Miloš Háčik

Higher monotonicity properties of zero points of the linear combination of the solution and its first derivative of the differential equation $y'' + q(t)y = 0$

Archivum Mathematicum, Vol. 10 (1974), No. 2, 67--77

Persistent URL: http://dml.cz/dmlcz/104819

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS,
X: 67—78, 1974

HIGHER MONOTONICITY PROPERTIES OF ZERO POINTS OF THE LINEAR COMBINATION OF THE SOLUTION AND ITS FIRST DERIVATIVE OF THE DIFFERENTIAL EQUATION $y'' + q(t)y = 0$

MILOŠ HÁČIK, Žilina

(Received June 25, 1973)

In paper [1] there has been deduced a simple, sufficient condition for the monotonicity of order n in the sequence of differences of zero points of linear combination of any solution and its first derivative of the differential equation

$$(q) \qquad y'' + q(t)y = 0$$

in the interval $I = (a, b)$, where $a < b$, $a \in E_1$, $b \in E_1^* = E_1 \cup (\infty)$.

In paper [4] J. Vosmanský has investigated the higher monotonicity properties of zero points and extremants of (q). By "*extremant*" of the function $y(t) \in C_2(I)$ we understand any number $\bar{t} \in I$ in which the function $y(t)$ acquires an locally extreme value (only proper local extremes are considered).

In this contribution there will be extended some results from paper [4] to the linear combination of the solution and its first derivative of (q).

1. In paper [2] M. Laitoch has deduced that if $y(t)$ is a solution of (q), where $q(t) \in C_2(I)$, $q(t) > 0$ for any $\bar{t} \in I$, then the function

$$(1) \qquad Y(t) = \frac{\alpha y + \beta y'}{\sqrt{\alpha^2 + \beta^2 q}},$$

where α, β are real numbers with the property $\alpha^2 + \beta^2 > 0$, is a solution of the differential equation

$$(Q) \qquad Y'' + Q(t)\,Y = 0,$$

where

$$(2) \qquad Q(t) = q + \frac{\alpha\beta q'}{\alpha^2 + \beta^2 q} + \frac{1}{2}\frac{\beta^2 q''}{\alpha^2 + \beta^2 q} - \frac{3}{4}\frac{\beta^4 q'^2}{(\alpha^2 + \beta^2 q)^2}$$

and conversely, if $\bar{Y}(t)$ is any solution of (Q), then there exists a solution $\bar{y}(t)$ of (q) such that

$$\frac{\alpha\bar{y} + \beta\bar{y}'}{\sqrt{\alpha^2 + \beta^2 q}} = \bar{Y}(t).$$

Definition 1: *The function $F(t)$ is said to be of the class $M_n(a, b)$ or monotonic of the order n in (a, b) if it has n $(\geqq 0)$ of continuous derivatives $F^{(0)}, F', F'', \ldots, F^{(n)}$ satisfying*

$$(3) \qquad (-1)^j F^{(j)}(t) \geqq 0 \quad j = 0, 1, 2, \ldots, n.$$

If the preceding inequalities are fulfilled for $j = 0, 1, 2, \ldots$, then the function $F(t)$ is called a completely monotonic in (a, b) and is denoted by $F(t) \in M_\infty\ (a, b)$.

Definition 2: *The function $F(t)$ is called the function of class $M_{n,m}(T_0, \infty)$ if there is $F(t) \in M_n(T_0, \infty)$ and $F(t)$ has for $t > T_0$ m derivatives for which there holds*

$$(4) \qquad F^{(i)}(t) \to 0 \text{ for } t \to \infty \qquad i = 0, 1, 2, \ldots, m.$$

(Evidently $F(t) \in M_{n,0}(T_0, \infty)$ implies $F(t) \in M_{n,n-1}(T_0, \infty)$ for $n \geqq 1$).

Lemma 1: *Let $q(t)$ possess a derivative $q'(t) \in M_{n+1}(0, \infty)$ $n \geqq 1$, $q(t) > 0$ for $t \in (0, \infty)$, $q(\infty) < \infty$ and let $Q(t)$ be defined by the formula (2). Let $\alpha^2 + \beta^2 > 0$, $\alpha\beta \leqq 0$.*
Then

$$(5) \qquad Q'(t) \in M_{n-1}(0, \infty)$$

$$(6) \qquad [q(t) - Q(t)] \in M_{n,0}(0, \infty)$$

$$(7) \qquad 0 \leqq \int^{\infty} [q(t) - Q(t)]\, \mathrm{d}t < \infty$$

where equality in (7) can hold only if $\beta = 0$ or $q(t) =$ const.

