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ON THE h-TOPOLOGY IN GROUPS

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This paper deals with the investigation of some properties of h-topologies in a given additive group G. These topologies fulfil conditions which are similar to those of topologies of topological groups G (see [2]). First, the properties of their bases about zero (i.e. their complete systems of neighbourhoods of zero), the lattice structure of the set of all h-topologies in G and a form of a low (an upper) h-modification of a given topology in G are described here.

Further, the investigation of the h-modifications of a given topology τ in the set G enables us to describe better the least upper (the greatest low) bound of the topology τ in a set of all topologies of topological groups G with another topological property respectively (see [1]).

By a topology there is meant topology in Bourbaki's sense in this paper. Without loss of generality we admit only open neighbourhoods.

1.

Definition. Let G be an additive group. A topology in G in which mappings $x + a$, $a + x$, $-x$ are homeomorphisms of G into itself for any $a \in G$ is called *an h-topology* in the group G. Furthermore, let us denote by $\mathfrak{B}$ ($\mathfrak{M}$) the system of all h-topologies in G (the set of all topologies in G).

The next propositions follow from properties of homeomorphisms and from characteristics of complete systems of neighbourhoods:

1.1. Let τ be a topology in a group G, Γ_x^* be a system of all neighbourhoods of an element $x \in G$ in the topology τ. Then τ is an h-topology in G if and only if for any a, $b \in G$ it holds:

1. $$\Gamma_b^* = \Gamma_a^* - a + b = b - a + \Gamma_a^*,$$
2. $$\Gamma_b^* = -\Gamma_b^*.$$

1.2. Let Γ be a basis about zero in an h-topology τ in a group G. It holds:

1. For any $U \in \Gamma$, $a \in G$ there exist neighbourhoods V_1, V_2, $V_3 \in \Gamma$ such that $U + a \supset a + V_1$, $a + U \supset V_2 + a$, $-U \supset V_3$.

2. If there exists $x \in G$ such that $x \in (U + a) \cap (V + b)$ or $x \in (a + U) \cap (V + b)$ or $x \in (a + U) \cap (b + V)$ respectively, where U, $V \in \Gamma$, a, $b \in G$ are arbitrary ele-

ments, then there exists a neighbourhood $W \in \Gamma$ with the property $W + x \subset \subset (U + a) \cap (V + b)$ or $W + x \subset (a + U) \cap (V + b)$ or $W + x \subset (a + U) \cap \cap (b + V)$ respectively.

1.3. If τ is an h-topology in a group G with a basis Γ about zero and if $M \subset G$ is a dense set in G, then the system $\Sigma = \{U + x : U \in \Gamma, x \in M\}$ is a basis of the topological space G with the topology τ.

Remark. By 1.3. an h-topology φ in the group G is uniquely defined by any complete system Γ of neighbourhoods of zero. We denote this fact by $\varphi = \tau(\Gamma)$.

1.4. Theorem. Let G be a group, Γ a system of subsets in G containing zero, and let Γ fulfil the next conditions:

1. The intersection of any two elements of Γ contains some element of Γ.
2. For any $U \in \Gamma$, $g \in G$ there exist $V_1, V_2, V_3 \in \Gamma$ such that $V_1 + g \subset g + U$, $g + V_2 \subset U + g$, $-V_3 \subset U$.
3. If there exists $x \in G$ such that $x \in (U + a) \cap (V + b)$ or $x \in (a + U) \cap (V + b)$ or $x \in (a + U) \cap (b + V)$ respectively, where $U, V \in \Gamma$, $a, b \in G$ are arbitrary elements, then a neighbourhood $W \in \Gamma$ exists such that $W + x \subset (U + a) \cap (V + b)$ or $W + x \subset (a + U) \cap (V + b)$ or $W + x \subset (a + U) \cap (b + V)$ respectively.

Then there exists precisely one h-topology in G in which Γ can be taken as a basis about zero.

Proof. If there exists some h-topology in G with a basis Γ about zero, then by 1.3. $\Sigma = \{U + g : g \in G, U \in \Gamma\}$ is its complete system of neighbourhoods. We shall show that Σ can be taken as a basis of some topology in G and the topology thus obtained is an h-topology in G uniquely defined by means of the complete system Γ of neighbourhoods of zero.

