffices to verify that A is a binary relation in the set $\bigcup A$. Thus, let $x, y \in G$, xAy hold; then from the symmetry of the relation A there follows yAx and from the transitivity xAx, yAy; thus $x, y \in \bigcup A$.

From the preceding consideration and from the fact that there exists a 1 - 1 correspondence between all partitions on a set and all equivalence relations in the same set, there follows the existence of a 1 - 1 correspondence between all partitions in the set G and all symmetric and transitive relations in G (cf. also [10], sec. 4 and 11). We shall find it useful to hold, if need be, the partitions in G for symmetric and transitive binary relations in G and vice versa.

0.2 Let (G, Ω) be a universal algebra with the system of operations Ω, and let A be a partition (symmetric and transitive binary relation) in the set G. We say that A is a *congruence in* the algebra (G, Ω) if for arbitrary n-ary $\omega \in \Omega$ there holds: $a_i, b_i \in G$ a_iAb_i $(i = 1, 2, \dots, n) \Rightarrow a_1 \dots a_n\omega Ab_1 \dots b_n\omega$. A congruence A in an algebra (G, Ω) will be called a *congruence on* the algebra (G, Ω) if A is a partition (equivalence relation) on the set G. The empty partition is a congruence in the algebra (and not a congruence on a nonempty algebra). It will be suitable to hold also the empty set for an algebra with an arbitrary system of operations Ω (though Ω contains nullary aperations). From this reason the empty set can be considered as a subalgebra of orbitrary algebra (G, Ω).

0.3 Notation:

$P(G)$ – system of all partitions (symmetric and transitive binary relations) in the set G.
$\Pi(G)$ – system of all partitions (equivalence relation) on the set G.
$\mathscr{K}(G)$ – system of all congruences in the algebra G.
$\mathscr{C}(G)$ – system of all congruences on the algebra G.

A number of papers have been devoted to the study of partitions in a set. From them there are quoted [2, 3, 4, 5, 7] used in the present paper. The subject of our interest will be to investigate the structure of the set $P(G)$ of all partitions in a set G, and the structure of the set $\mathscr{K}(G)$ of all congruences in G, where G will be a universal algebra or especially an Ω-group.

0.4 The known facts summarized in the following theorem will be used without any further quotations.

The set $\pi(G)$ of all partitions on a set G is a complete semimodular, relatively complemented lattice, [8] Th. 67. *The lattice $P(G)$ is complete, semimodular and upper-continuous, in general, it is not relatively complemented,* [5] Th. 4.5, 4.1 and 5.3. *$\pi(G)$ is a closed sublattice of $P(G)$. The set $\mathscr{C}(G)$ of all congruences on an algebra G is a closed sublattice of the lattice $\pi(G)$,* [8] Th. 84. *The lattice $\mathscr{C}(G)$, where G is an Ω-group, is modular,* [6] IV, 2.2. *The congruences on the group or on a relatively complemented lattice commute,* [8], p. 170, [9] § 4, Theor. 7. *If G is a lattice or an l-group, then $\mathscr{C}(G)$ is a distributive lattice,* [8] Th. 90, [1] XIV § 5, Th. 10.

0.5 The system $P(G)$ of all partitions (symmetric and transitive binary relation) in a set G is a complete lattice with respect to partial order defined as follows: $A \leqq B$ if $xAy \Rightarrow xBy$ $(A, B \in P(G))$. It is a matter of routine to prove that the greatest lower bound and the least upper bound in $P(G)$ are constructed in the following way [2] sections 13,14):

$$x(\bigwedge_{\alpha} {}_P A_\alpha)\, y \equiv xA_\alpha y \text{ for all } \alpha$$

$x(\bigvee_{\alpha} {}_P A_\alpha)\, y \equiv$ there exist elements $x_0 = x,\ x_1, \ldots, x_{n-1},\ x_n = y$ and indices $\alpha_1, \ldots, \alpha_n$ such that $x_0 A_{\alpha_1} x_1, \ldots, x_{n-1} A_{\alpha_n} x_n$.

As it is obvious the greatest lower bound and the least upper bound in $\pi(G)$ are constructed in the same way (see [4], 3.4 and 3.5, [3] I, 3.4 and 3.5, [11]).

0.6 When studying the structure of the set $\mathscr{K}(G)$ of all congruences in an algebra G we state first of all that $\mathscr{K}(G)$ is a complete lattice (1.1). The domain $\bigcup A$ of a congruence $A \in \mathscr{K}(G)$ is a subalgebra of G (1.3). The *nullblock* $A(0) = \{x \in G : xA0\}$ of a congruence A in an Ω-group is an ideal in $\bigcup A$ and there holds $A = \bigcup A/A(0)$ (1.4). In 1.5 and 1.6 there are described the domain and the nullblock of the greatest lower bound and the least upper bound of a congruence system in an algebra or in an

Ω-group, respectively. The remainder of the paper concerns the following problem. Let $\Phi(x_\alpha)$ and $\chi(y_\beta)$ be polynomials on a lattice (in indeterminates $\{x_\alpha\}$ and $\{y_\beta\}$, respectively), let $\{A_\alpha\}$ and $\{B_\beta\}$ be two systems of congruences in an Ω-group G. There are being looked for the conditions for the validity of implication $\Phi_P(A_\alpha) = \chi_P(B_\beta) \Rightarrow \Phi_{\mathscr{K}}(A_\alpha) = \chi_{\mathscr{K}}(B_\beta)$. For the particular polynomials $\Phi(x_\alpha) = \chi(x_\alpha) = \bigvee_\alpha x_\alpha$, the solution of the problem is given in section 1.7. A certain sufficient condition (distributivity of the lattice $\mathscr{K}(G)$) for the validity of the mentioned implication is found in sections 1.9, 1.10 and 1.11.

1.0 We shall investigate the structure of the set $\mathscr{K}(G)$ of all congruences in an algebra G. Some results will be derived only from the particular assumption that G is an Ω-group. An Ω-group is interpreted as a universal algebra whose operation system is the set Ω enlarged by one binary (group addition), one unary ($x \to -x$) and one nullary (0) operations.

