Archivum Mathematicum

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Archivum Mathematicum, Vol. 10 (1974), No. 1, 1--8

Persistent URL: http://dml.cz/dmlcz/104813

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS,
X: 1—8, 1974

DIRECT PRODUCTS OF HOMOMORPHIC MAPPINGS II

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(Received May 17, 1973)

In [4] it is proved that for direct products of so called pseudo-ordered algebras we can state the converse of the theorem on direct product of homomorphisms of the type "onto" (see [3] p. 127 or [4], Theorem 1). There exists very extensive class of algebras in which a more weak theorem than that holds. We can state also a similar assertion for homomorphic mapping of the type "into". These are the main golas of this paper. Apart from this, there is described one type of pseudo-ordered algebras (others are given in [4]) and there is given a characteristic of these algebras by means of a binary relation.

This paper is a continuation of [4], all conceptions and notations are taken from there.

1.

In whole this paper denotes the symbol $\mathfrak{A}$ a class of algebras with a zero element 0, binary operation $\oplus$ and a set of n-ary ($n \geqq 1$) operations Ω fulfilling identities:

(i) $a \oplus 0 = 0 \oplus a = a$ for each $A \in \mathfrak{A}$ and an arbitrary element $a \in A$.

(ii) $00 \dots 0\omega = 0$ for each $\omega \in \Omega$.
An algebra $A \in \mathfrak{A}$ is said to be *without zero-divisors* iff there exists $\Omega' \subseteq \Omega$, $\Omega' \neq \varnothing$ such that for each $\omega \in \Omega'$ there holds:

(iii) arity of ω is greater than 1 and $a_1 a_2 \dots a_n \omega = 0$ iff $a_i = 0$ for at least one $i \in \{1, \dots, n\}$.
Operations from Ω' are called *regular*. An algebra $A \in \mathfrak{A}$ is said to be *pseudo-ordered* if A is without zero-divisors and there exists $\Omega'' \subseteq \Omega'$, $\Omega'' \neq \varnothing$ such that for each $\omega \in \Omega''$ the following identity is true:

(iv) $a_1 a_2 \dots a_n \omega = a_i \alpha$, where $a\alpha = a$ or $\alpha \in \Omega$ and $a\alpha = 0$ iff $a = 0$; $i \in \{1, \dots, n\}$.
Then α is called the *operation corresponding to* $\omega \in \Omega''$.

Let $A_\tau \in \mathfrak{A}$ for $\tau \in T$, $A = \prod_{\tau \in T} A_\tau$ (the direct product of A_τ). By the symbol $\bar{A}_{\tau 0}$ (resp. $\overline{\prod_{\tau \in T'} A_\tau}$ for $T' \subseteq T$) we denote a subalgebra of A fulfilling $pr_{\tau 0} \bar{A}_{\tau 0} = A_{\tau 0}$, $pr_\tau \bar{A}_{\tau 0} = 0$ for $\tau \neq \tau_0$ (resp. $pr_{\tau 0}(\overline{\prod_{\tau \in T'} A_\tau}) = A_{\tau_0}$ for $\tau_0 \in T'$ and $pr_{\tau_1}(\overline{\prod_{\tau \in T'} A_\tau}) = 0$ for $\tau_1 \in T -$

$\smile T'$). An isomorphic mapping j of $\overline{\prod_{\tau \in T'} A_\tau}$ onto $\overline{\prod_{\tau \in T'} A_\tau}$ such that $pr_{\tau 0}(j(\overline{\prod_{\tau \in T'} A_\tau})) = pr_{\tau 0}(\prod_{\tau \in T'} A_\tau)$ for each $\tau_0 \in T'$ is called the *natural isomorphism*. The inverse mapping of j is called the natural isomorphism, too.

In [3] p. 127 the following theorem is stated:

Theorem 1. *Let $\mathfrak{B}$ be a class of algebras, A_τ, $B_\tau \in \mathfrak{B}$ and φ_τ be a homomorphic mapping of A_τ onto (resp. into) B_τ for $\tau \in T$. Then $\varphi = \prod_{\tau \in T} \varphi_\tau$ is a homomorphic mapping of $A = \prod_{\tau \in T} A_\tau$ onto (resp. into) $B = \prod_{\tau \in T} B_\tau$.*

Proof see [4], theorem 1.