Let $x \in G$ be an element such that $x \in (U + a) \cap (V + b)$, where $U, V \in \Gamma$, $a, b \in G$. The system Γ fulfils the condition 3. and therefore there exists $W \in \Gamma$ such that $W + x \subset (U + a) \cap (V + b)$. Hence Σ is a basis of some topology φ in the set G. Now let us show the topology φ is an h-topology in G. Let $x, y \in G$ and let U be an open set in φ containing x. Then $U_1 \in \Gamma$ exists such that $U_1 + x \subset U$. It holds $U_1 + x = U_1 + y - y + x = V_1 - y + x \subset U$, where V_1 is a neighbourhood of y. Further $U_2 \in \Gamma$ exists with the property $U \supset U_1 + x \supset x + U_2 = x - y + + y + U_2 = x - y + V_2$, where V_2 is a neighbourhood of y. If U' is an open set in the topology φ containing $-x$, then $U'_1 \in \Gamma$ exists such that $U' \supset U'_1 - x$. Moreover, there exist elements $U'_2, U'_3 \in \Gamma$ such that $-U'_2 \subset U'_1$, $U'_3 + x \subset x + U'_2$. Altogether $U' \supset U'_1 - x \supset -U'_2 - x = -(x + U'_2) \supset -(U'_3 + x) = -V'$, where V' is a neighbourhood of x. As the previous inclusions are valid for every $x, y \in G$, the mappings $x + c$, $c + x$ and $-x$ are homeomorphisms of the topological space G (with the topology φ) into itself for any $c \in G$, and hence φ is an h-topology in G.

Finally, we shall prove that the system Γ is a basis about zero in φ. Let W be an open set in φ containing zero. According to properties of Σ there exist $U \in \Gamma$, $a \in G$ such that $0 \in U + a \subset W$, and by the condition 3. there exists $V \in \Gamma$ such that $V = V + 0 \subset U' \cap (U + a) \subset W$, where $U' \in \Gamma$ is an arbitrary element. Hence Γ is a complete system of neighbourhoods of zero in the h-topology φ in G. As the system Σ can be taken as a basis of the topology $\tau(\Gamma)$, then $\varphi = \tau(\Gamma)$, i.e. the h-topology $\tau(\Gamma)$ is unique.

Definition. Let τ_1, τ_2 be topologies in the set G. We say that τ_1 is stronger than τ_2 (τ_2 is weaker than τ_1) if the identical mapping $G_1 \to G_2$, where G_i is the set G with the topology τ_i, $i = 1, 2$, is continuous.

1.5. Let G be a group and let $\tau(\Gamma_1)$, $\tau(\Gamma_2)$ be h-topologies in G. Then the following statements are equivalent:

1. $\tau(\Gamma_1)$ is weaker than $\tau(\Gamma_2)$.
2. For every $U \in \Gamma_1$ there exists a neighbourhood $V \in \Gamma_2$ such that $V \subset U$.
3. If Γ_i^* indicates the system of all neighbourhoods of zero in $\tau(\Gamma_i)$, $i = 1, 2$, then $\Gamma_1^* \subset \Gamma_2^*$.

Proof follows from the properties of comparable topologies.

