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SYSTEMS OF EQUATIONS OVER FINITE BOOLEAN ALGEBRAS

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It is possible to construct the theory of systems of Boolean equations on the base of a general theory of systems of equations given in [1] by J. Slominski. The general theory is raised on principle of a homomorphic mapping corresponding to the given system. The investigation of homomorhic mappings can be carried out by means of the so called matrix representaion in finite Boolean algebras (see [6]). The problem of solving the Boolean systems can be easy transformed to problem of extension of a given mapping to the homomorphism. This problem was solved for finite Boolean algebras in [7].

The presented theory solves the problem of existence and number of solutions of Boolean systems and gives a simple algorithm for solving these systems.

This paper is a continuation of papers [6] and [7], the main conceptions and notations are given there.

1.

Definition 1. Let m, n be positive integers and $X = \{x_1, \ldots, x_n\}$ $A = \{a_1, \ldots, a_m\}$ be sets. The free Boolean algebra freely generated by the set $X \cup A$ is denoted by $\boldsymbol{B}_{a,x}$. Each element of $\boldsymbol{B}_{a,x}$ is called a B-polynom. The elements $x_i \in X$ resp. $a_j \in A$ are called variables resp. coefficients.

The Boolean operation join is denoted by $+$, meet by . and the complement of an element $b \in \boldsymbol{B}_{a,x}$ is denoted by $\bar{b}$.

Every transformation of a given B-polynom by the Boolean operations and identities is called an elementary transformation. On the $\boldsymbol{B}_{a,x}$ there is given the relation of equivalence. The B-polynom Φ is equal to the B-polynom ψ if and only if there exists a finite sequence of elementary transformations which performs Φ onto ψ. This relation of equivalence is denoted by $=$.

The algebra $\boldsymbol{B}_{a,x}$ has just 2^{m+n} elements because it has just 2^{m+n} atoms, i.e. elementary conjuctive forms

$$\tilde{x}_1 \,.\, \tilde{x}_2 \,.\, \ldots \,.\, \tilde{x}_n \,.\, \tilde{a}_1 \,.\, \ldots \,.\, \tilde{a}_m,$$

where $\tilde{x}_i = x_i$ or $\bar{x}_i$, $\tilde{a}_j = a_j$ or $\bar{a}_j$.

Definition 2. Let E_0 be the Cartesian product $\boldsymbol{B}_{a,x} \times \boldsymbol{B}_{a,x}$. Each subset $E \subseteq E_0$ is called the system of Boolean equations (or simply Boolean system or B-system) with parameters $a_1, \ldots, a_m$. The elements of X are called unknowns of the B-system E. Each pair $\langle \Phi, \psi \rangle \in E$ will be called a B-equation.

A subalgebra of $\boldsymbol{B}_{a,x}$ generated by the set A is denoted by $\boldsymbol{B}_a$. The Boolean algebra $\boldsymbol{B}_a$ has 2^{2^m} elements.

Definition 3. The B-system $E_1 \subseteq \boldsymbol{B}_a \times \boldsymbol{B}_a$ is called a compatible system if and only if the relation $\langle \Phi, \psi \rangle \in E_1$ implies the equivalence $\Phi = \psi$.

Definition 4. A mapping h is said to be a characteristic mapping of a system E provided that:

(a) h is a homomorphic mapping of $\boldsymbol{B}_{a,x}$ into $\boldsymbol{B}_a$.

(b) $\langle \Phi, \psi \rangle \in E$ implies $h(\Phi) = h(\psi)$.

(c) $h \mid \boldsymbol{B}_a$ is a homomorphic mapping of $\boldsymbol{B}_a$ onto $h(\boldsymbol{B}_{a,x})$. A characteristic mapping h of a system E is called proper if instead of (c) the following condition holds:

(d) $h \mid \boldsymbol{B}_a$ is an identical isomorphic mapping of $\boldsymbol{B}_a$ onto $\boldsymbol{B}_a$.

It is obvious that (d) implies (c).

Definition 5. The congruence relation $\sim_h$ induced by the characteristic mapping h of E (obviously holding $\Phi \sim_h \psi$ for each $\langle \Phi, \psi \rangle \in E$) is called the regularizer of the system E.