In [4] there is proved that for direct products of pseudoordered algebras and for surjective homomorphisms the converse of the theorem 1 is true. For N-algebras and arbitrary homomorphisms only a weaker statement can be proved:

Theorem 2. *Let A_τ, $B_\sigma \in \mathfrak{A}$ be algebras without zero-divisors, $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$, T, s be finite sets, φ be a homomorphic mapping of A onto B. Then there exists a set I of indices α, and injective mappings of I into T and of I into 2^S assigning to each $\alpha \in I$ just one $\tau_\alpha \in T$ and $S_\alpha \subseteq S$ that:*

(1) $\bigcup_{\alpha \in I} S_\alpha = S$, $S_{\alpha'} \cap S_{\alpha''} = \varnothing$ *for* $\alpha', \alpha'' \in I$, $\alpha' \neq \alpha''$

(2) *If* $T^* = \{\tau_\alpha, \alpha \in I\}$, *then* $\varphi(A^*) = \varphi(A)$, *where* $A^* = \overline{\prod_{\alpha \in I} A_{\tau_\alpha}}$

(3) $\varphi \mid A^* = \prod_{\alpha \in I} f_\alpha$, *where* f_α *is a homomorphic mapping of* A_{τ_α} *onto* $\prod_{\sigma \in S_\alpha} B_\sigma$.

Proof. By the theorem 2 in [4], there exists just one $\tau_\sigma \in T$ such that $\varphi(\bar{A}_{\tau_\sigma}) \supseteq \bar{B}_\sigma$ for each $\sigma \in S$. Let $T^* = \{\tau_\alpha, \alpha \in I\}$ be a set of all these indices τ_σ without repetition (each τ_σ is in T^* only one times). Then $\varphi(\prod_{\alpha \in I} A_{\tau_\alpha}) \supseteq \prod_{\sigma \in S} B_\sigma = B = \varphi(A)$, thus $\varphi(A^*) = \varphi(A)$. It is obvious that card I $\leq$ card S. Denote S_α the set of all $\sigma \in S$ for which τ_α is the same (i.e. $\sigma \in S_\alpha \Rightarrow \bar{B}_\sigma \subseteq \varphi(\bar{A}_{\tau_\alpha})$). Evidently $\bigcup_{\alpha \in I} S_\alpha = S$. From 2 in [4] we obtain $S_{\alpha'} \cap S_{\alpha''} = \varnothing$. Let i_α be a natural isomorphism of $\overline{\prod_{\sigma \in S_\alpha} B_\sigma}$ onto $\prod_{\sigma \in S_\alpha} B_\sigma$, j_α be a natural isomorphism of A_{τ_α} onto $\bar{A}_{\tau_\alpha}$, then $f_\alpha = i_\alpha \cdot \varphi \mid \bar{A}_{\tau_\alpha} \cdot j_\alpha$ and $\varphi \mid A^* = \prod_{\alpha \in I} f_\alpha$.

Corollary 3. *Let A_τ, B_τ be algebras without zero-divisors, T finite set $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\tau \in T} B_\tau$ and φ be a homomorphic mapping of A onto B for which $\varphi(\bar{A}_\tau)$ is an algebra without zero-divisors for each $\tau \in T$. Then there exists a permutation π of the set T such that $\varphi = \prod_{\tau \in T} \varphi_\tau$, where φ_τ is a homomorphic maping of A_τ onto $B_\pi(\tau)$.*

Proof. For each $\tau \in T$, $\varphi(\bar{A}_\tau)$ is without zero-divisors, thus card $S_\alpha = 1$ for each

$\alpha \in I$. From the condition $\bigcup_{\alpha \in I} S_\alpha = S$ we obtain card I = card S, but $S = T$, thus $T^* = T$ and the assertion follows from the theorem 2.