1.6. Theorem. If G is a group, then the set $\mathfrak{B}$ of all h-topologies in G is a complete lattice.

Proof. Let $A = \{\tau_i \in \mathfrak{B} : i \in I\}$ and let Γ_i^* be the system of all neighbourhoods of zero in τ_i, $i \in I$. Let us denote by Q the set $\{\bigcap_{i \in I} U^i : U^i \in \Gamma_i^*, \text{card}\,\{i \in I : U^i \neq G\} < \aleph_0\}$. Since 1.2. holds for every system Γ_i^*, $i \in I$, the conditions of Theorem 1.4. are valid for Q, hence $\tau(Q)$ is an h-topology in G. We shall prove that $\tau(Q)$ is the least upper bound of the set A in the system $\mathfrak{B}$. From the definition of Q it follows $\Gamma_i^* \subset Q$, $i \in I$, and by 1.5. $\tau_i \leqq \tau(Q)$, $i \in I$, holds. Let $\varphi \in \mathfrak{B}$ be an h-topology with the property $\tau_i \leqq \varphi$, $i \in I$, and let R be some basis about zero in φ with the property $R \supset \Gamma_i^*$, $i \in I$. Any finite intersection of elements of $\bigcup_{i \in I} \Gamma_i^* \subset Q$ contains some element from R, thus $\varphi \geqq \tau(Q)$ (see 1.5.). The system Q is defined uniquely, hence $\tau(Q)$ is the least upper bound of A in $\mathfrak{B}$. The topology $\tau(\{G\})$ is clearly the weakest element of $\mathfrak{B}$ in G, i.e. $\mathfrak{B}$ is a complete lattice.

Definition. Let G be a group, $\mathfrak{B}$ a system of all h-topologies in G, τ a topology in the set G. We say that a topology τ^h is *an upper h-modification of the topology* τ if τ^h is the weakest element of the set $C_\tau^h = \{\varphi \in \mathfrak{B} : \varphi \geqq \tau\}$. We call τ_h *a low h-modification of the topology* τ if τ_h is the strongest topology in the set $B_\tau^h = \{\varphi \in \mathfrak{B} : \varphi \leqq \tau\}$.

Definition. Let G be a group, τ a topology in the set G. Let $\Delta_{\tau,g}^*$ denote the system of all neighbourhoods of an element $g \in G$ and let $h \in G$ be an arbitrary element. We define:

$$M_1(\tau, g, h) = \{-h + U_g - g + h : U_g \in \Delta^*_{\tau, g}\},$$
$$M_2(\tau, g, h) = \{h - g + U_g - h : U_g \in \Delta^*_{\tau, g}\},$$
$$M_3(\tau, g, h) = \{-h - U_{-g} - g + h : U_{-g} \in \Delta^*_{\tau, -g}\},$$
$$M_4(\tau, g, h) = \{h - g - U_{-g} - h : U_{-g} \in \Delta^*_{\tau, -g}\}.$$

1.7. Lemma. Let G be a group, φ be an h-topology in G, τ be a topology in the set G. Let Γ^* be the system of all neighbourhoods of zero in the topology φ. It holds:

1. $$\varphi \leqq \tau \Leftrightarrow \Gamma^* \subset \bigcap_{g \in G} \bigcap_{h \in G} \bigcap_{j=1}^{4} M_j(\tau, g, h).$$

2. $$\varphi \geqq \tau \Leftrightarrow \Gamma^* \supset \bigcup_{g \in G} \bigcup_{h \in G} \bigcup_{j=1}^{4} M_j(\tau, g, h).$$

Proof. Let us denote by Γ_g (Δ_g) the system of all neighbourhoods of $g \in G$ in the topology $\varphi(\tau)$.

$\Rightarrow$: If $\varphi \leqq \tau$, then $\Gamma_g \subset \Delta_g$, $g \in G$. From 1.1. it follows for any $g \in G$:

$$\Gamma_g = \Gamma_0 + g \Rightarrow \Gamma_0 \subset \Delta_g - g,$$
$$\Gamma_{-g} = -\Gamma_g = -(\Gamma_0 + g) = -g - \Gamma_0 \Leftrightarrow -g - \Gamma_0 \subset \Delta_{-g} \Rightarrow$$
$$\Rightarrow \Gamma_0 \subset -\Delta_{-g} - g.$$

Similarly it holds $\Gamma_0 \subset -g + \Delta_g$, $\Gamma_0 \subset -g - \Delta_{-g}$.

Now let us choose a fixed element $g \in G$. From 1.1. it follows for every $h \in G$:

$$\Gamma_g = \Gamma_h - h + g \Rightarrow \Gamma_h \subset \Delta_g - g + h \Rightarrow \Gamma_0 \subset -h + \Delta_g - g + h =$$
$$= M_1(\tau, g, h),$$
$$-\Gamma_{-g} = \Gamma_g = \Gamma_h - h + g \Rightarrow \Gamma_h \subset -\Delta_{-g} - g + h \Rightarrow$$
$$\Rightarrow \Gamma_0 \subset -h - \Delta_{-g} - g + h = M_3(\tau, g, h).$$

Similarly it holds: $\Gamma_0 \subset h - g - \Delta_{-g} - h = M_4(\tau, g, h)$,
$\Gamma_0 \subset h - g + \Delta_g - h = M_2(\tau, g, h)$.