Proof: Relation (5) is proved in Lemma 2 of paper [1]; from the proof of that lemma there also holds that $[q(t) - Q(t)] \in M_n(0, \infty)$. Since under assumption $q(\infty) = c < \infty$, so $q'(\infty) = 0$ and because $q'(t) \in M_{n+1}(0, \infty)$, it is evident that $q''(\infty) = 0$. Hence $[q(t) - Q(t)] \in M_{n,0}(0, \infty)$.
Because of identically holding

$$\int \frac{q''}{\alpha^2 + \beta^2 q}\, \mathrm{d}t = \frac{q'}{\alpha^2 + \beta^2 q} + \int \frac{\beta^2 q'^2}{(\alpha^2 + \beta^2 q)^2}\, \mathrm{d}t,$$

we have for $\beta \neq 0$

$$\int^{\infty} (q - Q)\, \mathrm{d}t = \int^{\infty} \left(\frac{3}{4} \frac{\beta^4 q'^2}{(\alpha^2 + \beta^2 q)^2} - \frac{1}{2} \frac{\beta^2 q''}{\alpha^2 + \beta^2 q} - \frac{\alpha\beta q'}{\alpha^2 + \beta^2 q} \right) \mathrm{d}t =$$

$$= \int^{\infty} \frac{3}{4} \left(\frac{\beta^2 q'}{\alpha^2 + \beta^2 q} \right)^2 \mathrm{d}t - \frac{1}{2} \left[\frac{\beta^2 q'}{\alpha^2 + \beta^2 q} \right]^{\infty} - \frac{1}{2} \int^{\infty} \left(\frac{\beta^2 q'}{\alpha^2 + \beta^2 q} \right)^2 \mathrm{d}t - \int^{\infty} \frac{\alpha\beta q'}{\alpha^2 + \beta^2 q}\, \mathrm{d}t =$$

$$= \frac{1}{4} \int^{\infty} \left(\frac{\beta^2 q'}{\alpha^2 + \beta^2 q} \right)^2 \mathrm{d}t - \frac{\alpha}{\beta} \int^{\infty} \frac{\beta^2 q'}{\alpha^2 + \beta^2 q}\, \mathrm{d}t - \frac{1}{2} \left[\frac{\beta^2 q'}{\alpha^2 + \beta^2 q} \right]^{\infty}.$$

It is evident that $\frac{\beta^2 q'}{\alpha^2 + \beta^2 q} \to 0$ for $t \to \infty$ and for sufficiently large t there holds

$$0 < \int_t^\infty \left(\frac{\beta^2 q'}{\alpha^2 + \beta^2 q} \right)^2 \mathrm{d}t < \int_t^\infty \frac{\beta^2 q'}{\alpha^2 + \beta^2 q} \, \mathrm{d}t = [\log(\alpha^2 + \beta^2 q)]_t^\infty < \infty.$$

From here there follows the validity of (7).

In the case $\beta = 0$ the assertion (7) is evident. If $q(t) = \text{const.}$, then $q(t) = Q(t)$ and the equality in (7) holds as well.

Lemma 2: *Let $q(t) > 0$ for $t \in (0, \infty)$, $q(\infty) \in M_{n+1}(0, \infty)$, $n \geqq 1$ and let $Q(t)$ be defined by formula (2). Let for constants α, β there hold $\alpha^2 + \beta^2 > 0$, $\alpha\beta \leqq 0$. Let $q'(t) \neq 0$ for any $t \in (0, \infty)$. Then there holds*

$$(8) \qquad (-1)^i Q^{(i+1)}(t) \geqq (-1)^i q^{(i+1)}(t) > 0 \qquad i = 0, 1, 2, \ldots, n-1,$$

where equality holds only if $\beta = 0$.

Proof: At first let $\beta \neq 0$ hold. As the assumptions of Lemma 1 are fulfilled, so there holds: $Q'(t) \in M_{n-1}(0, \infty)$ and $[q(t) - Q(t)] \in M_{n,0}(0, \infty)$ as well. Therefore if we choose in Lemma 0,3 of paper [4] pg. 40 $f(t) = q(t) - Q(t)$ and $c = 0$, we obtain the validity of the following inequality

$$(-1)^{i+1} [q(t) - Q(t)]^{(i+1)} > 0 \qquad \textit{for} \qquad i = 0, 1, 2, \ldots, n-1,$$

hence we have

$$(-1)^{i+1} q^{(i+1)}(t) > (-1)^{(i+1)} Q^{(i+1)}(t),$$

what is equivalent to the inequality

$$(-1)^i Q^{(i+1)}(t) > (-1)^i q^{(i+1)}(t).$$

Inequality $(-1)^i q^{(i+1)}(t) > 0$ follows directly from assumptions putting on the coefficient $q(t)$ of (q).

In the case $\beta = 0$, we have $q(t) \equiv Q(t)$ and equality holds in (8).

Thus, lemma is proved.