Corollary 4. *Let A_τ, B_τ be rings without zero-divisors, $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\tau \in T} B_\tau$, T be a finite set and φ be a homomorphic mapping of A onto B for which $(\bar{A}_\tau \cap ker\ \varphi)$ be a prime-ideal in $\bar{A}_\tau$ for each $\tau \in T$. Then $\varphi = \prod_{\tau \in T} \varphi_\tau$, where φ_τ is a homomorphic mapping of A_τ onto $B_{\pi(\tau)}$, π is a permutation of the index set T.*

Proof. Let $\bar{A}_\tau \cap ker\ \varphi$ be a prime-ideal in $\bar{A}_\tau$, then φ induces a congruence φ on $\bar{A}_\tau$ and the factor-ring $\bar{A}_\tau/\Theta$ is without zero-divisors (see [5]), $\varphi(\bar{A}_\tau)$ is isomorphic with $\bar{A}_\tau/\Theta$. Then $\varphi(\bar{A}_\tau)$ is without zero-divisors, too. From corollary 3 we obtain the assertion.

Corollary 5. *Let A_τ, B_τ be simple-rings (or fields) for $\tau \in T$, φ be a surjective homomorphic mapping of the direct product $A = \prod_{\tau \in T} A_\tau$ onto direct product $B = \prod_{\tau \in T} B_\tau$, T finite. Then $\varphi = \prod_{\tau \in T} \varphi_\tau$, where φ_τ is a surjective homomorphic mapping of A_τ onto $B_{\pi(\tau)}$, π is a permutation of the index set T.*

Proof. The mapping $\varphi \mid \bar{A}_\tau$ is a homomorphic mapping, $ker\ \varphi \mid \bar{A}_\tau = \bar{A}_\tau \cap ker\ \varphi$, but $ker\ \varphi \mid \bar{A}_\tau$ is an ideal in $\bar{A}_\tau$. Fields and simple-rings have no proper ideals, then $\bar{A}_\tau \cap ker\ \varphi$ is the zero $0 \in \bar{A}_\tau$ or whole $\bar{A}_\tau$ for each $\tau \in T$, but they are primeideals.

From the theorem 2 in [4] there follows an analog of the classical Krull-Remark-Schmidt theorem (for rings see [5]):

Corollary 6. *Let A_τ, A_γ be algebras without zero-divisors for $\tau \in T$, $\gamma \in \Gamma$, and $\prod_{\tau \in T} A_\tau \prod_{\gamma \in \Gamma} A_\gamma$, be isomorphic algebras.*
where A is an N-albegra.

Then card Γ = card T *and there exists a permutation π of the set T such that $\gamma \in \Gamma \Rightarrow A_\gamma = A_{\pi(\tau)}$ for just one $\tau \in T$; in other words, the direct decomposition of an algebra A in algebras without zero-divisors is uniquely determined up to order of direct factors.*

Proof. Let $\prod_{\tau \in T} A_\tau$, $\prod_{\gamma \in \Gamma} A_\gamma$, be isomorphic then there exists an isomorphic mapping φ of $\prod_{\tau \in T} A_\tau$ onto $\prod_{\gamma \in \Gamma} A_\gamma$ and isomorphic mapping φ^{-1} of $\prod_{\gamma \in \Gamma} A_\gamma$ onto $\prod_{\tau \in T} A_\tau$, $\varphi\varphi^{-1} = \varphi^{-1}\varphi = id_A$. By the theorem 2 in [4] there exists just one A_τ for each A_γ such that $\varphi(\bar{A}_\tau) \subseteq \bar{A}_\gamma$ and just one $A_{\gamma'}$ for each A_τ such that $\varphi^{-1}(\bar{A}_{\gamma'}) \supseteq \bar{A}_\tau$. Thus, $\bar{A}_{\gamma'} = \varphi\varphi^{-1}(\bar{A}_{\gamma'}) \supseteq \varphi(\bar{A}_\tau) \supseteq \bar{A}_\gamma$, but $\bar{A}_{\gamma'} \supseteq \bar{A}_\gamma$ is impossible for $\gamma \neq \gamma'$. Then $\gamma = \gamma'$ and $\bar{A}_{\gamma'} = \bar{A}_\gamma = \varphi(\bar{A}_\tau)$. We obtain $\varphi^{-1}(\bar{A}_\gamma) = \bar{A}_\tau$ analogously.