1.1 *Let (G, Ω) be an algebra. Then $\mathscr{K}(G)$ is a complete lattice with respect to the order given by inclusion (of binary relations). For $\{A_\alpha\} \in \mathscr{K}(G)$ there holds $\bigwedge_{\alpha}{}_{\mathscr{K}} A_\alpha = \bigwedge_{\alpha}{}_P A$. If G is an Ω-group, then the set of all nonempty congruences in G is a closed sublattice of the lattice $\mathscr{K}(G)$.*

Proof. The first statement will be proved by showing that $A = \bigwedge_{\alpha}{}_P A_\alpha$ belongs to $\mathscr{K}(G)$. The statement regarding Ω-groups follows from the fact that $G_{\min}$ (= partition containing only one block $\{0\}$) is the least nonempty congruence in G.

If A is the empty partition, then $A \in \mathscr{K}(G)$. If A is nonempty, let $\omega \in \Omega$ be an n-ary operation ($n \geqq 1$), $a_i A a_i'$, $i = 1, 2, \ldots, n$. Then $a_i A_\alpha a_i'$ for all α and $i = 1, 2, \ldots, \ldots, n$ hence $a_1 \ldots a_n \omega A_\alpha a_1' \ldots a_n' \omega$ and thus $a_1 \ldots a_n \omega A a_1' \ldots a_n' \omega$. $A \in \mathscr{K}(G)$ is proved.

1.2 *Let (G, Ω) be an algebra, $\{A_\alpha\} \in \mathscr{K}(G)$. Then $\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha = \bigvee_{\gamma}{}_P B_\gamma$, where by B_γ is meant the congruence $A_{\alpha_1} \vee_{\mathscr{K}} \ldots \vee_{\mathscr{K}} A_{\alpha_n}$ for arbitrary finite choice $A_{\alpha_1}, \ldots, A_{\alpha_n}$ in $\{A_\alpha\}$.*

Proof. Since $\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha = \bigvee_{\gamma}{}_{\mathscr{K}} B_\gamma \geqq \bigvee_{\gamma}{}_P B_\gamma \geqq A_\alpha$ for all α, it is sufficient to prove that $A = \bigvee_{\gamma}{}_P B_\gamma$ is a congruence in G. Let us take an operation in Ω, for the sake of simplicity a binary one and let us donete it by o. A similar proof can be given for the operations of other arity ($\geqq 1$). Thus it will be proved xAx', $yAy' \Rightarrow (x \circ y) A (x' \circ y')$. From the definition of $\vee_P$ we get:

$xAx' \equiv$ there exist $x_1, \ldots, x_{n-1} \in G$, $B_{\gamma_1}, \ldots, B_{\gamma_n} \in \{B_\gamma\}$ so that

$$x B_{\gamma_1} x_1 B_{\gamma_2} x_2 \ldots x_{n-1} B_{\gamma_n} x',$$

$yAy' \equiv$ there exist $y_1, \ldots, y_{m-1} \in G$, $B_{\delta_1}, \ldots, B_{\gamma_m} \in \{B_\gamma\}$ so that

$$yB_{\delta_1}y_1B_{\gamma_2}y_2 \ldots y_{m-1}B_{\delta_m}y'.$$

Hence

$$(x \circ y)(B_{\gamma_1} \vee_{\mathscr{K}} B_{\delta_1})(x_1 \circ y)(B_{\gamma_2} \vee_{\mathscr{K}} B_{\delta_1})(x_2 \circ y) \ldots$$
$$\ldots (x_{n-1} \circ y)(B_{\gamma_n} \vee_{\mathscr{K}} B_{\delta_1})(x' \circ y)(B_{\gamma_n} \vee_{\mathscr{K}} B_{\delta_1})(x' \circ y_1) \ldots$$
$$\ldots (x' \circ y_{m-1})(B_{\gamma_n} \vee_{\mathscr{K}} B_{\delta_m})(x' \circ y').$$

1.2.0 Corollary. *Let (G, Ω) be an algebra, $\{A_\alpha\}$ an up-directed subset of $\mathscr{K}(G)$. Then $\bigvee_{\mathscr{K}\,\alpha} A_\alpha = \bigvee_{P\,\alpha} A_\alpha$.*

1.2.1 Definition. If A is a partition in a set G, then the set $\{x \in G : xAx\}$ is denoted by $\bigcup A$ (see 0.1) and is called a *domain* of the partition A.

1.3 $\bigcup A = \{x \in G : y \in G$ *exists such that* $xAy\}$ *for arbitrary partition A in the set G. Hence A is a partition on $\bigcup A$. Let (G, Ω) be an algebra. For every $A \in \mathscr{K}(G)$, $\bigcup A$ is a subalgebra of G.*

Proof. The inclusion $\{x \in G :$ there exists $y \in G$ such that $xAy\} \subseteqq \{x \in G : xAx\}$ results from the fact that the relation xAy implies yAx and both imply xAx. The inverse inclusion is evident. The other statements are obvious, too.

1.3.1 Let (G, Ω) be an algebra. As it was said above, the empty set is also included among subalgebras of the algebra G. The subalgebra generated in G by a subset $\mathfrak{C} \subseteqq G$ is denoted by $\langle\mathfrak{C}\rangle$ and in case $\mathfrak{C} = \emptyset$, by $\langle\mathfrak{C}\rangle$, is meant the empty subalgebra in G.

Let G be an Ω-group, $\mathfrak{A}$ an Ω-subgroup of G. On the basis of the above agreement concerning subalgebras, $\mathfrak{A} = \emptyset$ is not excluded. The ideal in $\mathfrak{A}$ generated by a set $\mathfrak{C} \subseteqq \mathfrak{A}$ is denoted by $\langle\langle\mathfrak{C}\rangle\rangle_{\mathfrak{A}}$ and in case $\mathfrak{C} = \emptyset$, by $\langle\langle\mathfrak{C}\rangle\rangle_{\mathfrak{A}}$, is meant the empty set. If $\mathcal{O} \neq A \in \mathscr{K}(G)$, let us denote $A(0) = \{x \in G : xA0\}$. The set $A(0)$ is called a *null-block* of the congruence A. The same terminology will be used also in case $A \in P(G)$, $0 \in \bigcup A$.

1.4 *Let G be an Ω-group, $\mathcal{O} \neq A$ a congruence in G. Then $A(0)$ is an ideal in $\bigcup A$, $A(0) \neq \emptyset$ and $A = \bigcup A/A(0)$. The empty congruence A in G gives $\bigcup A = \emptyset$ and* $A(0) = \emptyset$. For formal reason, it is writen also in this case $A = \bigcup A/A(0)$, and $A(0)$ is considered as an ideal in $\bigcup A$.