Lemma 3: *Let $q(t) > 0$ for any $t \in (a, b)$, $a < b$ $a, b \in \bar{E}_1 = E_1 \cup (-\infty) \cup (\infty)$. Let α, $\beta (\neq 0)$ be real constants. Then between every two neighbouring zero points of arbitrary non-trivial solution $y(t)$ of (q) there lies only one zero point of the function $\alpha y + \beta y'$.*

Proof: Let $t_1, t_2 \in (a, b)$ be two neighbouring zero points of a solution $y(t)$ of (q), i.e. $y(t_1) = 0$, $y(t_2) = 0$. It is evident that $\alpha y(t_1) + \beta y'(t_2) \neq 0$ and $\alpha y(t_2) + \beta y'(t_2) \neq$

$\neq 0$ as well. Suppose that the function $\alpha y + \beta y'$ has no zero point in the interval (t_1, t_2). We can easy derive a relation

$$\left(\frac{y}{\alpha y + \beta y'}\right)' = \frac{\beta(qy^2 + y'^2)}{(\alpha y + \beta y')^2} \tag{9}$$

which holds for any t fulfilling $\alpha y + \beta y' \neq 0$. After integrating (9) from t_1 to t_2 we obtain zero on the left-hand side and with regard to $q(t) > 0$ we have a number different from zero on the righthand side which is a contradiction. Thus, we have found that between two neighbouring zero points t_1, t_2 of any non-trivial solution $y(t)$ of (q) there lies at least one zero point of the function $\alpha y + \beta y'$. If two zero points lie between t_1 and t_2 — denote them by τ and $\bar{\tau}$ — we can easily prove in a similar way as above, considering the relation

$$\left(\frac{\alpha y + \beta y'}{y}\right)' = \frac{-\beta(y'^2 + qy^2)}{y^2}$$

that between τ, $\bar{\tau}$ there lies at least one zero point $\bar{t}$ of the solution $y(t)$. Hence we have

$$t_1 < \tau < \bar{t} < \bar{\tau} < t_2$$

which is impossible because t_1 and t_2 are by assumption two neighbouring zero points of $y(t)$. Thus, the proof of the lemma is complete.

Definition 3: *Let $\{t_k\}$ denote the sequence and $\Delta^n t_k$ an n-th difference of the sequence $\{t_k\}$ so that*

$$\Delta^0 t_k = t_k,\ \Delta t_k = t_{k+1} - t_k, \ldots, \Delta^n t_k = \Delta^{n-1} t_{k+1} - \Delta^{n-1} t_k$$
$$k = 0, 1, 2, \ldots\ n = 1, 2, \ldots.$$

The sequence $\{t_k\}$ is called monotonic of order n if

$$(-1)^j \Delta^j t_k \geqq 0 \qquad k = 0, 1, 2, \ldots, j = 1, 2, \ldots, n.$$

If $n = \infty$, then the sequence $\{t_k\}$ is called a completely monotonic one.

Theorem 1: *Let the function $q(t)$ possess a derivative $q'(t) \in M_{n+1}$ $(0, \infty)$, $n \geqq 2$, $q(t) > 0$, $q'(t) \neq 0$ for $t \in (0, \infty)$ and $q(\infty) < \infty$. Let $\{t_k\}$ denote the sequence of zeros of any solution $y(t)$ of (q) and $\{T_k\}$ the sequence of zeros of the linear combination $\alpha y + \beta y'$ of the same solution, whereby for constants α, β there holds $\alpha^2 + \beta^2 > 0$, $\alpha\beta \leqq 0$. If there is $t_0 > T_0 > 0$, then there holds*

$$(-1)^i \Delta^i (t_k - T_k) > 0 \qquad i = 0, 1, 2, \ldots, n\ k = 0, 1, 2, \ldots. \tag{10}$$

Proof: Let the function $Q(t)$ be defined by the formula (2). Because $q'(\infty) = q''(\infty) = 0$, so it holds $q(\infty) = Q(\infty)$. By Lemma 1 and Lemma 2 $[q(t) - Q(t)] \in M_{n,0}$ $(0, \infty)$, $Q'(t) \in M_{n-1}$ $(0, \infty)$ and $q(t) \neq Q(t)$ holds for $t \in (0, \infty)$. The conditions of Theorem 4; 1 and Theorem 4; 2 of paper [4], where n is replaced by $n - 1$, are therefore fulfilled. From the introductory remarks and from the formula (1) it is evident that the sequence $\{T_k\}$ is a sequence of zeros of a suitable solution of (Q) and conversely. Therefore because of Theorem 4; 2 and Remark 4; 1 of paper [4] pg. 57–58 and by using (7) we obtain that for any mumber T_0 there exists a number t_0 (sufficiently large) such that (10) holds and $(t_k - T_k) \to \gamma$ as $k \to \infty$, where $\gamma(\geqq 0)$ is constant. The number t_0 is given by the relation (4; 24) of paper [4].