Definition 6. Each set $\{F_1, \ldots, F_n\}$ of B-polynoms $F_i \in \boldsymbol{B}_a$ is called a solution of the B-system E if the substitution of F_i instead of x_i in all places in Φ, ψ implies

$$\Phi(F_1, \ldots, F_n, a_1, \ldots, a_m) \sim_h \psi(F_1, \ldots, F_n, a_1, \ldots, a_m)$$

for each $\langle \Phi, \psi \rangle \in E$, where $\sim_h$ is a regularizer of the B-system E. If $\sim_h$ is equal to $=$, the solution is called proper.

Accordingly, the solution is proper iff the characteristic mapping corresponding to the regularizer $\sim_h$ is proper. It is a solution by a classical definition.

Definition 7. Let $\sim_1, \sim_2$ be two congruences on $\boldsymbol{B}_{a,x}$. We define the ordering: $\sim_1 \leq \sim_2$ iff for arbitrary elements $a, b \in \boldsymbol{B}_{a,x}$ the implication $a \sim_1 b \Rightarrow a \sim_2 b$ holds.

A regularizer of E which is minimal with respect to the ordering $\leq$ on the set of all regularizers of E is called a minimal regularizer of E (see [1]).

Definition 8. The B-systems $E \subseteq E_0$, $E' \subseteq E_0$ are equivalent iff they have identical set of solutions.

Theorem 1. Let $E = \{\langle \Phi_1, \psi_1 \rangle, \ldots, \langle \Phi_k, \psi_k \rangle\}$ be a B-system of $\boldsymbol{B}_{a,x}$; π be a permutation of the set $\{1, \ldots, k\}$, f be a B-polynom of k variables and E^* be a compatible B-system of $\boldsymbol{B}_a$. Then the system E is equivalent with systems $E' =$

$= \{\langle \Phi_{\pi(1)}, \psi_{\pi(1)}\rangle, \ldots \langle \Phi_{\pi(k)}, \psi_{\pi(k)}\rangle\}$, $E'' = E \cup E^*$, $E''' = E \cup \{\langle f(\Phi_1, \ldots, \Phi_k)$ $f(\psi_1, \ldots, \psi_k\}$.

Proof. The equivalence of E, E' is obvious. Equivalence of E, E''' follows from the fact that the regularizer is a congruence. Equivalence of E, E'' follows from the relation $= \leqq \sim_h$.

Theorem 2. Each characteristic mapping h of B-system E induces a solution of the B-system E.

Proof. Let h be a characteristic mapping of E and $\sim_h$ be the regularizer induced by h. Denote $h(x_i) = C_i \in \boldsymbol{B}_a$. By the definition 4 (c) or (d) each element of $h(\boldsymbol{B}_{a,x})$ has a preimage in $\boldsymbol{B}_a$. Let F_i be an element from $\boldsymbol{B}_a$ fulfilling $h(F_i) = G_i$, thus $x_i \sim_h$ $\sim_h F_i \in \boldsymbol{B}_a$. From $h(\Phi) = h(\psi)$ and $x_i \sim_h F_i$ it follows

$$\Phi(F_1, \ldots, F_n, a_1, \ldots, a_m) \sim_n \psi(F_1, \ldots, F_n, a_1, \ldots, a_m)$$

for each $\langle \Phi, \psi \rangle \in E$. Thus $\{F_1, \ldots, F_n\}$ is a solution of E.

Theorem 3. Let h_1 be a characteristic mapping of the B-system E and $\sim_1$ be a corresponding regularizer. Let h_2 be a homomorphic mapping of $\boldsymbol{B}_{a,x}$ into $\boldsymbol{B}_a$ and corresponding congruence relation $\sim_2$ fulfil $\sim_1 \leqq \sim_2$. Then h_2 is a characteristic mapping of E and the set R_1 of all solutions of E induced by h_1 is a subset of the set R_2 of all solutions of E induced by h_2.

Proof. Relation $\sim_1 \leqq \sim_2$ implies the condition (b) of definition 4, validity of (a), (c) is evident, thus h_2 is a characteristic mapping of E. Relation $\{F_1, \ldots, F_n\} \in$ $\in R_1$ holds iff for each $\langle \Phi, \psi \rangle \in E$ there holds $\Phi(F_1, \ldots, F_n, a_1, \ldots, a_m) \sim_1 \psi(F_1, \ldots,$ $\ldots, F_n, a_1, \ldots, a_m)$. This relation and $\sim_1 \leqq \sim_2$ imply

$$\Phi(F_1, \ldots, F_n, a_1, \ldots, a_m) \sim_2 \psi(F_1, \ldots, F_n, a_1, \ldots, a_m)$$

for each $\langle \Phi, \psi \rangle \in E$. Accordingly $\{F_1, \ldots, F_n\} \in R_2$, i.e. $R_1 \subseteq R_2$.