By the inclusions above it is clear that the first proposition of this theorem is valid. The second relation can be proved quite analogously.

$\Leftarrow$: If $\Gamma_0 \subset \bigcap_{g \in G} \bigcap_{h \in G} \bigcap_{j=1}^{4} M_j(\tau, g, h)$, then $\Gamma_0 \subset M_1(\tau, g, 0) = \{U - g : U \in \Delta_g\}$ for every $g \in G$. Hence $\Gamma_g \subset \Delta_g$, i.e. $\varphi \leqq \tau$. The proof of the second assertion is analogous.

1.8. Theorem. If G is a group and τ a topology in the set G, then the low h-modification τ_h of the topology τ exists and $\tau_h = \bigvee_{\mathfrak{M}} B^h_\tau = \bigvee_{\mathfrak{B}} B^h_\tau$ holds.

Proof. Let us denote by $E(\tau)$ the set $\bigcap_{g\in G}\bigcap_{h\in G}\bigcap_{j=1}^{4} M_j(\tau, g, h)$, and by $\Gamma^*_{\varphi,g}$ the system of all neighbourhoods of an element $g \in G$ in a topology φ. Let φ be an h-topology in G. Then by 1.7. the relation $\varphi \leqq \tau$ holds if and only if $\Gamma^*_{\varphi,0} \subset E(\tau)$. Instead of the system $\Gamma^*_{\varphi,0}$ in the last inclusion there may equivalently stay any complete system of neighbourhoods of zero in the topology φ.

Now let $B^h_\tau = \{\varphi_i \in \mathfrak{B} : i \in I\}$. Let us write for brevity Γ_i instead of $\Gamma^*_{\varphi i,0}$. By the proof of Theorem 1.6. the system $Q = \{\bigcap_{i\in I} U^i : U^i \in \Gamma_i, \text{card}\,\{i \in I : U^i \neq G\} < \aleph_0\}$ is a basis about zero in the h-topology $\tau(Q)$ in G and $\tau(Q)$ is the least upper bound in $\mathfrak{B}$ of the set B^h_τ. Hence $\tau_i \leqq \tau(Q)$, $i \in I$. Furthermore, $\Gamma_i \subset E(\tau)$ for any $i \in I$. As any finite intersection of elements of $M_j(\tau, g, h)$ belongs to the same set for arbitrary $g, h \in G$, $j \in \{1, 2, 3, 4\}$, then also $E(\tau)$ has this property. Hence $Q \subset E(\tau)$, i.e. $\tau(Q) \leqq \tau$. Thus it holds $\tau_h = \tau(Q) = \bigvee_{\mathfrak{B}} B^h_\tau$.

Finally, when we prove $\tau(Q) \leqq \sigma$ for any $\sigma \in \mathfrak{M}$ such that $\varphi_i \leqq \sigma$, $i \in I$, we shall show $\tau_h = \tau(Q) = \bigvee_{\mathfrak{B}} B^h_\tau = \bigvee_{\mathfrak{M}} B^h_\tau$. So let $\varphi_i \leqq \sigma$, $i \in I$. Then for arbitrary element $i \in I$, $g \in G$ there holds $\Gamma^*_{\sigma,g} \supset \Gamma^*_{\varphi i,g} = \Gamma^*_{\varphi i,0} + g$, and thus $\Gamma^*_{\sigma,g} \supset (\bigcup_{i\in I} \Gamma_i) + g$. Hence $\Gamma^*_{\sigma,g} \supset Q + g$ for every $g \in G$. However $Q + g$ is the complete system of neighbourhoods of the element g in the topology $\tau(Q)$, thus $\tau(Q) \leqq \sigma$.