Since $q(t) > 0$, by Lemma 3, we get that between any pair of neighbouring zeros of arbitrary non-trivial solution $y(t)$ of (q) there lies the only zero point of the function $\alpha y + \beta y'$ and therefore the constant $\gamma > 0$. If $\{t_k\}$ (resp. $\{T_k\}$) denotes the sequence of zeros of $y(t)$ (resp. the sequence of zeros of the function $\alpha y + \beta y'$) where $y(t)$ is the above mentioned solution of (q), then $t_k > T_k$ for any k. Therefore the relation (10) holds and Theorem 1 is proved.

Remark 1: Let assumptions of Theorem 1 be fulfilled. Let $\{\bar{t}_k\}$ denote the sequence of zeros of either the same solution $y(t)$ of (q), for which the linear combination $\alpha y + \beta y'$ has the sequence of zeros $\{T_k\}$, or the other solution $\bar{y}(t)$ of the same differential equation (q). From Remark 5; 3 of paper [4] it follows that if t_0 is greater than or equal to a solution of equation (4; 11) of paper [4] (which is also designated by t_0), then there holds

$$(\overline{10}) \qquad (-1)^i \Delta^i (\bar{t}_k - T_k) > 0 \qquad i = 0, 1, 2, \ldots, n \;\; k = 0, 1, 2, \ldots.$$

However, the relation $(\overline{10})$ holds if t_0 is greater than or equal to the first zero of the solution $y(t)$, but this zero point must be greater than the zero point T_0 of $\alpha y + \beta y'$.

Lemma 4: *Let* $q_\lambda(t) > 0$, $q'_\lambda(t) \in M_{n+1}(0, \infty)$, $n \geqq 1$, $\lambda = 1, 2$ *and let* $[q_1(t) - q_2(t)] \in M_{n+2}(0, \infty)$. *Let further* $\alpha\beta \leqq 0$ *and also*

$$(11) \qquad Q_\lambda(t) = q_\lambda + \frac{\alpha\beta q'_\lambda}{\alpha^2 + \beta^2 q_\lambda} + \frac{1}{2} \frac{\beta^2 q''_\lambda}{\alpha^2 + \beta^2 q_\lambda} - \frac{3}{4} \frac{\beta^4 q'^2_\lambda}{(\alpha^2 + \beta^2 q_\lambda)^2}.$$

Then the function $[Q_1(t) - Q_2(t)]$ *is of the class* $M_n(0, \infty)$.

Proof: According to the supposition of Lemma 4 it holds for $t > 0$ and $i = 0, 1, \ldots, n + 1$

$$(12) \qquad q_1(t) \leqq q_2(t)$$

and

$$(13) \qquad 0 \leqq (-1)^i q_1^{(i+1)}(t) \leqq (-1)^i q_2^{(i+1)}(t)$$

thus, there holds $|q_1^{(i+1)}(t)| \leqq |q_2^{(i+1)}(t)|$. Hence it follows that

$$(14) \qquad \left| \frac{q_1^{(i+1)}(t)}{\alpha^2 + \beta^2 q_1} \right| \leqq \left| \frac{q_2^{(i+1)}(t)}{\alpha^2 + \beta^2 q_2} \right| \qquad i = 0, 1, \ldots, n + 1.$$

According to Lemma 2 of paper [5] the relation

$$(15)\qquad \left|\left[\frac{q_\lambda^{(i)}}{\alpha^2+\beta^2 q_\lambda}\right]^{(p)}\right| = \sum_{k=1}^{N}\left|C_k \frac{q_\lambda^{(\nu_1,k)}}{\alpha^2+\beta^2 q_\lambda}\cdots\frac{q_\lambda^{(\nu_1,k)}}{\alpha^2+\beta^2 q^\lambda}\right|,$$

holds for $p + i < n + 2$, where C_k, N, l, $\nu_{j,k}$ are suitable numbers, whose detailed specification is not necessary in our case. From term-by-term comparison of the expression (15) for the functions $q_1(t)$, $q_2(t)$ because of (14) there follows the inequality

$$(16)\qquad \left|\left(\frac{q_1^{(i)}}{\alpha^2+\beta^2 q_1}\right)^{(p)}\right| \leqq \left|\left(\frac{q_2^{(i)}}{\alpha^2+\beta^2 q_2}\right)^{(p)}\right|.$$

Furthermore, we have the identity

$$(17)\left\{\left(\frac{q_\lambda}{\alpha^2+\beta^2 q_\lambda}\right)'\frac{q'_\lambda}{\alpha^2+\beta^2 q_\lambda}\right\}^{(p)} = \sum_{i=0}^{p}\binom{p}{i}\left(\frac{q'_\lambda}{\alpha^2+\beta^2 q_\lambda}\right)^{(i+1)}\cdot\left(\frac{q'_\lambda}{\alpha^2+\beta^2 q_\lambda}\right)^{(p-i)}.$$