Theorems 2 and 3 give a method for the finding of solutions of the B-system E by means of characteristic mappings of this system. The B-system E is solved if we find all characteristic mappings h_i of E whose regularizers are minimal. Then a set of solutions of E is the set of all $\{F_1, \ldots, F_n\}$, where $F_j \in h^{-1}h(x_j)$, $F_j \in \boldsymbol{B}_a$ and h is arbitrary homomorphism whose congruence $\sim_h$ fulfils $\sim_h \geqq \sim_{h_i}$ (we can write $h \geqq h_i$ iff $\sim_h \geqq \sim_{h_i}$). If $\sim_{h_i}$ is equal to $=$, the solution of E induced by h_i is proper.

2.

Investigations of solutions of B-systems we shall deal by means of an isomorphic representation of $\boldsymbol{B}_{a,x}$. Each Boolean algebra having 2^m elements is isomorphic with the direct power $\{0, 1\}^m$ of two-elements Boolean algebra $\{0, 1\}$ by Birkhoff's theorem. Let us denote $\{0, 1\}2^{m+n}$ by $\mathfrak{M}_{a,x}$ and $\{0, 1\}2^{m}$ by $\mathfrak{M}_{a}$. Elements of $\mathfrak{M}_{a,x}$ (resp. $\mathfrak{M}_a$) are called 2^{m+n}-dimensional (resp. 2^m-dimensional) B-vectors*). The Boolean algebra $\boldsymbol{B}_{a,x}$ is isomorphic with $\mathfrak{M}_{a,x}$, $\boldsymbol{B}_a$ with $\mathfrak{M}_a$. Let us fix the isomorphism i of $\boldsymbol{B}_{a,x}$ onto $\mathfrak{M}_{a,x}$ such that it holds:

$$i(x_j) = (11\ldots100\ldots011\ldots100\ldots0\ \ .\ \ .\ \ .\ \ 11\ldots100\ldots0)$$

for $j = 1, \ldots, n$, where each group of 1 or 0 has 2^{j-1} elements and

$$i(a_k) = (11\ldots100\ldots011\ldots100\ldots0\ \ .\ \ .\ \ .\ \ 11\ldots100\ldots0)$$

for $k = 1, \ldots, m$, where each group of 1 or 0 has 2^{n+k-1} elements.

For $\boldsymbol{B}_a$, $\mathfrak{M}_a$ we fix the isomorphism j of $\boldsymbol{B}_a$ onto $\mathfrak{M}_a$ for which $j(a_k) = (11\ldots100\ldots011\ldots100\ldots0\ .\ .\ .\ 11\ldots100\ldots0)$ for $k = 1, \ldots m$, where each group of 1 or 0 has 2^{k-1} elements.

Obviously $i(0) = (00\ldots0)$, $j(0) = (00\ldots0)$. They are called the zero-vectors of $\mathfrak{M}_{a,x}$ or $\mathfrak{M}_a$ respectively; $i(J) = (11\ldots1)$, $j(J) = (11\ldots1)$, they are called the unit-vectors of $\mathfrak{M}_{a,x}$ or $\mathfrak{M}_a$ respectively. The element of $\mathfrak{M}_{a,x}$ isomorphic to B-polynom $\Phi \in \boldsymbol{B}_{a,x}$(by i) will be denoted by Φ again.

We shall consider now a finite B-system E of $\boldsymbol{B}_{a,x}$ given by relations:

$$(E)\qquad \begin{gathered}\Phi_i(x_1, \ldots, x_n, a_1, \ldots, a_m) = \psi_i(x_1, \ldots, x_n, a_1, \ldots a_m)\\ i = 1, \ldots, k\end{gathered}$$

We shall determine a characteristic mapping of (E) in "B-vectors representation", i.e. a homomorphic mapping h of $\mathfrak{M}_{a,x}$ into $\mathfrak{M}_a$ fulfilling $h(\Phi_i) = h(\psi_i)$ for $i = 1, \ldots, k$ and $h_\gamma(\mathfrak{M}_{a,x}) = h(\mathfrak{N})$, where $\mathfrak{N}$ is a subalgebra of $\mathfrak{M}_{a,x}$ generated by B-vectors $\{a_1, \ldots, a_m\}$ of $\mathfrak{M}_{a,x}$. This mapping corresponding to the characteristic mapping of (E) in the given isomorphic representation.