1.9. Theorem. Let G be a group, τ be a topology in the set G. Then the upper h-modification τ^h of the topology τ exists and there holds $\tau^h = \bigwedge_{\mathfrak{M}} C^h_\tau = \bigwedge_{\mathfrak{B}} C^h_\tau$.

Proof. Let us denote by $D'(\tau)$ the system $\bigcup_{g\in G}\bigcup_{h\in G}\bigcup_{j=1}^{4} M_j(\tau, g, h)$, and by $\Gamma^*_{\varphi,g}$ the set of all neighbourhoods of an element $g \in G$ in the topology φ. Then for an h-topology $\varphi \in \mathfrak{B}$ the inequality $\varphi \geqq \tau$ holds if and only if $\Gamma^*_{\varphi,0} \supset D'(\tau)$. Thus the set C^h_τ is described.

Let $C^h_\tau = \{\varphi_i \in \mathfrak{B} : i \in I\}$ and let us write for brevity Γ_i instead of $\Gamma^*_{\varphi i,0}$. By the properties of the systems $\Gamma_i, i \in I$, the set $P = \bigcap_{i\in I} \Gamma_i$ fulfils the conditions of Theorem 1.4. Obviously, the topology $\tau(P)$ is the greatest low bound in $\mathfrak{B}$ of the topologies φ_i, $i \in I$. Namely, if $\tau(S) \in \mathfrak{B}$, $\tau(S) \leqq \varphi_i$ for any $i \in I$, then by 1.5. there holds $S \subset \subset \bigcap_{I\in i} \Gamma_i = P$, i.e. $\tau(S) \leqq \tau(P)$. Since the system P is determined uniquely, the assertion above holds. Moreover, P is the set of all neighbourhoods of zero in the topology $\tau(P)$. Since $\varphi_i \geqq \tau$, $i \in I$, then $\Gamma_i \supset D'(\tau)$, $i \in I$, and thus $P = \bigcap_{i\in I} \Gamma_i \supset D'(\tau)$, i.e. $\tau(P) \geqq \tau$. Hence $\tau^h = \tau(P) = \bigwedge_{\mathfrak{B}} C^h_\tau$.

Finally we prove that for any topology $\sigma \in \mathfrak{M}$ such that $\sigma \geqq \varphi_i$, $i \in I$, there holds $\sigma \leqq \tau(P)$. Let σ be a topology with those properties. Then $\Gamma^*_{\sigma,g} \subset \Gamma^*_{\varphi_i,g} = \Gamma^*_{\varphi_i,0} + + g = \Gamma_i + g$, where $i \in I, g \in G$ are arbitrary elements. Hence $\Gamma^*_{\sigma,g} \subset (\bigcap_{i\in I} \Gamma_i) + g =$

$= P + g$, where $g \in G$ is an arbitrary element. Since $P + g$ is a basis about g in the topology $\tau(P)$, then $\sigma \leqq \tau(P)$ and we have shown $\bigwedge_{\mathfrak{B}} C_\tau^h = \bigwedge_{\mathfrak{M}} C_\tau^h$.

1.10. Corollary. The complete lattice $\mathfrak{B}$ of all h-topologies in a group G is a closed sublattice in $\mathfrak{M}$.

Proof. Let $A \subset \mathfrak{B}$ be an arbitrary set. If we denote by B the set $\{\tau \in \mathfrak{B} : \tau \leqq \bigvee_{\mathfrak{M}} A\}$, then $A \subset B \subset \mathfrak{B}$ and $\bigvee_{\mathfrak{M}} B = \bigvee_{\mathfrak{M}} A$. Hence $\bigvee_{\mathfrak{B}} A \geqq \bigvee_{\mathfrak{M}} A = \bigvee_{\mathfrak{M}} B = \bigvee_{\mathfrak{B}} B \geqq \bigvee_{\mathfrak{B}} A$ (see 1.8.), i.e. $\bigvee_{\mathfrak{M}} A = \bigvee_{\mathfrak{B}} A$, thus $\mathfrak{B}$ is a closed upper sublasttice in $\mathfrak{M}$. In the same way (see 1.9.) we can prove that $\mathfrak{B}$ is a low sublattice in $\mathfrak{M}$, hence $\mathfrak{B}$ is a closed sublattice in $\mathfrak{M}$.