From lemma 2 of paper [5] it follows that $\dfrac{q'_\lambda}{\alpha^2+\beta^2 q_\lambda} \in M_{n+1}(0, \infty)$ and because of this all members of the sum on the right-hand side of (17) have the same sign, namely $(-1)^{p+1}$. Then because of (16) there holds

$$(18)\quad \left|\left\{\left(\frac{q'_1}{\alpha^2+\beta^2 q_1}\right)'\frac{q'_1}{\alpha^2+{}^2\beta\, q_1}\right\}^{(p)}\right| \leqq \left|\left\{\left(\frac{q'_2}{\alpha^2+\beta^2 q_2}\right)'\frac{q'_2}{\alpha^2+\beta^2 q_2}\right\}^{(p)}\right|.$$

For the $(i + 1)$-th derivative $i = 0, 1, 2, \ldots, n - 1$ of the function $Q_\lambda(t)$ we obtain

$$Q_\lambda^{(i+1)} = q^{(i+1)} + \alpha\beta\left(\frac{q'_\lambda}{\alpha^2+\beta^2 q_\lambda}\right)^{(i+1)} - \frac{3}{2}\beta^4\left\{\left(\frac{q'_\lambda}{\alpha^2+\beta^2 q_\lambda}\right)'\frac{q'_\lambda}{\alpha^2+\beta^2 q_\lambda}\right\}^{(i)} +$$

$$(19)\qquad + \frac{1}{2}\beta^2\left\{\frac{q''_\lambda}{\alpha^2+\beta^2 q_\lambda}\right\}^{(i+1)}.$$

Using term-by-term comparison of the right sides of (19) for $\lambda = 1, 2$ and considering that $\alpha\beta \leqq 0$ we get because of (16) and (18)

$$(20)\qquad |Q_1^{(i+1)}(t)| \leqq |Q_2^{(i+1)}(t)| \qquad i = 0, 1, 2, \ldots, n-1.$$

Because by lemma 1, $Q'_\lambda(t) \in M_{n-1}(0, \infty)$, we obtain from (20) the inequalities

$$(21)\qquad 0 \leqq (-1)^i Q_1^{(i+1)}(t) \leqq (-1)^i Q_2^{(i+1)}(t) \qquad i = 0, 1, \ldots, n-1.$$

The inequality $0 \leqq Q_2(t) \leqq Q_1(t)$ follows with regard to (14) directly, using term-by-term comparison [11] for $\lambda = 1, 2$ and Lemma 2 of paper [3]. Thus the proof is finished.

Lemma 5: *Let the conditions of Lemma 4 be satisfied and let $q_1(t) \neq q_2(t)$ for all $t \in (0, \infty)$. Then there holds*

$$(22) \qquad (-1)^i Q_1^{(i)}(t) > (-1)^i Q_2^{(i)}(t) \qquad i = 0, 1, \dots, n; \; t \in (0, \infty).$$

Proof: Since $q_1'(t) < q_2'(t)$ for $t \in (0, \infty)$, because of the conditions of Lemma 5, we have $q_1'(t) < q_2'(t)$ and by lemma 0; 3 of paper [4] also

$$(23) \qquad (-1)^i q_1^{(i)}(t) > (-1)^i q_2^{(i)}(t) \geqq 0$$

for $i = 0, 1, \dots, n + 1$, $t \in (0, \infty)$, and therefore the non-strict inequalities in (14), (16), (18), (20) and non-strict inequalities on the right side of (21) can be replaced by strict ones with the possible exception of the inequalities for the highest derivatives. From (19) it may be seen, however, that in (20) and in the right side of (21) the sharp inequality also remains for $i = n - 1$; thus (22) holds for $i = 1, 2, \dots, n$. The validity of (22) for $i = 0$ may be seen directly from (11) by means of the relations (14).

Theorem 2: *Let the function $q_1(t)$, resp. $q_2(t)$ possess a derivative $q_1'(t) \in M_{n+1}(0, \infty)$, resp. $q_2'(t) \in M_n(0, \infty)$ $n \geqq 3$ and further let $[q_1(t) - q_2(t)] \in M_{n+1}(0, \infty)$, $q_1(t) >$ $> q_2(t) \geqq 0$ hold for $t \in (0, \infty)$ and $q_1(\infty) = q_2(\infty) < \infty$.*

Let $\{t_{\lambda,k}\}$ for $\lambda = 1, 2$ denote the sequence of zeros of any solution $y_\lambda(t)$ of the differential equation

$$(q_\lambda) \qquad y_\lambda'' + q_\lambda(t)\, y = 0$$

and $\{T_{\lambda,k}\}$ denote the sequence of zeros of the function $\alpha\bar{y} + \beta\bar{y}'$, where $\bar{y}_\lambda(t)$ is either the solution $y_\lambda(t)$ of (q_λ) or any other solution of (q_λ) and α, β are constants with the properties $\alpha^2 + \beta^2 > 0$, $\alpha\beta \leqq 0$.