Proof. Since by 1.3 $A \neq \mathcal{O}$ is a congruence on the Ω-group $\bigcup A$, the statement follows from [6] III, 2.5.

1.5 *Let (G, Ω) be an algebra, $\{A_\alpha\} \subseteqq \mathscr{K}(G)$ or $\subseteqq P(G)$, respectively. Then there holds with respect both to $\mathscr{K}$ and P: $\bigcup(\bigwedge_\alpha A_\alpha) = \bigcap_\alpha(\bigcup A_\alpha)$. If G is an Ω-group, then* *when using notation* $\mathfrak{B} = \bigcap_\alpha (\mathsf{U}A_\alpha)$, $\mathfrak{C} = \bigcap_\alpha [A_\alpha(0)]$ *there holds* $(\bigwedge_{\alpha\mathscr{K}} A_\alpha)(0) = (\bigwedge_{\alpha P} A_\alpha)(0) = \mathfrak{C}$ and $\bigwedge_{\alpha\mathscr{K}} A_\alpha = \mathfrak{B}/\mathfrak{C} = \bigwedge_{\alpha P} A_\alpha$.

Proof. Because of $\mathsf{U}A_\alpha \supseteq \mathsf{U}(\bigwedge_\alpha A_\alpha)$ for all α ($\bigwedge_\alpha$ refers both to $\mathscr{K}$ and to P – see 1.1), we shall have $\mathsf{U}(\bigwedge_\alpha A_\alpha) \subseteq \bigcap_\alpha(\mathsf{U}A_\alpha)$. The reverse inclusion follows from the following: $x \in \bigcap_\alpha(\mathsf{U}A_\alpha) \Rightarrow x \in \mathsf{U}A_\alpha$ for all $\alpha \Rightarrow xA_\alpha x$ for all $\alpha \Rightarrow x(\bigwedge_\alpha A_\alpha)x \Rightarrow x \in \mathsf{U}(\bigwedge_\alpha A_\alpha)$.

If now G is an Ω-group there holds:
$x \in (\bigwedge_\alpha A_\alpha)(0) \Leftrightarrow x(\bigwedge_\alpha A_\alpha)\,0 \Leftrightarrow xA_\alpha 0$ for all $\alpha \Leftrightarrow x \in A_\alpha(0)$ for all $\alpha \Leftrightarrow x \in \bigcap_\alpha A_\alpha(0)$.
The remaining statement results from 1.1 and 1.4.

1.5.1 Let A be a binary relation in a set $G, \mathfrak{C} \subseteq G$. The *intersection* of the relation A and the set $\mathfrak{C}$ is denoted by $A \sqcap \mathfrak{C}$ and defined by the rule $A \sqcap \mathfrak{C} = A \cap (\mathfrak{C} \times \mathfrak{C})$ (or $x(A \sqcap \mathfrak{C})\, y \equiv x, y \in \mathfrak{C}, xAy$) [3] I, 2.3, [4] 2.3. If A is a partition in G, so is $A \sqcap \mathfrak{C}$. If G is an algebra, $\mathfrak{C}$ a subalgebra in G and A a congruence in G, then $A \sqcap \mathfrak{C}$ is a congruence in G (also in $\mathfrak{C}$).

1.6 *Let* (G, Ω) *be an algebra'* $\{A_\alpha\} \subseteq \mathscr{K}(G)$ *or* $\subseteq P(G)$. *Then* $\mathsf{U}(\bigvee_{\alpha\mathscr{K}} A_\alpha) = \langle \bigcup_\alpha (\mathsf{U}A_\alpha)\rangle$ *or* $\mathsf{U}(\bigvee_{\alpha P} A_\alpha) = \bigcup_\alpha(\mathsf{U}A_\alpha)$, *respectively. If* G *is an* Ω*-group,* $\{A_\alpha\} \subseteq \mathscr{K}(G)$, *then* $(\bigvee_{\alpha\mathscr{K}} A_\alpha)(0) = \langle\langle \bigcup_\alpha (A_\alpha(0))\rangle\rangle_{\mathfrak{A}} = \langle (\bigvee_{\alpha P} A_\alpha)(0)\rangle_{\mathfrak{A}}$, *where* $\mathfrak{A} = \langle \bigcup_\alpha(\mathsf{U}A_\alpha)\rangle$.

Proof. Let $A = \bigvee_{\alpha\mathscr{K}} A_\alpha$ and $C = A \sqcap \mathfrak{A}$. C is a congruence in the algebra G. Since $A \geqq A_\alpha$ for all α, we have $\mathsf{U}A \supseteq \mathfrak{A}$ and therefore $\mathsf{U}C = \mathsf{U}A \cap \mathfrak{A} = \mathfrak{A}$. If $xA_\alpha y$ for some α, then xAy, $x, y \in \mathsf{U}A_\alpha \subseteq \mathfrak{A}$ so xCy and hence $C \geqq A_\alpha$ for all α, thus $C \geqq A$. From this we conclude $\mathfrak{A} = \mathsf{U}C \supseteq \mathsf{U}A$, thus $\mathfrak{A} = \mathsf{U}A$.

The second equality: evidently $\mathsf{U}(\bigvee_{\alpha P} A_\alpha) \supseteq \mathsf{U}A_\alpha$ for all α, thus $\mathsf{U}(\bigvee_{\alpha P} A_\alpha \supseteq \bigcup_\alpha(\mathsf{U}A_\alpha)$. Conversely, $x \in \mathsf{U}(\bigvee_{\alpha P} A_\alpha) \Rightarrow x(\bigvee_{\alpha P} A_\alpha)\, x \Rightarrow$ there exist $x_1, \ldots, x_{n-1} \in G$, $A_{\alpha_1}, \ldots, A_{\alpha_n} \in \{A_\alpha\}$ such that $xA_{\alpha_1}x_1A_{\alpha_2}x_2 \ldots x_{n-1}A_{\alpha_n}x \Rightarrow xA_{\alpha_1}x_1 \Rightarrow x \in \mathsf{U}A_{\alpha_1} \Rightarrow x \in \bigcup_\alpha(\mathsf{U}A_\alpha)$. By this the reverse inclusion is proved and so the equality $\mathsf{U}(\bigvee_{\alpha P} A_\alpha) = \bigcup_\alpha(\mathsf{U}A_\alpha)$.