From the corollary 6 we get a generalization of the first part of the theorem 4.1. in [1].

2.

Lemma A. *Let $A \in \mathfrak{A}$, O_A be a zero-element of A and φ be a homomorphic mapping of A into $B \in \mathfrak{A}$. Then $\varphi(0_A)$ is a zero-element of $\varphi(A)$ and we have no other zero-elements in $\varphi(A)$.*

Proof. Let $b \in \varphi(A)$, let $a \in A$ and $\varphi(a) = b$. Then $b \oplus \varphi(0_A) = \varphi(a) \oplus \varphi(0_A) = \varphi(a \oplus 0_A) = \varphi(a) = b$, analogously $\varphi(0_A) \oplus b = b$. Let ω be an arbitrary n-ary operation from Ω, then $\varphi(0_A)\,\varphi(0_A) \ldots \varphi(0_A)\ \omega = \varphi(0_A 0_A \ldots 0_A \omega) = \varphi(0_A)$, Thus, $\varphi(0_A)$ is a zero of $\varphi(A)$. The unicity of the zero $\varphi(0_A)$ is evident.

Lemma B. *Let $A \in \mathfrak{A}$ be a pseudo-ordered algebra, φ be a homomorphic mapping of A into $B \in \mathfrak{A}$ fulfilling $\varphi(0_A) = 0_B$. Then $\varphi(A)$ is a pseudo-ordered algebra and $\varphi(A) \in \mathfrak{A}$.*

Proof. Let $b_1, \ldots, b_n \in \varphi(A)$, $b_i \neq 0_B$ and let $a_1, \ldots, a_n \in A$, $\varphi(a_i) = b_i$ and ω be an arbitrary operation from Ω''. Then $b_1 \ldots b_n\omega = \varphi(a_1) \ldots \varphi(a_n)\,\omega = \varphi(a_1 \ldots \ldots a_n\omega) = \varphi(a_i\alpha) = \varphi(a_i)\,\alpha = b_i\alpha$ for some $i \in \{1, \ldots, n\}$. By the lemma A we have $\varphi(0_A)\,\alpha = \varphi(0_A)$. Let $b_j = 0_B = \varphi(0_A)$ for some j, then $b_1 \ldots 0_B \ldots b_n\omega = \varphi(a_1 \ldots \ldots 0_A \ldots a_n\omega) = \varphi(0_A) = 0_B$. Conversely, let $b_1 \ldots b_n\omega = 0_B$, then $0_B = \varphi(a_1 \ldots a_n\omega) = \varphi(a_i)\,\alpha = b_i\alpha$, then $b_i = 0_B$ for some $i \in \{1, \ldots, n\}$. The assertion is evident.

Lemma C. *Let $A \in \mathfrak{A}$ be a pseudo-ordered algebra and $\alpha = id_A$ the corresponding operation for each $\omega \in \Omega''$. Let φ be a homomorphic mapping of A into $B \in \mathfrak{A}$. Then $\varphi(A)$ is a pseudo-ordered algebra with the same Ω''.*

Proof. Let $b_i \in \varphi(A)$, $b_i \neq \varphi(0_A)$ for $i = 1, \ldots, n$. Analogously as in the proof of Lemma B we obtain $b_1 \ldots b_n\omega = b_i$. Let $b_i = \varphi(0_A)$, then $b_1 \ldots b_n\omega = \varphi(a_1 \ldots \ldots 0_A \ldots a_n\omega) = \varphi(0_A)$. Conversely, let $b_1 \ldots b_n\omega = \varphi(0_A)$, then $\varphi(0_A) = \varphi(a_1 \ldots \ldots a_n\omega) = \varphi(a_i) = b_i$ for some $i \in \{1, \ldots, n\}$. The assertion is evident.

Theorem 7. *Let $A_\tau, B_\sigma \in \mathfrak{A}$ be algebras without zero-divisors, T, S be finite sets, $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$, let φ be a homomorphic mapping of A into* B *and let at least one of the following conditions be true:*

(I) $\varphi(0_A) = 0_B$

(II) *A_τ, B_σ are pseudo-ordered and there exists $\omega_0 \in \Omega''$ such that the corresponding operation $\alpha = id$ (for each $\tau \in T$, $\sigma \in S$).*

Then for each $\sigma \in S$ we have $pr_\sigma\varphi(A) = pr_\sigma\varphi(0_A)$ or there exists just one $\tau_\sigma \in T$ such that $pr_\sigma\varphi(A) = pr_\sigma\varphi(\bar{A}_{\tau_\sigma})$.