Then for any fixed choice of $T_{2,0} > 0$ and any numbers $\gamma \in \langle 0, \infty)$, $\delta \in \langle 0, \infty)$ there exist numbers $T_{1,0}$, $t_{1,0}$ sufficiently large so that

$$(24) \qquad (-1)^i \Delta_i(T_{1,k} - T_{2,k}) > 0 \qquad i = 0, 1, \dots, n - 2, k = 0, 1, \dots.$$

$$(-1)^{n-1} \Delta^{n-1}(T_{1,k} - T_{2,k}) \geqq 0,$$

$$(25) \qquad (T_{1,k} - T_{2,k}) \to \gamma \qquad \textit{for} \qquad k \to \infty$$

and furthermore

$$(26) \qquad (-1)^i \Delta^i(t_{1,k} - T_{2,k}) > 0 \qquad i = 0, 1, \dots, n - 1, k = 0, 1, \dots$$

$$(27) \qquad (t_{1,k} - T_{2,k}) \to \delta \qquad \textit{for} \qquad k \to \infty$$

if and only if the integral $\int^{\infty} [q_1(t) - q_2(t)]\, dt$ converges.

Proof: Let the functions $Q_1(t)$, $Q_2(t)$ be defined by the formula (11). According to the hypotheses $q_1(\infty) = q_2(\infty) < \infty$. Therefore $q_\lambda'(\infty) = q_\lambda''(\infty) = 0$, $\lambda = 1, 2$

and $q_1(\infty) = q_2(\infty) = Q_1(\infty) = Q_2(\infty)$. By Lemma 1 we have $Q_1'(t) \in M_{n-1}(0, \infty)$, $Q_2'(t) \in M_{n-2}(0, \infty)$. By Lemmas 1 and 4 there holds for $t \in (0, \infty)$ and $i = 0, 1, \ldots, n-1$

$$(28) \qquad (-1)^i Q_1^{(i)}(t) > (-1)^i Q_2^{(i)}(t).$$

The conditions of Theorems 4; 1 and 4; 2 of [4], where Q is changed to Q_2, q to Q and n to $n-2$ are therefore satisfied. The integral $\int^{\infty} [q_\lambda(t) - Q_\lambda(t)]\,dt$ converges by Lemma 1. Furthermore the integral $\int^{\infty} [Q_1(t) - Q_2(t)]\,dt$ converges if and only if the integral $\int^{\infty} [q_1(t) - q_2(t)]\,dt$ converges. From Theorems 4; 1 and 4; 2 of paper [4], the assertions (24) and (25) follow. The relations (26) and (27) follow also from Theorems 4; 1 and 4; 2 and Remark 4; 1 of paper [4], where q is changed to q_1 and Q to Q_2. The integral $\int^{\infty} [q_1(t) - Q_2(t)]\,dt$ converges because of Lemma 1 if and only if the integral $\int^{\infty} [q_1(t) - q_2(t)]\,dt$ does so.

Remark 2: If in the assumptions of Theorem 2 there holds $q_1'(t) \in M_{n+2}(0, \infty)$, then (24) holds in the form

$$(29) \qquad (-1)^i (T_{1,k} - T_{2,k}) > 0 \qquad i = 0, 1, 2, \ldots, n-1 \; k = 0, 1, \ldots,$$

this follows from Remark 4; 1 of paper [4] because in this case $Q_1'(t) \in M_n(0, \infty)$.

Remark 3: If $\gamma > 0$, $\delta > 0$, then because of Theorem 4; 2 of [4] the identity $q_1(t) \equiv q_2(t)$ for $t \in (a, \infty)$ a > 0 can occurr provided $q_1'(t) = 0$ for $t \in (0, \infty)$.

2. In this part we give an application of the preceding results to the Bessel differential equation

$$(30_\nu) \qquad y'' + \left\{1 - \frac{\nu^2 - \frac{1}{4}}{t^2}\right\} y = 0, \qquad t > 0,$$

the fundamental system of solutions of which is formed by the functions $\left(\frac{1}{2}\pi t\right)^{\frac{1}{2}} J_\nu(t)$, $\left(\frac{1}{2}\pi t\right)^{\frac{1}{2}} Y_\nu(t)$.