We shall not differ between the characteristic mapping of (E) and this mapping h of $\mathfrak{M}_{a,x}$ into $\mathfrak{M}_a$ corresponding to the characteristic mapping in given representation. By the theorem 2 in [6] there exists just one B-matrix of the type $2^{m+n}/2^m$ representing the characteristic mapping.

Theorem 4. Let C be a B-matrix representing a characteristic mapping of B-system E. Let $f_j^{(i)}$ (resp. $g_j^{(i)}$) be the j-th coordinate of B-vector Φ_i (resp. ψ_i). If there exists an index $i \in \{1, \ldots, k\}$ such that $f_j^{(i)} \neq g_j^{(i)}$, then all elements in the j-th row of C are equal to zero (so called "zero row").

*) See [6]; $\mathfrak{M}_{a,x}$ is $\mathfrak{M}_{m+n}$, $\mathfrak{M}_a$ is $\mathfrak{M}_m$ from this paper and conception of B-vector is identical.

Proof. Let for example $f_j^{(i)} = 1$, $g_j^{(i)} = 0$ and $c_{js} = 1$ for an index $s \in \{1, \ldots, 2^m\}$, where c_{js} is the element of B-matrix C in the j-th row and the s-th column. Then:

$$h(\Phi_i) = (t_1, \ldots, t_{s-1}, 1, t_{s+1}, \ldots, t_{2^m}),$$

$h(\psi_i) = (v_1, \ldots, s_{s-1}, 0, v_{s+1}, \ldots, v_{2^m})$ because C has at most one unity in each column by the theorem 1 in [6]. Accordingly $h(\Phi_i) \neq h(\psi_i)$ which is a contradiction.

Let us denote by the r-th section a sequence

$$\langle f^{(i)}_{r.2^n+1}, f^{(i)}_{r.2^n+2}, \ldots, f^{(i)}_{r.2^n+2^n} \rangle$$

of coordinates of B-vector Φ_i or a sequence

$$\langle c_{r.2^n+1}, c_{r.2^n+2}, \ldots, c_{r.2^n+2^n} \rangle$$

of rows of B-matrix C, where $r = 0, 1, \ldots, 2^m - 1$.

Theorem 5. Let C be a B-matrix representing a characteristic mapping of B-system E. There are not two different non-zero rows in an arbitrary section of C.

Proof. Let there be a unity in the r-th section of C in the t-th row and v-th column and in the s-th row and w-th column, $t \neq s$. By the theorem 1 in [6] we have $v \neq w$ because h is a homomorphic mapping. Then an image of B-vector $(\underbrace{00\ldots0}_{t\text{-1 zero-elements}}10\ldots0) \in$ $\in \mathfrak{M}_{a,x}$ is equal to $b = (b_1, \ldots, b_{v-1}, 1, b_{v+1}, \ldots, b_{w-1}, 0, b_{w+1}, \ldots, b_{2^m})$. But an image of each B-vector of $\mathfrak{N}$ which has unity in the r-th section has the v-th and w-th coordinates equal to 1 and an image of each B-vector of $\mathfrak{N}$ which has not unity in the r-th section has the v-th and w-th coordinates equal to 0. Accordingly, there does not exist a B-vector of $\mathfrak{N}$ whose image is equal to b, i.e. $h(\mathfrak{M}_{a,x}) \neq h(\mathfrak{N})$ which is a contradiction with the definition 4 (c).

Theorem 6. A B-matrix C of the type $2^{m+n}/2^m$, having at most one unity in each column, represents a characteristic mapping of B-system E if and only if it holds:

(a) if there exists an index $i \in \{1, \ldots, k\}$ such that $f_j^{(i)} \neq g_j^{(i)}$, then C has in the j-th row only 0.

(b) C has not two different non-zero rows in an arbitrary section of C.

Proof. Necessity follows from the theorems 4 and 5. Sufficiency: Let C fulfil assumptions of the theorem 6. Then C represents a homomorphic mapping of $\mathfrak{M}_{a,x}$ into $\mathfrak{M}_a$. (a) implies $h(\Phi_i) = h(\psi_i)$ for $i = 1, 2, \ldots, k$, (b) implies $h(\mathfrak{M}_{a,x}) = h(\mathfrak{N})$. q.e.d.