Proof. The inclusion $pr_\sigma\varphi(A) \supseteq pr_\sigma\varphi(\bar{A}_\tau)$ is evident for each $\tau \in T$ and $\sigma \in S$.

Let $pr_{\sigma_0}\varphi(A) \neq pr_{\sigma_0}\varphi(0_A)$ for $\sigma_0 \in S$ and suppose that it does not exist $\tau_0 \in T$ with $pr_{\sigma_0}\varphi(\bar{A}_{\tau_0}) \supseteq pr_{\sigma_0}\varphi(A)$. Then there exists a set $T' \subseteq T$ such that

$$pr_{\sigma_0}\varphi(\overline{\prod_{\tau \in T'} A_\tau}) \supseteq pr_{\sigma_0}\varphi(A),$$

because for $T' = T$ it is true. Accordingly, card $T' > 1$. Let $\tau_1, \tau_2 \in T'$, $\tau_1 \neq \tau_2$.

(a) Let there exist $\bar{a}_1 \in \bar{A}_{\tau_1}$, $\bar{a}_2 \in \bar{A}_{\tau_2}$ such that

$$pr_{\sigma_0}\varphi(\bar{a}_1) \neq pr_{\sigma_0}\varphi(0_A) \neq pr_{\sigma_0}\varphi(\bar{a}_2).$$

If the condition (I) is fulfilled, then $\varphi(\bar{a}_1\bar{a}_2 \dots \bar{a}_2\omega) = \varphi(0_A) = 0_B$ for an arbitrary n-ary ω which is the direct product of regulary operations from Ω', but $pr_\sigma\varphi(\bar{a}_1)$ $pr_{\sigma_0}\varphi(\bar{a}_2) \dots pr_{\sigma_0}\varphi(\bar{a}_2)\,\omega \neq pr_{\sigma_0}\varphi(0_A) = pr_{\sigma_0}0_B$ which is a contradiction. If the condition (II) is fulfilled, then $\varphi(\bar{a}_1\bar{a}_2 \dots \bar{a}_2\omega_0) = \varphi(0_A)$, but $pr_{\sigma_0}\varphi(\bar{a}_1)\, pr_{\sigma_0}\varphi(\bar{a}_2) \dots$ $\dots pr_{\sigma_0}\varphi(\bar{a}_2)\,\omega_0 = pr_{\sigma_0}\varphi(\bar{a}_i) \neq pr_{\sigma_0}\varphi(0_A)$ which is a contradiction again.

(b) Let the assumption (a) be not true, then there exists $\tau_0 \in T$ such that $pr_{\sigma_0}\varphi(\bar{A}_\tau) =$ $= pr_{\sigma_0}\varphi(0_A)$ for $\tau \neq \tau_0$. Let $b_{\sigma_0} \in pr_{\sigma_0}\varphi(A)$, $b_{\sigma_0} \neq pr_{\sigma_0}\varphi(0_A)$. Let us choose arbitrary $a \in A$ fulfilling $pr_{\sigma_0}\varphi(a) = b_{\sigma_0}$. By the lemma A we have $a \neq 0_A$. We can write $a = \overline{a(\tau_0)} \oplus c$, where $pr_{\tau_0}c = 0$ (and $pr_\tau \overline{a(\tau_0)} = 0$ for $\tau \neq \tau_0$). Then $\varphi(c) = \varphi(0_A)$ by the assumption (b), and: $pr_{\sigma_0}\varphi(a) = pr_{\sigma_0}\varphi(\overline{a(\tau_0)} \oplus c) = pr_{\sigma_0}\varphi(\overline{a(\tau_0)}) \oplus pr_{\sigma_0}\varphi(0_A)$, by the lemma A we obtain $pr_{\sigma_0}\varphi(a) = pr_{\sigma_0}\varphi(\overline{a(\tau_0)})$. From this it is obvious that $pr_{\sigma_0}\varphi(A) \subseteq pr_{\sigma_0}\varphi(\bar{A}_{\tau_0})$, contrary to the assumption of the proof.