Let G be now an Ω-group, $\{A_\alpha\} \subseteq \mathscr{K}(G)$, $J = \langle\langle \bigcup_\alpha (A_\alpha(0))\rangle\rangle_{\mathfrak{A}}$. By 1.4 $A(0)$ is an ideal in $\mathfrak{A}$, $A(0) \supseteq A_\alpha(0)$ for all α thus $A(0) \supseteq J$. Again by 1.4, we have $A = \mathfrak{A}/A(0) \geqq \mathfrak{A}/J \geqq \mathsf{U}A_\alpha/A_\alpha(0) = A_\alpha$ for all α which implies $A \geqq \mathfrak{A}/J \geqq A$. We have obtained the equalities $\mathfrak{A}/A(0) = A = \mathfrak{A}/J$ hence $A(0) = J$.

The remaining equality may be obtained from that proved above as follows:

$$(\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha)(0) \supseteq (\bigvee_{\alpha}{}_{P} A_\alpha)(0) \supseteq A_\alpha(0) \text{ for all } \alpha \Rightarrow$$

$$\Rightarrow (\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha)(0) \supseteq \langle\langle(\bigvee_{\alpha}{}_{P} A_\alpha)(0)\rangle\rangle_{\mathfrak{A}} \supseteq \langle\langle\bigcup_{\alpha}(A_\alpha(0))\rangle\rangle_{\mathfrak{A}} = (\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha)(0) \Rightarrow$$

$$\Rightarrow (\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha)(0) = \langle\langle(\bigvee_{\alpha}{}_{P} A_\alpha)(0)\rangle\rangle_{\mathfrak{A}}.$$

1.6.1 Remark. From Theorem 1.6 it is evident that the lattice $\mathscr{K}(G)$ is not a sublattice of the lattice $P(G)$. Indeed, $\mathsf{U}(A \vee_{\mathscr{K}} B) = \langle \mathsf{U}A \cup \mathsf{U}B \rangle$ holds for $A, B \in \mathscr{K}(G)$ while $\mathsf{U}(A \vee_P B) = \mathsf{U}A \cup \mathsf{U}B$.

Using 1.6 and 1.4, we have the following.

1.6.2 Corollary. *Let G be an Ω-group, $\{A_\alpha\} \subseteq \mathscr{K}(G)$, $\mathfrak{A} = \langle \bigcup_{\alpha}(\mathsf{U}A_\alpha)$, $J = \langle\langle\bigcup_{\alpha}(A_\alpha(0))\rangle\rangle_{\mathfrak{A}}$. Then $\bigvee_{\mathscr{K}} A_\alpha = \mathfrak{A}/J$.*

1.7 *Let G be an Ω-group, $\{A_\alpha\}$, $\{B_\beta\}$ systems in $\mathscr{K}(G)$, $\mathfrak{L}_1 = \mathsf{U}(\bigvee_{\alpha}{}_{P} A_\alpha)$, $\mathfrak{L}_2 = \mathsf{U}(\bigvee_{\beta}{}_{P} B_\beta)$.*

If it is true

a) $\langle \mathsf{U}(\bigvee_{\alpha}{}_{P} A_\alpha) = \langle \mathsf{U}(\bigvee_{\beta}{}_{P} B_\beta)\rangle$

and at the same time one of the conditions b, b′, b″:

b) $\mathfrak{L}_1 \sqcap \bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha = \mathfrak{L}_2 \sqcap \bigvee_{\beta}{}_{\mathscr{K}} B_\beta$

b′) $\mathfrak{L}_1 \cap (\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha)(0) = \mathfrak{L}_2 \cap (\bigvee_{\beta}{}_{\mathscr{K}} B_\beta)(0)$

b″) $(\bigvee_{\alpha}{}_{P} A_\alpha)(0) = (\bigvee_{\beta}{}_{P} B_\beta)(0)$,

then $\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha = \bigvee_{\beta}{}_{\mathscr{K}} B_\beta$.

Proof. If we prove 1) $\mathsf{U}(\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha) = \mathsf{U}(\bigvee_{\beta}{}_{\mathscr{K}} B_\beta)$ $(=\mathfrak{A})$ and 2) $(\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha)(0) = (\bigvee_{\beta}{}_{\mathscr{K}} B_\beta)(0)$ $(=J)$, then by 1.6.2

$$\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha = \mathfrak{A}/J = \bigvee_{\beta}{}_{\mathscr{K}} B_\beta.$$

The equality 1) follows from a) and 1.6. We shall prove 2). First, it is clear that $b \Rightarrow b'$.

Next, from the relations

$(\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha)(0) \supseteq \mathfrak{L}_1 \cap (\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha)(0) \supseteq (\bigvee_{\alpha}{}_{P} A_\alpha)(0) \supseteq A_\alpha(0)$ for all α there follows (by 1.6)

$$(\underset{\alpha}{\bigvee_{\mathcal{K}}} A_\alpha)(0) \supseteq \langle\langle \mathfrak{L}_1 \cap (\underset{\alpha}{\bigvee_{\mathcal{K}}} A_\alpha)(0)\rangle\rangle_{\mathfrak{A}} \supseteq \langle\langle(\underset{\alpha}{\bigvee_{P}} A_\alpha)(0)\rangle\rangle_{\mathfrak{A}} \supseteq \langle\langle \underset{\alpha}{\bigcup}(A_\alpha(0))\rangle\rangle_{\mathfrak{A}} = $$
$$= (\underset{\alpha}{\bigvee_{\mathcal{K}}} A_\alpha)(0)$$

thus $(\underset{\alpha}{\bigvee_{\mathcal{K}}} A_\alpha)(0) = \langle\langle \mathfrak{L}_1 \cap (\underset{\alpha}{\bigvee_{\mathcal{K}}} A_\alpha)(0)\rangle\rangle = \langle\langle(\underset{\alpha}{\bigvee_{P}} A_\alpha)(0)\rangle\rangle_{\mathfrak{A}}$. If b′) or b'') holds, then from the preceding equalities and from analogical equalities for B_β there follows 2).