1.11. Let G be a group, τ a topology in the set G. Let us denote by $D(\tau)$ the least system fulfilling the conditions of Theorem 1.4. and containing the set $\bigcup_{g \in G} \bigcup_{h \in G} \bigcup_{j=1}^{4} M_j(\tau, g, h)$. Then the upper h-modification τ^h of the topology τ is an h-topology in G which is determined by the basis $D(\tau)$ about zero.

Proof. Making use of the notation in the proof of Theorem 1.9. P is the set of all neighbourhoods of zero in the h-topology $\tau(P) = \bigwedge_{\mathfrak{B}} C_\tau^h$, thus $P \supset D'(\tau)$. According to its definition $D(\tau)$ is a complete system of neighbourhoods of zero in some h-topology $\tau(D(\tau))$. Since $D(\tau) \supset D'(\tau)$, then $\tau(D(\tau)) \geqq \tau$ (see 1.7.) and $\tau(D(\tau)) \in C_\tau^h$. But $P \supset D(\tau)$, $\tau(P) = \bigwedge_{\mathfrak{B}} C_\tau^h$, thus $\tau(P) = \tau(D(\tau))$.

1.12. Theorem. Let G be a group, τ be a topology in the set G. If we denote by $E(\tau)$ the system $\bigcap_{g \in G} \bigcap_{h \in G} \bigcap_{j=1}^{4} M_j(\tau, g, h)$, then the low h-modification τ_h of the topolgy τ is an h-topology in G which is determined by the basis $E(\tau)$ about zero.

Proof. Making use of the notation in the proof of Theorem 1.9. the set Q is a basis about zero in the h-topology $\tau(Q) = \bigvee_{\mathfrak{B}} B_\tau^h$ in G and $Q \subset E(\tau)$ holds. We shall show that $E(\tau)$ satisfies the conditions of Theorem 1.4.

Let $U, V \in E(\tau)$ be arbitrary elements. Then $U, V \in M_j(\tau, g, h)$ for any $g, h \in G, j \in \{1, 2, 3, 4\}$. Let $g, h \in G$ be arbitrary fixed elements and let $j = 1$. Then $U = -h + U_g - g + h$, $V = -h + V_g - g + h$, where U_g, V_g belong to $\Gamma_{\tau,g}^*$ (see 1.9.). Evidently $-h + (U_g \cap V_g) - g + h = U \cap V \in M_1(\tau, g, h)$ and similarly for $j = 2, 3, 4$. Now let $U \in E(\tau)$, $x \in G$. Put $j = 1$ and denote by $E_j(\tau)$ the system $\bigcap_{g \in G} \bigcap_{h \in G} M_j(\tau, g, h)$. Obviously $U \in E_1(\tau)$ and for any $h, g \in G$ $U = -h + U_g - g + h$ holds. Thus $U + x = x - x - h + U_g - g + h + x = x - h' + U_g - g + h' = x + U$, where $h' = h + x$. Similarly for $j = 2, 3, 4$.

Further, we can express the element $U \in E(\tau)$ in the form of each the set $M_j(\tau, g, h)$, where $g, h \in G$, $j \in \{1, 2, 3, 4\}$. It holds $U = -h + U_g - g + h = -[(-h) - (-g) - U_{-(-g)} - (-h) = -U'$, where U' is an expression of U in the form of $M_4(\tau, -g, -h)$.

Let $U' \in E(\tau)$, $g \in G$ be arbitrary elements. Then $U' + g = g + U'$ are neighbourhoods of g in τ such that $-U' = U'$. Now let $U, V \in E(\tau)$, $a, b \in G$ be arbitrary elements. Hence $(U + a) \cap (V + b) = (a + U) \cap (V + b) = (a + U) \cap (b + V)$ are open sets in τ. Let $x \in (U + a) \cap (V + b)$. Then $W = (U + a - x) \cap \cap (V + b - x)$ is a neighbourhood of zero in τ. According to properties of $U, V \in \in E(\tau)$ it is $-W = W$ and, moreover, $W + g = g + W = g - W = -W + g$ are neighbourhoods of g in τ, where $g \in G$ is an arbitrary element. Therefore $W_g = = h + W - h + g = g - h + W + h = g - h - W + h = h - W - h + g$ is a neighbourhood of a fixed element $g \in G$ for every $h \in G$. Hence $W \in E(\tau)$ such that $W + x = (U + a) \cap (V + b)$. In the whole $E(\tau)$ fulfils the conditions of Theorem 1.4. and it is a basis about zero in some h-topology $\tau(E(\tau))$ in G.