Corollary 8. *Let A_τ, $B_\sigma \in \mathfrak{A}$ be algebras without zero-divisors, T, S be finite sets, $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$, be a homomorphic mapping of A into B and let at least one of the conditions (I), (II) of the theorem 7 be true. Let S' be the least subset of S such that*

$$b \in \varphi(A) \Rightarrow pr_\sigma b = pr_\sigma\varphi(0_A) \quad \text{for} \quad \sigma \in S - S'.$$

Let $S' \neq \varnothing$. Then there exists a set Γ of indices γ such that to each $\gamma \in \Gamma$ corresponds just one $\tau_\gamma \in T$ and $S_\gamma \subseteq S$ fulfilling:

(1) $\bigcup_{\gamma \in \Gamma} S_\gamma = S'$, $S_{\gamma'} \cap S_{\gamma''} = \varnothing$ *for* $\gamma', \gamma'' \in \Gamma$, $\gamma' \neq \gamma''$, *and* $\tau_{\gamma'} \neq \tau_{\gamma''}$

(2) $T^* = \{\tau_\gamma, \gamma \in \Gamma\}$, *then* $\varphi(A) = \varphi(A^*)$, *where* $A^* = \overline{\prod_{\gamma \in \Gamma} A_{\tau_\gamma}}$

(3) $\varphi \mid A^* = \prod_{\gamma \in \Gamma} \varphi_\gamma$, *where* φ_γ *is a homomorphic mapping of* A_{τ_γ} *onto* B_{S_γ} *and* $B_{S_\gamma} =$ $= \{b \in \varphi(A);\ pr_\sigma b = pr_\sigma\varphi(0_A)$ *for* $\sigma \in S - S_\gamma\}$.

Proof. By the theorem 7, there exists just one $\tau_\sigma \in T$ for each $\sigma \in S$ such that $pr_\sigma\varphi(A) = pr_\sigma\varphi(\bar{A}_{\tau_\sigma})$. Let us denote by T^* the set of all these pairwise different τ_σ. For each $\tau_0 \in T^*$ we denote S_0 a subset of all σ, for which

$$\sigma \in S_0 \Rightarrow pr_\sigma\varphi(A) = pr_\sigma\varphi(\bar{A}_{\tau_0}).$$

Then $\varphi(\bar{A}_{\tau_0}) = B_{S_0}$ (by notation of the theorem 8). Let us denote $T^* = \{\tau_\gamma, \gamma \in \Gamma\}$. Then by the theorem 7, to each $\tau_\gamma \in T^*$ there corresponds just one $S_\gamma \subseteq S$ and $S_{\gamma'} \cap \cap S_{\gamma''} = \emptyset$ for $\gamma' \neq \gamma''$ and $\bigcup_{\gamma \in \Gamma} S_\gamma = S'$. By the theorem 7 we obtain $\varphi(A^*) = \varphi(A)$. Let j_{S_γ} be a natural isomorphism of $\bar{B}_{S_\gamma}$ onto $\prod_{\sigma \in S_\gamma} B_\sigma$ and i_γ be a natural isomorphism of A_{τ_γ} onto $\bar{A}_{\tau_\gamma}$, then $\varphi_\gamma = j_{S_\gamma} \cdot \varphi \mid \bar{A}_{\tau_\gamma} \cdot i_\gamma$. It is evident that $\varphi \mid A^* = \prod_{\gamma \in \Gamma} \varphi_\gamma$.

This corollary is more weak than the converse of Theorem 1 for homomorphisms of the type "into" fulfilling (I) or (II). However, we can easy show an example when the converse of Theorem 1 for homomorphisms of the type "into" is not generally true (not even for pseudo-ordered algebras).