1.7.1 Remark. In 1.7 it is possible to put the following weaker conditions instead of b′) or b″), respectively:

$$\langle\langle \mathfrak{L}_1 \cap (\underset{\alpha}{\bigvee_{\mathcal{K}}} A_\alpha)(0)\rangle\rangle_{\mathfrak{A}} = \langle\langle \mathfrak{L}_2 \cap (\underset{\beta}{\bigvee_{\mathcal{K}}} B_\beta)(0)\rangle\rangle_{\mathfrak{A}}$$

or

$$\langle\langle(\underset{\alpha}{\bigvee_{P}} A_\alpha)(0)\rangle\rangle_{\mathfrak{A}} = \langle\langle(\underset{\beta}{\bigvee_{P}} B_\beta)(0)\rangle\rangle_{\mathfrak{A}}.$$

1.7.2 Corollary. *Let G be an Ω-group, $\{A_\alpha\}$, $\{B_\beta\}$ systems in $\mathcal{K}(G)$. Then*

$$\underset{\alpha}{\bigvee_{P}} A_\alpha = \underset{\beta}{\bigvee_{P}} B_\beta \Rightarrow \underset{\alpha}{\bigvee_{\mathcal{K}}} A_\alpha = \underset{\beta}{\bigvee_{\mathcal{K}}} B_\beta.$$

In the proof of 1.7 there was proved the following statement completing the second part of 1.6:

1.7.3 *Let G be an Ω-group, $\{A_\alpha\}$ a system in $\mathcal{K}(G)$, $\mathfrak{L} = \underset{\alpha}{\bigcup}(\underset{}{\bigvee_{P}} A_\alpha)$, $\mathfrak{A} = \langle \underset{\alpha}{\bigcup}(\bigvee_{P} A_\alpha)\rangle$.* Then

$$(\underset{\alpha}{\bigvee_{\mathcal{K}}} A_\alpha)(0) = \langle\langle \mathfrak{L} \cap (\underset{\alpha}{\bigvee_{\mathcal{K}}} A_\alpha)(0)\rangle\rangle_{\mathfrak{A}}.$$

1.8 Definition. Let Ω be a system of operations, I a nonempty set, $\{x_\alpha : \alpha \in I\}$ a set of some elements. A *polynomial* (over Ω in indeterminates x_α $(\alpha \in I)$) is defined (by way of finite induction) as follows:

a) Every x_α $(\alpha \in I)$ and every symbol of nullary operation in Ω is a polynomial.

b) If $\omega \in \Omega$ is an n-ary operation, $n \geqq 1$, $v_1, \ldots, v_n$ polynomials, then $v_1 \ldots v_n\omega$ is a polynomial.

If (G, Ω) is a complete lattice, we admit even infinite lattice operations.

Let a_α be an element of an algebra (G, Ω) for every $\alpha \in I$. By the *value* of a polynomial $\Phi(x_\alpha : \alpha \in I)$ in a_α $(\alpha \in I)$, is meant an element in G which we have got by substituting a_α for x_α and by replacing the symbols of nullary operations by corresponding elements of the algebra G and by applying the operations ω, as prescribed in (G, Ω). We denote by $\Phi_G(a_\alpha : \alpha \in I)$ (or briefly by $\Phi_G(a_\alpha)$; similarly $\Phi(x_\alpha)$).

1.8.1 Let us recall Corollary 1.7.2 to Theorem 1.7: For congruences A_α, B_β in an Ω-group there holds $\underset{\alpha}{\bigvee_{P}} A_\alpha = \underset{\beta}{\bigvee_{P}} B_\beta \Rightarrow \underset{\alpha}{\bigvee_{\mathcal{K}}} A_\alpha = \underset{\beta}{\bigvee_{\mathcal{K}}} B_\beta$.

It is a question whether there does not hold the following more general theorem:

(1.8,1) Let Ω' denote the system of the lattice operations, let $\Phi(x_\alpha)$ and $\chi(y_\beta)$ be polynomials over Ω'. If A_α and B_β are two systems of congruences in an Ω-group G then (1.8,2) $\Phi_P(A_\alpha) = \chi_P(B_\beta) \Rightarrow \Phi_{\mathscr{K}}(A_\alpha) = \chi_{\mathscr{K}}(B_\beta)$.

The meaning of such a theorem consists in the possibility of transferring the first equality in (1.8,2) from $P(G)$ to $\mathscr{K}(G)$. A special attention should be paid (and this is in possibilities of the theorem) to transferring the identity from $P(G)$ to $\mathscr{K}(G)$. In more details:

If the equality $\Phi(x_\alpha) = \chi(x_\alpha)$ holds for all systems $\{A_\alpha\} \subseteqq P(G)$ (or $\subseteqq \mathscr{K}(G)$) according to $P(G)$, then it holds also for all systems $\{A_\alpha\} \subseteqq \mathscr{K}(G)$ according to $\mathscr{K}(G)$.

The rest of paragraph is devoted to the just mentioned problem.

1.9 Let the lattice of all subalgebras of an algebra (G, Ω) and the lattice of all subsets of a set G be denoted by the symbol $S = S(G)$ and $M = M(G)$, respectively. Let the symbol Ω' denote as above the system of the lattice operations.

1.9.1 *Let $\Phi(x_\alpha)$ be a polynomial over Ω' in indeterminates x_α. Let (G, Ω) be an algebra, A_α congruences in G. Then*

$$\bigcup \Phi_{\mathscr{K}}(A_\alpha) = \Phi_s(\bigcup A_\alpha) \supseteqq \bigcup \Phi_P(A_\alpha) = \Phi_M(\bigcup A_\alpha).$$

Remark. Because of 1.3 $\bigcup \Phi_{\mathscr{K}}(A_\alpha)$ is a subalgebra in (G, Ω) and thus

$$\bigcup \Phi_{\mathscr{K}}(A_\alpha) \supseteqq \langle \bigcup_P(A_\alpha)\rangle.$$

Proof. $\Phi(x_\alpha) = \bigvee_\beta \Psi^\beta(x_\alpha)$ or $= \bigwedge_\beta \Psi^\beta(x_\alpha)$ is satisfied for suitable polynomials $\Psi^\beta(x_\alpha)$ ($\beta \in B$, card $B \geqq 2$). Let us assume by way of induction that for every β there holds

$$\bigcup \Psi^\beta_{\mathscr{K}}(A_\alpha) = \Psi^\beta_S(\bigcup A_\alpha) \supseteqq \bigcup \Psi^\beta_P(A_\alpha) = \Psi^\beta_M(\bigcup A_\alpha). \tag{1.9,1}$$