Let $\{t_{\nu,k}\}$ denote a sequence of zeros of any non-trivial solution $y_\nu(t)$ of (30_ν) and let $\{T_{\nu,k}\}$ denote a sequence of zeros of the function $\alpha \tilde{y}_\nu + \beta \tilde{y}_\nu$, where $\tilde{y}_\nu(t)$ is either the above-mentioned solution $y_\nu(t)$ or any other non-trivial solution of (30_ν), where α, β are real constants with property $\alpha^2 + \beta^2 > 0$.

Let $\{\bar{t}_{\nu,k}\}$, resp. $\{\bar{T}_{\nu,k}\}$ denote a sequence of zeros of the solution $\bar{y}_\nu(t)$ of (30_ν), resp. zeros of the function $\alpha \bar{y}_\nu + \beta \bar{y}_\nu$, $\alpha^2 + \beta^2 > 0$, whereby $\bar{y}_\nu(t)$ may but need not be identical with the above-mentioned solution $y_\nu(t)$.

Lee Lorch and P. Szego showed (see [3] pg. 63, 72) that if $t_{\nu,0} < \bar{t}_{\nu,0}$, then for $|\nu| > \frac{1}{2}$ the sequence $\{\bar{t}_{\nu,k} - t_{\nu,k}\}$ is completely monotonic and there holds

(31) $$(-1)^i \Delta^i(\bar{t}_{\nu,k} - t_{\nu,k}) > 0 \qquad i, k = 0, 1, 2, \ldots .$$

In paper [5] J. Vosmanský showed that for the sequences of differences of extremants there holds also a relation analogous to (31).

According to the assertion of the corollary of Theorem of paper [1] we obtain directly that in case $\alpha\beta \leqq 0$ there holds also

(32) $$(-1)^i \Delta^i(\bar{T}_{\nu,k} - T_{\nu,k}) > 0 \qquad i = 0, 1, 2, \ldots$$

In paper [3] it is further shown that for $\nu > \mu > \frac{1}{2}$ and any fixed sequences $\{t_{\mu,k}\}$, $\{t_{\nu,k}\}$ there exists an integer r such that

(33) $$(-1)^i \Delta^i(t_{\mu,k+r} - t_{\nu,k}) > 0 \qquad i, k = 0, 1, 2, \ldots$$

In paper [4] pg. 66 it is shown that the relation (33) follows directly from Theorem 4; 1 of paper [4] because in case $\nu > \mu \geqq \frac{1}{2}$ there holds

(34) $$q_\mu(t) - q_\nu(t) = (\nu^2 - \mu^2)\frac{1}{t^2} \in M_\infty(0, \infty); \qquad 0 < \int^{\infty} [q_\mu(t) - q_\nu(t)]\, \mathrm{d}t < \infty.$$

The following theorem includes all preceding results and also Theorem 6;1 of paper [4] as special cases.

Theorem 3: *Let ν, μ be any real numbers satisfying $\nu > \mu \geqq \frac{1}{2}$. Let $\{t_{\nu,k}\}$, resp. $\{t_{\mu,k}\}$ denote a sequence of zeros of any solution $y_\nu(t)$, resp. $y_\mu(t)$ of the differential equation (30_ν), resp. (30_μ). Let $\{\bar{t}_{\mu,k}\}$ denote a sequence of zeros of the solution $\bar{y}_\mu(t)$ of (30_μ), where $\bar{y}_\mu(t)$ may but need not be identical with $y_\mu(t)$. Let $\{T_{\nu,k}\}$, resp. $\{T_{\mu,k}\}$ denote a sequence of zeros of the function $\alpha\bar{y}_\nu + \beta\bar{y}'_\nu$, resp. $\alpha\bar{y}_\mu + \beta\bar{y}'_\mu$, where α, β are real constants with the properties $\alpha^2 + \beta^2 > 0$, $\alpha\beta \leqq 0$. Let $\gamma_\nu = \nu^2 - \frac{1}{4}$, $\gamma_\mu = \mu^2 - \frac{1}{4}$.*

Then for any pair of numbers $\lambda \in (0, \infty)$ and $t_{\nu,0} \in (0, \infty)$, resp. $T_{\nu,0} \in (\gamma_\mu, \infty)$ there exists a number $t_{\mu,0}$, resp. $T_{\mu,0}$, $\bar{t}_{\mu,0}$ such that

(35) $$(-1)^i \Delta^i(t_{\mu,k} - t_{\nu,k}) > 0 \qquad i, k = 0, 1, 2, \ldots$$
$$(t_{\mu,k} - t_{\nu,k}) \to \lambda \qquad \text{for } k \to \infty,$$

$$(36) \qquad (-1)^i \Delta^i(\bar{t}_{\mu,k} - T_{\nu,k}) > 0 \qquad i, k = 0, 1, 2, \ldots$$
$$(\bar{t}_{\mu,k} - T_{\nu,k}) \to \lambda \qquad \textit{for } k \to \infty,$$

$$(37) \qquad (-1)^i \Delta^i(T_{\mu,k} - T_{\nu,k}) > 0 \qquad i, k = 0, 1, 2, \ldots$$
$$(T_{\mu,k} - T_{\nu,k}) \to \lambda \qquad \textit{for } k \to \infty.$$

The numbers $t_{\mu,0}$, $\bar{t}_{\mu,0}$, $T_{\mu,0}$ *are the solutions of the corresponding equations* (4;24), (4;11) *resp. of paper* [4].