From Lemmas B, C, Theorem 4 in [4] and Corollary 8 there follows directly:

Corollary 9. *Let A_τ, B_τ be completely ordered groups or chains with the maximal element or chains with the minimal element and φ be a supremum and infimum preserving homomorphic mapping of $A = \prod_{\tau \in T} A_\tau$ into $B = \prod_{\tau \in T} B_\tau$, where T is a finite index set. Then there exist a set $T^* \subseteq T$ such that $\varphi(A) = \varphi(A^*)$, where $A^* = \overline{\prod_{\tau \in T^*} A_\tau}$ and $\varphi(A) = \prod_{\gamma \in \Gamma} B^{(\gamma)}$, where $B^{(\gamma)}$ is a completely ordered group or chain with the maximal element or chain with the minimal element for each $\gamma \in \Gamma$, respectively, and $\varphi \mid A^* = \prod_{\gamma \in \Gamma} \varphi_\gamma$, where φ_γ is order preserving homomorphic mapping of A_{τ_γ} onto $B^{(\gamma)}$.*

3.

In [13] it is proved that for direct products of pseudo-ordered algebras is true the converse of the theorem 1 for mappings of the type "onto". Let us introduce a new conception:

Definition. *Let $A \in \mathfrak{A}$ be an algebra, let $\omega \in \Omega$. The operation ω is said to be* weak-commutative *iff the following identity for each $a, b \in A$ holds: $ab \dots b\omega = ba \dots a\omega$.*

It is clear that for binary operations the conceptions of weak-commutativity and commutativity are equivalent.

Definition. *A binary relation R on an algebra $A \in \mathfrak{A}$ is said to be* weak-antisymmetric *iff $\langle a, b\rangle \in R$ and $\langle b, a\rangle \in R$ imply $a\alpha = b\alpha$, where $a\alpha = a$ or $\alpha \in \Omega$ is a unary operation for which $a\alpha = 0$ iff $a = 0$. A binary relation R is called the* pseudo-ordering *on A iff it is reflexive, weak-antisymmetric and complete on A.*

Theorem 10. *Let $A \in \mathfrak{A}$ be a pseudo-ordered algebra and let there exist a weak-commutative operation $\omega \in \Omega''$. Then there exists a pseudo-ordering on A.*

Proof. Let $A \in \mathfrak{A}$ be a pseudo-ordered algebra and $\omega \in \Omega''$ be weak-commutative. Introduce the relation P:

$$\langle a, b\rangle \in P \quad \text{iff} \quad ab \dots b\omega = a\alpha.$$

From $aa \dots a\omega = a\alpha$ we obtain a reflexivity of P. For each $a, b \in A$ we have $ab \dots \dots b\omega = a\alpha$ or $ab \dots b\omega = b\alpha$, thus, P is complete. If $\langle a, b\rangle \in P$ and $\langle b, a\rangle \in P$, then $ab \dots b\omega = a\alpha$, $ba \dots a\omega = b\alpha$ and from weak-commutativity we obtain $a\alpha = b\alpha$; accordingly, P is a pseudo-ordering.

Theorem 11. *Let A be an algebra with a zero-element* 0 *and a set Ω of n-ary operations ω fulfilling* $00 \dots 0\omega = 0$ *for each $\omega \in \Omega$. Let an antisymmetrical pseudo-ordering P be defined on A. Then A is the pseudo-ordered algebra with a commutative binary operation $\omega_0 \in \Omega''$.*

Proof. Let us define operations $\oplus$ and ω_0 by the following way: $a, b \in A$, then $\langle a, b\rangle \in P$ iff $a \oplus b = b \oplus a = b$ and $ab\omega_0 = ba\omega_0 = a\alpha$, where α is the identity on A. It implies $0 \oplus 0 = 0$, $00\omega_0 = 0$ and from completeness of P we obtain $a_1 a_2 \omega_0 = a_i$ for $i = 1$ or 2. Summary, $\omega_0 \in \Omega''$ is commutative and A is a pseudo-ordered algebra with 0, $\oplus$ and the set of operations $\Omega \cup \{\omega_0\}$.

It is clear that each homomorphism of an algebra A with pseudo-odering P preserving P preserves operations $\oplus$ and ω_0, too.