Further, by 1.7. $\tau(E(\tau)) \leqq \tau$, i.e. $\tau(E(\tau)) \in B_\tau^h$. Since $Q \subset E(\tau)$ and $\tau(Q) = \bigvee_{\mathfrak{B}} B_\tau^h$, then $\tau(Q) = \tau(E(\tau))$. Thus the low h-modification τ_h of the topology τ is the h-topology in G determined by the basis $E(\tau)$ about zero. $E(\tau)$ is evidently the system of all neighbourhoods of zero in τ_h.

2.

Let G be a group, τ a topology in the set G. Let us denote by $\mathfrak{G}$ the set of all topologies of topological groups G. Let a topology $\varphi \in \mathfrak{G}$ be called a *g-topology*. Evidently, every g-topology in G is an h-topology simultaneously, i.e. $\mathfrak{G} \subset \mathfrak{B}$.

Now we define the *g-modifications of the topology* τ: A topology τ^g (τ_g) is called *the upper* (*low*) *g-modification* of the topology τ if it is the weakest (strongest) element in the system of all g-topologies which are stronger (weaker) than the given topology τ.

Let x denote a topological property, X the set of all g-topologies with the property x in G. Let us denote by C_τ^x (B_τ^x) the set $\{\varphi \in X : \varphi \geqq \tau\}$ $(\{\varphi \in X : \varphi \leqq \tau\})$, and by $\tau^x(\tau_x)$ the weakest (strongest) element of the set $C_\tau^x(B_\tau^x)$ provided that they exist.

Further, let us denote by $inf_x C_\tau^x$ the strongest element of the set $J_\tau^x = \{\varphi \in X : : \varphi \leqq \psi \textit{ for any } \psi \in C_\tau^x\}$, and by $sup_x B_\tau^x$ the weakest element of the set $L_\tau^x = \{\varphi \in X : : \varphi \geqq \psi \textit{ for any } \psi \in B_\tau^x\}$, provided that $J_\tau^x \neq \Phi$, $L_\tau^x \neq \Phi$ and provided that those elements exist.

The following assertions are valid:

2.1. If there exists one of topologies τ^x, $(\tau^h)^x$ or τ_x, $(\tau_h)_x$ respectively, then the second topology exists and it holds $\tau^x = (\tau^h)^x$ or $\tau_x = (\tau_h)_x$ respectively. Especially, $\tau_g = (\tau_h)_g$ (these topologies exist in any case), and if there exists either τ^g or $(\tau^h)^g$, then also the second topology exists and $\tau^g = (\tau^h)^g$.

2.2. Following statements are equivalent:

1. τ^x exists.
2. $inf_x\, C_\tau^x \geqq \tau$.
3. $inf_x\, C_\tau^x = \bigwedge_{\mathfrak{B}} C_\tau^x$.

The dual assertion holds for the topology τ_x.

Remark. Obviously it holds $inf_x\, C_\tau^x = \bigwedge_x C_\tau^x (sup_x\, B_\tau^x = \bigvee_x B_\tau^x)$, if X forms a low (upper) lattice.

2.3. The topology τ^x (τ_x) exists for any $\tau \in \mathfrak{M}$ for which $C_\tau^x \neq \Phi$ $(B_\tau^x \neq \Phi)$, if and only if the system X forms a closed low (upper) sublattice in $\mathfrak{B}$.

So, the describing of properties of a basis about zero in an h-topology in G enables us to investigate on what conditions the g-modifications of a given topology in the set G would be connected, compact, Hausdorff, etc.

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