The case $\Phi(x_\alpha) = \bigvee_\beta \Psi^\beta(x_\alpha)$. By 1.6 and by induction hypothesis we have got

$$\bigcup \Phi_{\mathscr{K}}(A_\alpha) = \bigcup(\bigvee_{\beta}{}_{\mathscr{K}} \Psi^\beta_{\mathscr{K}}(A)) = \langle \bigcup_\beta(\bigcup \Psi^\beta_{\mathscr{K}}(A_\alpha)\rangle =$$
$$= \langle \bigcup_\beta(\Psi^\beta_S(\bigcup A_\alpha))\rangle = \bigvee_{\beta}{}_S \Psi^\beta_S(\bigcup A_\alpha) = \Phi_S(\bigcup A_\alpha),$$

thus

$$\bigcup \Phi_{\mathscr{K}}(A_\alpha) = \Phi_s(\bigcup A_\alpha).$$

Further

$$\bigcup \Phi_P(A_\alpha) = \bigcup[\bigvee_{\beta}{}_P \Psi^\beta_P(A_\alpha) = \bigcup_\beta[\bigcup \Psi^\beta_P(A_\alpha)].$$

By induction hypothesis, the last member is obtained in the set

$$\langle \bigcup_{\beta}(\bigcup \Psi^{\beta}_{\mathscr{K}}(A_{\alpha}))\rangle = \bigcup(\bigvee_{\beta}{}_{\mathscr{K}}\Psi^{\beta}_{\mathscr{K}}(A_{\alpha})) = \bigcup \Phi_{\mathscr{K}}(A_{\alpha}).$$

Hence

$$\bigcup \Phi_{\mathscr{K}}(A_{\alpha}) \supseteq \bigcup \Phi_{P}(A_{\alpha}).$$

Finally by 1.6 and by induction hypothesis

$$\bigcup \Phi_{P}(A_{\alpha}) = \bigcup(\bigvee_{\beta}{}_{P}\Psi^{\beta}_{P}(A_{\alpha})) = \bigcup_{\beta}(\bigcup \Psi^{\beta}_{P}(A_{\alpha})) = \bigcup_{\beta}(\Psi^{\beta}_{M}(\bigcup A_{\alpha})) = \Phi_{M}(\bigcup A_{\alpha})$$

hence

$$\bigcup \Phi_{P}(A_{\alpha}) = \Phi_{M}(\bigcup A_{\alpha})$$

is fulfilled.

The case $\Phi(x_{\alpha}) = \bigwedge_{\beta} \Psi^{\beta}(x_{\alpha})$. By 1.5 one gets

$$\bigcup \Phi_{\mathscr{K}}(A_{\alpha}) = \bigcup[\bigwedge_{\beta}{}_{\mathscr{K}}\Psi^{\beta}_{\mathscr{K}}(A_{\alpha})] = \bigcap_{\beta}[\bigcup \Psi^{\beta}_{\mathscr{K}}(A_{\alpha})].$$

By induction assumption, the last member equals

$$\bigwedge_{\beta}{}_{S}[\Psi^{\beta}_{S}(\bigcup A_{\alpha})] = \Phi_{S}(\bigcup A_{\alpha})$$

and contains the set

$$\bigcap_{\beta}[\bigcup \Psi^{\beta}_{P}(A_{\alpha})] = \bigcup[\bigwedge_{\beta}{}_{P}\Psi^{\beta}_{P}(A_{\alpha})] = \bigcup \Phi_{P}(A_{\alpha}).$$

Thus it is proved

$$\bigcup \Phi_{\mathscr{K}}(A_{\alpha}) = \Phi_{s}(\bigcup A_{\alpha}) \supseteq \bigcup \Phi_{P}(A_{\alpha}).$$

Finally, by 1.5 and by induction assumption there holds

$$\bigcup \Phi_{P}(A_{\alpha}) = \bigcup[\bigwedge_{\beta}{}_{P}\Psi^{\beta}_{P}(A_{\alpha})] = \bigcap_{\beta}[\bigcup \Psi^{\beta}_{P}(A_{\alpha})] = \bigcap_{\beta}[\Psi^{3}_{M}(\bigcup A_{\alpha})] =$$
$$= \bigwedge_{\beta}{}_{M}[\Psi^{\beta}_{M}(\bigcup A_{\alpha})] = \Phi_{M}(\bigcup A_{\alpha}).$$

This verifies

$$\bigcup \Phi_{P}(A_{\alpha}) = \Phi_{M}(\bigcup A_{\alpha}).$$

1.10 *Let $\Phi(x_{\alpha})$ be a polynomial over Ω' in indeterminates $x_{\alpha}(\alpha \in I)$. Let (G, Ω) be an algebra, $A_{\alpha}(\alpha \in I)$ congruences in G. Let $\mathscr{K}(G)$ be distributive if only the finite lattice operations appear in Φ, let it be completely distributive otherwise. Then*

$$\bigcup \Phi_{\mathscr{K}}(A_{\alpha}) = \Phi_{s}(\bigcup A_{\alpha}) = \langle \bigcup \Phi_{P}(A_{\alpha})\rangle, \qquad \bigcup \Phi_{P}(A_{\alpha}) = \Phi_{M}(\bigcup A_{\alpha}).$$

Proof. Our aim is to prove the equality $\bigcup \Phi_{\mathscr{K}}(A_{\alpha}) = \bigcup \Phi_{P}(A_{\alpha})$. The other equalities follow directly from 1.9.1. As in the proof to 1.9.1 let $\Phi(x_{\alpha}) = \bigvee_{\beta} \Psi^{\beta}(x_{\alpha})$ or $= \bigwedge_{\beta} \Psi^{\beta}(x_{\alpha})$ be for suitable polynomials $\Psi^{\beta}(x_{\alpha})$ $(\beta \in B)$.