Corollary 12. *Let A be a completely ordered algebra with zero* 0 *and a set Ω of operations fulfilling* $00 \dots 0\omega = 0$ *for each $\omega \in \Omega$. Then A is a pseudo-ordered algebra.*

It follows directly from the theorem 11 because each complete ordering is a pseudo-ordering. From the theorem 11 we obtain:

Theorem 13. *Let A, B algebras with a zero* 0 *and with the same set Ω of n-ary operations fulfilling* $00 \dots 0\omega = 0$ *for each $\omega \in \Omega$.* Let P be a pseudo-ordering on A, Q pseudo-ordering on B and let φ be a homomorphic mapping of A into B fulfilling $\varphi(0) = 0$ and preserving pseudo-ordering (i.e. $\langle a, b\rangle \in P \Rightarrow \langle\varphi(a), \varphi(b)\rangle \in Q$). Then $\varphi(A)$ is a pseudo-ordered algebra and φ preserves $\oplus$ and ω_0 (introduced in the proof of the theorem 11).

Corollary 14. *Let A_τ, B_τ be algebras of the same class of algebras with zero* 0 *and a set Ω of operations fulfilling* $00 \dots 0\omega = 0$ *for each $\omega \in \Omega$, let P_τ (resp. Q_τ) be a pseudo-ordering on A_τ (resp. B_τ) and R (resp. S) be a direct product of P_τ (resp. Q_τ), i.e. $\langle a, b\rangle \in R$ iff $\langle a(\tau), b(\tau)\rangle \in P_\tau$ for each $\tau \in T$, T finite and $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\tau \in T} B_\tau$. Let φ be a homomorphic mapping of A onto B preserving the boundary of R. Then $\varphi = \prod_{\tau \in T} \varphi_\tau$, where φ_τ is a homomorphic mapping of A_τ onto $B_{\pi(\tau)}$ preserves pseudo-ordering and π is a permutation of T.*

Remark. We say that φ *preserves the boundary* of the relation $P = \prod_{\tau \in T} P_\tau$ if φ preserves the direct product of the operations $\oplus$ and ω_0 introduced in the proof of the theorem 11.

This corollary follows directly from the theorem 13 and the theorem 7 in [4].

Corollary 15. *The converse of the theorem* 1 *for o-homomorphisms of the type "onto" is true for direct products of completely ordered algebras with* 0 *and a set* Ω *of n-ary operations fulfilling* $00 \dots 0\omega = 0$, *if* φ *preserves supremum and infimum.*

Theorem 16. *Each cyclically ordered set (see to* [2]) *is a pseudo-ordered algebra.*

Proof. Let A be a cyclically ordered set, fix $a_0 \in A$. Then the set $A - \{a_0\}$ is completely ordered. This ordering S is induced by a cyclical ordering — see [2], and S is uniquely corresponding to cyclical ordering on A and conversely. Let us extend S to S' by the following way: $S' = S$ on $A - \{a_0\}$, $\langle a_0, a\rangle \in S'$ and $\langle a, a_0\rangle \notin S'$ for each $a \in A - \{a_0\}$. Then S' is uniquely corresponding to S and to cyclical ordering on A, too. By the Theorem 4 in [4], A is a pseudo-ordered algebra.

Let A_τ be cyclically ordered set for each $\tau \in T$. We can introduce so called partially cyclical ordering C on $A = \prod_{\tau \in T} A_\tau$ by the rule: $\langle a, b, c\rangle \in C$ iff $\langle a(\tau), b(\tau), c(\tau)\rangle \in C_\tau$, where C_τ is a cyclical ordering on A_τ. From the Theorem 16 and the Theorem 7 in [4] there follows:

Corollary 17. *Let* A_τ, B_τ *be cyclically ordered sets,* T *finite,* $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\tau \in T} B_\tau$ *and* S *be a partial ordering which is the direct product of complete orderings* S'_τ *correspoding to* C_τ *by the proof of the Theorem* 16. *Let* φ *be a homomorphic mapping of* A *onto* B *preserving binary operations supremum and infimum of the partial ordering* S. *Then there exists a permutation* π *of the index set* T *that* $\varphi = \prod_{\tau \in T} \varphi_\tau$, *where* φ_τ *is a homomorphic mapping of* A_τ *onto* $B_{\pi(\tau)}$ *preserving the cyclical ordering.*

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