Let us consider the case when h is a proper characteristic mapping of E. Then $h(a_i) = a_i$. The B-matrix C representing a proper characteristic maping of E is quasidiagonal (see Fig. 1), i.e. all elements out of frames are equal to 0 and in frame there is a section of column.

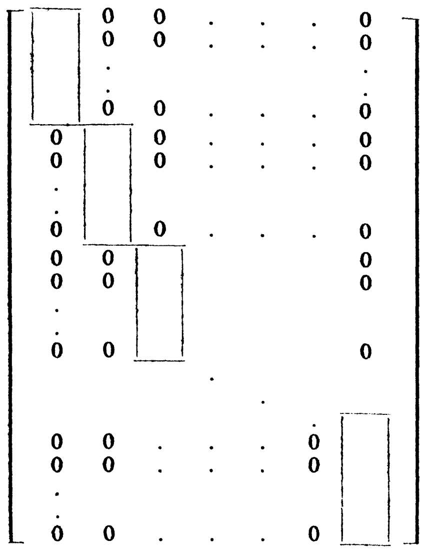

Fig. 1

In this case we must fill in each column just one "diagonal" element. These elements must fulfil the assumptions of theorem 6. We can fill 0 in these sections by theorem 4 comparing coordinates in all pairs $\langle \Phi_i, \psi_i \rangle$ of B-vectors corresponding to equations of E.

If h is not a proper characteristic mapping, we can not assume $h(a_i) = a_i$. We can only fill in B-matrix C zero rows by theorem 4. Other elements are equal to 1 or 0 but they must fulfill assumption of the theorem 6.

Let us fill in B-matrix C zero row if $f_j^{(i)} \neq g_j^{(i)}$ for at least one index i and unit row if $f_j^{(i)} = g_j^{(i)}$ for all $i = 1, \ldots, k$. The matrix constructed by this way is called the matrix of solutions of B-system E.

Definition 9. By a section decomposition of matrix of solutions C we understand the set $\{C_1, \ldots, C_s\}$ of all B-matrices of the type identical with the type of C such that:

(a) each C_i has in each section at most one unit row

(b) if C has in the p-th row only zero elements, each C_i has in the p-th row only zero elements

(c) each C_i has only zero rows and unit rows

(d) $C_1 + C_2 + \ldots + C_s = C$ (the sum of B-matrices is defined in [6]).

It is easy to show that all B-matrices representing all characteristic mappings of E which regularizers are minimal are included in decompositions (defined in [6]) of B-matrices $C_1, \ldots, C_s$ forming a section decomposition of matrix of solutions of E.

Moreover, each B-matrix from decompositions of a section decomposition of matrix of solutions represents a characteristic mapping of E. All other characteristic mappings of E are represented by matrices which are obtained from matrices of decompositions of section decomposition of matrix of solutions by substitution 0 instead 1 respectively in all unit elements.

Theorem 7. The B-system E has a proper solution if and only if the matrix of solutions of E has at least one unit row in each section.

Proof. If E has a proper solution, then there exists a quasidiagonal matrix C' with non-zero columns, i.e. decompositions of section decomposition of matrix of solutions have in each section non-zero elements and the statement of the theorem holds. Conversely, if matrix of solutions fulfils assumption of the theorem, decompositions of a section decomposition of this matrix contain a quasidiagonal matrix of desirable property.

It is easy to show the following.

Theorem 8. Let the matrix of solutions of given B-system E has in the j-th section just k_j unit rows. Then the B-system E has just $s = k_1 \,.\, k_2 \ldots k_2$ different proper solutions.

Theorem 9. Let the matrix C of solutions of B-system E have in the j-th section just k_j unit rows, $p_j = \max(k_j, 1)$, $r_j = \min(k_j, 1)$ $q = 2^m \,.\, \min(1, \sum_{i=1}^{2^m} r_i)$. Then the B-system E has just

$$s = p_1 \,.\, p_2 \,.\, \ldots \,.\, p_{2^m} \,.\, (\sum_{i=1}^{2^m} r_i)^{2^m} \,.\, (2^q - 1) + 1 \qquad \text{solutions.}$$

Proof. The section decomposition of matrix of solutions C contains only matrices $C_1, \ldots, C_s$, where $s = p_1 \,.\, p_2 \ldots p_{2^m}$ (by theorem 8). Each C_i, $i = 1, \ldots, s$, contains just $\sum_{i=1}^{2^m} r_i$ unities in each column (it has 2^m columns), thus (see to [6]) their decompositions contain $p_1 \ldots p_{2^m} \,.\, (\sum_{i=1}^{2^m} r_i)^{2^m}$ matrices. Each matrix of decompositions of section decomposition of C contains q unities, i.e. we receive 2^q B-matrices replacing 1 by 0. Disregarding the zero matrix, we receive $2^q - 1$ matrices.