Proof: *Since* $q_\nu'(t) \in M_\infty(0, \infty)$, $q_\mu'(t) \in M_\infty(0, \infty)$, $q_\nu(\infty) = q_\mu(\infty) = 1$ and (34) holds, the conditions of Theorems 4;1 and 4;2 of paper [4] are satisfied for $n = \infty$ and q changed to q_ν, Q changed to q_μ. Hence (35) follows directly. Assertions (36) and (37) follow analogously from Theorem 2, where the interval $(0, \infty)$ of the variable t is replaced by interval (γ_μ, ∞).

Remark 4: *For the Bessel equation the function* $Q_\nu(t)$ *is of the form*

$$Q_\nu(t) = 1 - \frac{\gamma_\nu}{t^2} + \frac{2\beta\gamma_\nu\left(\alpha - \frac{\beta}{t}\right)}{t^3\left[\alpha^2 + \beta^2 - \frac{\beta^2\gamma_\nu}{t^2}\right]} - \frac{3\beta^4\gamma_\nu^2}{t^6\left[\alpha^2 + \beta^2 - \frac{\beta^2\gamma_\nu}{t^2}\right]}.$$

Remark 5: Provided $\mu > \frac{1}{2}$, it is possible to put $\mu = \nu$ in the preceding theorem, so that the relations (35)–(37) are then related to the sequences of zeros of the same or different solutions of the same Bessel equation, resp. the sequences of zeros of the functions $\alpha\bar{y}_\nu + \beta\bar{y}_\nu'$, where $\bar{y}_\nu(t)$ are the same or different solutions of the same Bessel equation with exception of the case $\lambda = 0$ in (35) and (37), when $\{t_{\nu,k}\} = \{t_{\mu,k}\}$ and $\{T_{\mu,k}\} = \{T_{\nu,k}\}$.

Theorem 4: *For* $\nu > \frac{1}{2}$ *let* $\{t_{\nu,k}\}$ *denote the sequence of zeros of any non-trivial solution* $y_\nu(t)$ *of* (30_ν). *Let* $\{T_{\nu,k}\}$ *denote the sequence of zeros of the function* $\alpha y_\nu + \beta y_\nu'$, *where* $y_\nu(t)$ *is the abovementioned solution and* α, β *are real constants with the properties* $\alpha\beta \leqq 0$, $\beta \neq 0$. *If* $t_0 > T_0 > \gamma_\nu$, $\left(\gamma_\nu = \nu^2 - \frac{1}{4}\right)$, *then*

$$(38) \qquad (-1)^i \Delta^i(t_{\nu,k} - T_{\nu,k}) > 0 \qquad i, k = 0, 1, \ldots$$

Proof: Theorem 4 is a direct consequence of Theorem 1 for $n = \infty$ and $t \in (\gamma_\nu, \infty)$ because $q_\nu'(t) \in M_\infty(\gamma_\nu, \infty)$, $q_\nu(t) > 0$ $t > \gamma_\nu$ *and* $q_\nu(\infty) = 1$.

Concluding this paper I should like to express my gratitude to RNDr. Jaromír Vosmanský, CSc. for his valuable remarks and advice.

REFERENCES

[1] Háčik, M.: *Contribution to the monotonicity of the sequence of zero points of integrals of the differential equation* $y'' + q(t)y = 0$ *with regard to the basis* (α, β)
Archivum Mathematicum (Brno) (to appear)

[2] Laitoch, M.: *L'équation associée dans la théorie des transformations des équations différentielles du second ordre.*
Acta Univ. Palack. Olomucensis 1963 TOM 12 pp. 45—62

[3] Lorch, L. and Szego, P.: *Higher monotonicity properties of certain Sturm-Liouville functions.*
Acta Math., 109 1963 pp. 55—73

[4] Vosmanský, J.: *Monotonic properties of zeros and extremants of the differential equation* $y'' + q(t)y = 0.$
Archivum Mathematicum (Brno) T 6 1970 pp. 37—73

[5] Vosmanský, J.: *The monotonicity of extremants of integrals of the differential equation* $y'' + q(t)y = 0.$
Archivum Mathematicum (Brno) T 2 1966 pp. 105—111

M. Háčik
010 88 Žilina, Marxa-Engelsa 25
Czechoslovakia