Induction hypothesis:

$$\mathsf{U}\Psi^{\beta}_{\mathscr{K}}(A_\alpha) = \langle \mathsf{U}\Psi^{\beta}_{P}(A_\alpha)\rangle \qquad \text{for all } \beta \in B.$$

The case $\Phi(x_\alpha) = \bigvee_\beta \Psi^\beta(x_\alpha)$. By 1.6 there holds

$$(1.10,1) \quad \langle \mathsf{U}\Phi_P(A_\alpha) = \langle \bigcup_\beta [\mathsf{V}_P \Psi^\beta_P(A_\alpha)] = \langle \bigcup_\beta [\mathsf{U}\Psi^\beta_P(A_\alpha)]\rangle \subseteq \langle \bigcup_\beta \langle \mathsf{U}\Psi^\beta_P(A_\alpha)\rangle\rangle.$$

Last but one member contains the set $\langle \mathsf{U}\Psi^\beta_P(A_\alpha)\rangle$ for all $\beta \in B$ therefore also the set $\langle \bigcup_\beta \langle \mathsf{U}\Psi^\beta_P(A_\alpha)\rangle\rangle$. Conseqently, it is possible to replace in (1.10,1) the inclusion by the equality. By induction hypothesis, the last member in (1.10,1) equals

$$\langle \bigcup_\beta (\mathsf{U}\Psi^\beta_{\mathscr{K}}(A_\alpha))\rangle = \mathsf{U}[\bigvee_\beta{}_{\mathscr{K}} \Psi^\beta_{\mathscr{K}}(A_\alpha)] = \mathsf{U}\Phi_{\mathscr{K}}(A_\alpha).$$

The case $\Phi(x_\alpha) = \bigwedge_\beta \Psi^\beta(x_\alpha)$. For suitable polynomials $\Psi^{\beta,\gamma}(x_\alpha)$ $(\beta \in B, \gamma = \Gamma)$ there holds $\Phi(x_\alpha) = \bigwedge_\beta \bigvee_\gamma \Psi^{\beta,\gamma}(x_\alpha)$. If Φ contains only finite operations, the sets B, Γ are finite. From the (complete) distributivity of the lattice $\mathscr{K}(G)$ there follows

$$\bigwedge_\beta{}_{\mathscr{K}} \bigvee_\gamma{}_{\mathscr{K}} \Psi^{\beta,\gamma}_{\mathscr{K}}(A_\alpha) = \bigvee_{f \in \Gamma^B}{}_{\mathscr{K}} \bigwedge_\beta{}_{\mathscr{K}} \Psi^{\beta,f(\beta)}_{\mathscr{K}}(A_\alpha).$$

If we denote $\bigwedge_\beta \Psi^{\beta,f(\beta)}(x_\alpha) = T^f(x_\alpha)$, we have $\Phi_{\mathscr{K}}(A_\alpha) = \bigvee_f{}_{\mathscr{K}} T^f_{\mathscr{K}}(A_\alpha)$ and we have reduced the discussion to the preceding case. In the present situation the induction hypothesis will concern the polynomials T^f $(f \in \Gamma^B)$ instead of Ψ^β. There holds (in the third equality we use the complete distributivity of the lattice $M(G)$, in the seventh one, the induction hypothesis):

$$\langle \mathsf{U}\Phi_P(A_\alpha)\rangle = \langle \mathsf{U} \bigwedge_\beta{}_P \bigvee_\gamma{}_P \Psi^{\beta,\gamma}_P(A_\alpha)\rangle = \langle \bigcap_\beta \bigcup_\gamma \mathsf{U}\Psi^{\beta,\gamma}_P(A_\alpha)\rangle =$$

$$= \langle \bigcup_{f \in \Gamma^B} \bigcap_\beta \mathsf{U}\Psi^{\beta,f(\beta)}_P(A_\alpha)\rangle = \langle \bigcup_f \langle \bigcap_\beta \mathsf{U}\Psi^{\beta,f(\beta)}_P(A_\alpha)\rangle\rangle =$$

$$= \langle \bigcup_f \langle \mathsf{U} \bigwedge_\beta{}_P \Psi^{\beta,f(\beta)}_P(A_\alpha)\rangle\rangle = \langle \bigcup_f \langle \mathsf{U}T^f_P(A_\alpha)\rangle\rangle =$$

$$= \langle \bigcup_f \mathsf{U}T^f_{\mathscr{K}}(A_\alpha)\rangle = \mathsf{U} \bigvee_f{}_{\mathscr{K}} T^f_{\mathscr{K}}(A_\alpha) = \mathsf{U}\Phi_{\mathscr{K}}(A_\alpha).$$

The theorem is proved.

1.10.1 Corollary. *Let $\Phi(x_\alpha)$ and $\chi(y_\beta)$ be polynomials over Ω' in indeterminates x_α $(\alpha \in I)$ and y_β $(\beta \in J)$, respectively. Let (G, Ω) be an algebra, A_α, B_β congruences in G. Let the lattice $\mathscr{K}(G)$ be distributive if only finite lattice operations appear in Φ and χ, and let it be completely distributive otherwise.*

Then there holds

$$\langle \mathsf{U}\Phi_P(A_\alpha)\rangle = \langle \mathsf{U}\chi_P(B_\beta) \Rightarrow \mathsf{U}\Phi_{\mathscr{K}}(A_\alpha) = \mathsf{U}\chi_{\mathscr{K}}(B_\beta).$$

Proof follows from the fact that by 1.10 there is $\bigcup\Phi_{\mathscr{K}}(A_\alpha) = \langle\bigcup\Phi_P(A_\alpha)\rangle$ and similarly for $\chi(B_\beta)$.

1.11 *Let the conditions of Theorem* **1.10** *be fulfilled. Then*

$$\langle\langle(\Phi_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} = (\Phi_{\mathscr{K}}(A_\alpha))(0), \text{ where } \mathfrak{A} = \bigcup\Phi_{\mathscr{K}}(A_\alpha).$$

Proof is analogous to that of 1.10. First, let $\Phi(x_\alpha) = \bigvee_\beta \Psi^\beta(x_\alpha)$ for suitable polynomials $\Psi^\beta(x_\alpha)$ $(\beta \in B)$. The induction hypothesis

$$(\Psi^\beta_{\mathscr{K}}(A_\alpha))(0) = \langle\langle(\Psi^\beta_P(A_\alpha))(0)\rangle\rangle\mathfrak{B}_\beta, \quad \text{where} \quad \mathfrak{B}_\beta = \bigcup\Psi^\beta_{\mathscr{K}}(A_\alpha).$$

There holds (the second equality by induction hypothesis and the third one by 1.6)

$$\langle\langle(\Phi_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} = \langle\langle(\bigvee_{P\,\beta}\Psi^\beta_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} \supseteq \langle\langle\bigcup_\beta \langle\langle(\Psi^\beta_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{B}_\beta}\rangle\rangle_{\mathfrak{A}} =$$
$$= \langle\langle\bigcup_\beta(\Psi^\beta_{\mathscr{K}}(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} = (\bigvee_{\mathscr{K}\,\beta}\Psi^\beta_{\mathscr{K}}(A_\alpha))(0) =$$
$$= (\Phi_{\mathscr{K}}(A_\alpha))(0) \supseteq \langle\langle(\Phi_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}}.$$