Accordingly, we receive at all $p_1 \ldots p_{2^m} (\sum_{i=1}^{2^m} r_i)^{2^m} \,.\, (2^q - 1) + 1$ B-matrices representing all characteristic mappings of E by the theorem 6.

Theorem 10. Each B-system E has at least one solution.

Proof. The zero matrix (it contains only zero elements) fulfils assumptions of the theorem 6, thus the zero-homomorphism h_0 with $h_0(\mathfrak{M}_{a,x}) = \{o\}$ is a characteristic mapping of E (o is the zero-vector). Then the corresponding regularizer is induced by the ideal $I = \mathfrak{M}_{a,x}$ (or $I = \boldsymbol{B}_{a,x}$) and the set $\{F_1, \ldots, F_n\}$, where F_i is an arbitrary B-polynom of $\boldsymbol{B}_a$, is a solution of E.

These theorems form a complete theory of solution of Boolean equations over finite Boolean algebras. We can state when the given B-system has proper solutions and enumerate the number of them, we can enumerate number of all solutions and by a simple algorithm (constructing a matrix of solutions, section decomposition, decompositions and substitution 1 by 0) construct matrices representing all characteristic mappings. If h is a characteristic mapping of E, we can determine a solution from relations $h(x_i) = h(F_i)$, $F_i \in \boldsymbol{B}_a$. Corresponding regularizer, i.e. congruence relation replacing the equivalence, is induced by an ideal I (i.e. the set of all B-vectors of $\mathfrak{M}_{a,x}$ fulfilling $h(I) = o$). Finally, each B-system has a solution.

The solving of a given B-system can demonstrated by an example.

Example. Consider the B-system

$$\begin{aligned} (x_1 + x_2)\,\bar{a}_1 a_2 + \bar{x}_1 x_2 a_1 a_2 &= \bar{x}_1 x_2 \\ x_1 x_2 \bar{a}_1 &= (x_1 + \bar{x}_2)\,\bar{a}_1 \bar{a}_2 \qquad (E') \\ a_1(\bar{x}_1 x_2 + x_1 \bar{x}_2) &= a_1 \end{aligned}$$

We can write vertically the B-vectors $\Phi_1, \psi_1, \Phi_2, \psi_2, \Phi_3, \psi_3$, corresponding to equations of E' and by theorems 4 and 6 construct the matrix of solutions $\boldsymbol{C}$ of $\boldsymbol{E}'$.

$$\begin{aligned} i(x_1) &= (1010101010101010) \\ i(x_2) &= (1100110011001100) \\ i(a_1) &= (1111000011110000) \\ i(a_2) &= (1111111100000000) \end{aligned}$$

$$j(a_1) = (1010) \qquad j(a_2) = (1100)$$

Φ_1	Ψ_1	Φ_2	Ψ_2	Φ_3	Ψ_3	
0	0	0	0	0	1	1-st section
1	1	0	0	1	1	
0	0	0	0	1	1	
0	0	0	0	0	1	
1	0	1	0	0	0	2-nd section
1	1	0	0	0	0	
1	0	0	0	0	0	
0	0	0	0	0	0	
0	0	0	0	0	1	3-rd section
0	1	0	0	1	1	
0	0	0	0	1	1	
0	0	0	0	0	1	
0	0	1	1	0	0	4-th section
0	1	0	0	0	0	
0	0	0	1	0	0	
0	0	0	1	0	0	

$C =$

0	0	0	0
1	1	1	1
1	1	1	1
0	0	0	0
0	0	0	0
1	1	1	1
0	0	0	0
1	1	1	1
0	0	0	0
0	0	0	0
1	1	1	1
0	0	0	0
1	1	1	1
0	0	0	0
0	0	0	0
0	0	0	0

From this $k_1 = 2,\ k_2 = 2,\ k_3 = 1,\ k_4 = 1$

$$p_1 = 2,\ p_2 = 2,\ p_3 = 1,\ p_4 = 1$$

$$r_1 = 1,\ r_2 = 1,\ r_3 = 1,\ r_4 = 1$$

From theorem 8 it follows that the B-system E' has just 4 proper solutions. Corresponding B-matrices C_1', C_2', C_3', C_4' form a decomposition of the quasidiagonal matrix C' (derived from C):

$$C' = \begin{bmatrix} 0 & & & \\ 1 & & & \\ 1 & & & \\ 0 & & & \\ & 0 & & \\ & 1 & & \\ & 0 & & \\ & 1 & & \\ & & 0 & \\ & & 0 & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix}, \text{ i.e. } C_1' = \begin{bmatrix} 0 & & & \\ 1 & & & \\ 0 & & & \\ 0 & & & \\ & 0 & & \\ & 1 & & \\ & 0 & & \\ & 0 & & \\ & & 0 & \\ & & 0 & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix} \quad C_2' = \begin{bmatrix} 0 & & & \\ 1 & & & \\ 0 & & & \\ 0 & & & \\ & 0 & & \\ & 0 & & \\ & 0 & & \\ & 1 & & \\ & & 0 & \\ & & 0 & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix}$$

$$C_3' = \begin{bmatrix} 0 & & & \\ 0 & & & \\ 1 & & & \\ 0 & & & \\ & 0 & & \\ & 1 & & \\ & 0 & & \\ & 0 & & \\ & & 0 & \\ & & 0 & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix} \quad C_4' = \begin{bmatrix} 0 & & & \\ 0 & & & \\ 1 & & & \\ 0 & & & \\ & 0 & & \\ & 0 & & \\ & 0 & & \\ & 1 & & \\ & & 0 & \\ & & 0 & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix}$$

We can determine solutions. From C_1':

$$x_1 \to (0\ 0\ 1\ 1),\ \bar{a}_2 \to (0\ 0\ 1\ 1) \qquad \text{from this} \quad x_1 = \bar{a}_2$$

$$x_2 \to (1\ 1\ 0\ 1),\ \bar{a}_1 + a_2 \to (1\ 1\ 0\ 1) \qquad x_2 = \bar{a}_1 + a_2$$

is the 1-st solution. Other proper solutions are obtained from C_2', C_3', C_4' analogously:

$$x_1 = \bar{a}_1 \qquad x_1 = a_1 + \bar{a}_2 \qquad x_1 = \bar{a}_2$$

$$x_2 = a_1 s_2 + \bar{a}_1\bar{a}_2 \qquad x_2 = \bar{a}_1 \qquad x_2 = \bar{a}_1\bar{a}_2$$

Further we can construct a section decomposition of C (4 matrices form it). Let us choose from this section decomposition for example this matrix:

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The decomposition of C_1 is formed by $(\sum_{i=1}^{2^2} r_i)^{2^2} = 4^4 = 256$ B-matrices. Let us choose from them for example the following one:

$$C_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix C_{11} represents a characteristic mapping h of E' with

$$\left.\begin{array}{ll} h(x_1) = (1\ 1\ 1\ 1) & h(J) = (1\ 1\ 1\ 1) \\ h(x_2) = (0\ 0\ 1\ 0) & h(\bar{a}_1\bar{a}_2) = (0\ 0\ 1\ 1) \end{array}\right\} \quad x_1 \sim_h J,\ x_2 \sim_h \bar{a}_1\bar{a}_2$$

We can determine the ideal corresponding to congruence $\sim_h$. But for the investigation of solutions it is only intersection of this ideal with $\boldsymbol{B}_a$ necessary. The intersection is equal to $\{0, \bar{a}_1 a_2\}$. We can say that $x_1 = J$, $x_2 = \bar{a}_1\bar{a}_2$ is a „conditional" solution of E′ iff the condition $\bar{a}_1 a_2 = 0$ holds.

If we replace for example the unity in the first column by 0, we obtain the solution $x_1 \sim_{h'} (\bar{a}_1 + \bar{a}_2)$, $x_2 \sim_{h'} \bar{a}_1\bar{a}_2$ and $\sim_{h'}$ on $\boldsymbol{B}_a$ is given by the ideal $\{0, \bar{a}_1 a_2\}$.

All other solutions of E' can be determined analogously.

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