The last equality is true because for arbitrary $Q_\beta \in \mathscr{K}(G)$ there is

$$(\bigvee_{\mathscr{K}\,\beta}Q_\beta)(0) \supseteq (\bigvee_{P\,\beta}Q_\beta)(0), \; (\bigwedge_{\mathscr{K}\,\beta}Q_\beta)(0) = (\bigwedge_{P\,\beta}Q_\beta)(0).$$

Hence

$$\langle\langle(\Phi_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} = (\Phi_{\mathscr{K}}(A_\alpha))(0). \tag{1.11,1}$$

The second case $\Phi(x_\alpha) = \bigwedge_\beta \Psi^\beta(x_\alpha)$. For suitable polynomials $\Psi^{\beta,\gamma}(x_\alpha)$ there holds $\Phi(x_\alpha) = \bigwedge_\beta\bigvee_\gamma \Psi^{\beta,\gamma}(x_\alpha)$.

From the distributivity of the lattice $\mathscr{K}(G)$ there follows

$$\bigwedge_{\mathscr{K}\,\beta}\bigvee_{\mathscr{K}\,\gamma}\Psi^{\beta,\gamma}(A_\alpha) = \bigvee_{\mathscr{K}\, f\in\Gamma^B}\bigwedge_{\mathscr{K}\,\beta}\Psi^{\beta,f(\beta)}_{\mathscr{K}}(A_\alpha).$$

When denoting $\bigwedge_\beta \Psi^{\beta,f(\beta)}(x_\alpha) = T^f(x_\alpha)$, it will be $\Phi_{\mathscr{K}}(A_\alpha) = \bigvee_{\mathscr{K}\,f} T^f_{\mathscr{K}}(A_\alpha)$ and the discussion is reduced to the previous case. In the present situation the induction hypothesis will concern the polynomials $T^f(x_\alpha)$ instead of $\Psi^\beta(x_\alpha)$. We have got

$$\langle\langle(\Phi_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} = \langle\langle(\bigwedge_{P\,\beta}\bigvee_{P\,\gamma}\Psi^{\beta,\gamma}_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} \supseteq$$
$$\supseteq \langle\langle\bigcap_\beta\bigcup_\gamma(\Psi^{\beta,\gamma}_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} = \bigcup_{f\in\Gamma^B}\bigcap_\beta(\Psi^{\beta,f(\beta)}_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} =$$
$$= \langle\langle\bigcup_f(\bigwedge_\beta\Psi^{\beta,f(\beta)}_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} = \langle\langle\bigcup_f(T^f_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} =$$
$$= \langle\langle\bigcup_f(T^f_{\mathscr{K}}(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}} = (\Phi_{\mathscr{K}}(A_\alpha))(0) \supseteq \langle\langle(\Phi_P(A_\alpha))(0)\rangle\rangle_{\mathfrak{A}}.$$

So the equality (1.11,1) is proved.

1.11.1 Corollary. *Let the conditions of Theorem* **1.10.1** *be satisfied. Then there holds*

$$\Phi_P(A_\alpha) = \chi_P(B_\beta) \Rightarrow \Phi_{\mathscr{K}}(A_\alpha) = \chi_{\mathscr{K}}(B_\beta).$$

Proof. By 1.10.1 there holds $\mathfrak{A} = \bigcup \Phi_{\mathscr{K}}(A_\alpha) = \bigcup \chi_{\mathscr{K}}(B_\beta)$ and by 1.11.1 $(\Phi_{\mathscr{K}}(A_\alpha))(0) = (\chi_{\mathscr{K}}(B_\beta))(0)$ $(= J)$. This implies (by 1.4)

$$\Phi_{\mathscr{K}}(A_\alpha) = \chi_{\mathscr{K}}(B_\beta).$$

1.12 *Theorems* **1.10** *and* **1.11** *are not valid if we omit the hypothesis of (complete) distributivity of the lattice* $\mathscr{K}(G)$.

Proof. Let A, B, C, D be four distinct lines in the plane passing through the origin. Each of them represents a one-element partition in the plane G considered as the additive group of ordered couples of real numbers. All these four partitions belong to $\mathscr{K}(G)$. For $\Phi(x, y, u, v) = (x \vee y) \wedge (u \vee v)$ there clearly holds $\Phi_{\mathscr{K}}(A, B, C, D) = G_{\max}$, thus $\bigcup \Phi_{\mathscr{K}}(A, B, C, D) = G = (\Phi_{\mathscr{K}}(A, B, C, D))(0)$ whereas

$$\Phi_P(A, B, C, D) = \{0\}, \quad \text{thus} \quad \langle \bigcup \Phi_P(A, B, C, D) \rangle = \{0\},$$
$$\langle\langle(\Phi_P(A, B, C, D))(0)\rangle\rangle_{\mathfrak{A}} = \{0\} \quad (\text{where } \mathfrak{A} = \bigcup \Phi_{\mathscr{K}}(A, B, C, D) = G).$$

REFERENCES

[1] Birkgoff G.: *Teorija struktur.* Moskva 1952

[2] Borůvka O.: *Teorie rozkladů v množině.* Publ. Fac. Sci. Univ. Brno, No. 278 (1946), 1—37

[3] Borůvka O.: *Grundlagen der Gruppoid- und Gruppentheorie.* Berlin 1960

[4] Borůvka O.: *Základy teorie grupoidů a grup.* ČSAV Praha 1962

[5] Draškovičová H.: *The lattice of partitions in a set.* Acta Fac. Rer. Nat. Univ. Comen. Math. 24 (1970), 37—65

[6] Kuroš A. G.: *Lekcii po obščej algebre.* Moskva 1962

[7] Ore O.: *Theory of equivalence relations.* Duke Math. J. 9 (1942), 573—627

[8] Szász G.: *Théorie des treillis.* Budapest 1971

[9] Skornjakov L. A.: *Elementy teorii struktur.* Moskva 1970

[10] Sturm T.: *Verbände von Kernen isotoner Abbildungen.* Czechosl. Math. J. 22 (97) 1972, 126—144

[11] Schmidt E. T.: *Kongruenzrelationen algebraischer Strukturen.* Berlin 1969.

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