Pavol Šoltés

A remark on the oscillatory behavior of solutions of differential equations of order 3 and 4

Archivum Mathematicum, Vol. 9 (1973), No. 3, 115--118

Persistent URL: http://dml.cz/dmlcz/104801

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS,
IX: 115—118, 1973

A REMARK ON THE OSCILLATORY BEHAVIOUR OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF ORDER 3 AND 4

P. Šoltés, Košice

(Received February 5, 1973)

Consider a differential equation of the form

$$y''' + q(x)y' + r(x)y = f(x) \tag{1}$$

with $q(x) \in C_1 \langle x_0, \infty)$, $r(x) \in C_0 \langle x_0, \infty)$, $f(x) \in C_0 \langle x_0, \infty)$ and $\int_{x_0}^{\infty} |f(t)| \, dt < \infty$ where $x_0 \in (-\infty, \infty)$. Suppose further that

$$F(y(x)) = y(x)y''(x) + \frac{1}{2} q(x) y^2(x) - \frac{1}{2} y'^2(x),$$

where $y(x)$ is a solution of (1). Then we have

Theorem 1. (*Theorem 4 in* [3]). *Let for any* $x \in \langle x_0, \infty)$ *the following condition hold:*

$$q(x) \geqq 0, \; r(x) \geqq k_1 > 0, \; 2r(x) - q'(x) - 1 \geqq k_2 > 0, \; \left| \int_{x_0}^{x} f(t) dt \right| \leqq K < \infty.$$

If $y(x)$ *is a solution of* (1) *such that*

$$F(y(x_0)) + \frac{1}{2} \int_{x^0}^{\infty} f^2(t) \, dt \leqq 0,$$

then $y(x)$ *is oscillatory or* $\lim_{x \to \infty} y(x) = 0$.

The following result is of the similar character

Theorem 2. *For any* $x \in \langle x_0, \infty)$, *let the following conditions hold:*

$$q(x) \geqq 0 \text{ and } 2r(x) - q'(x) - |f(x)| \geqq 0.$$

If

$$\int_{x_0}^{\infty} q(t) \, dt = +\infty, \tag{2}$$

then a solution $y(x)$ *of* (1) *which satisfies*

$$F(y(x_0)) + \frac{1}{2} \int_{x_0}^{\infty} |f(t)| \, dt \leqq 0$$

is oscillatory or $\lim_{x \to \infty} y(x) = 0$.

Proof. Suppose that the hypotheses hold and that $y(x)$ is not oscillatory. Thus there exists a number $x_1 \geqq x_0$ such that $y(x) \neq 0$ for $x \in \langle x_1, \infty)$. Them from (1) we have

$$F(y(x)) + \int_{x_0}^{x} \left[r(t) - \frac{1}{2} q'(t) - \frac{1}{2} | f(t) |\right] y^2(t) \, dt \leqq F(y(x_0)) + \frac{1}{2} \int_{x_0}^{x} | f(t) | \, dt$$

and therefore

$$y(x)y''(x) - \frac{1}{2} y'^2(x) \leqq -\frac{1}{2} q(x) \, y^2(x). \tag{3}$$

From (3) we get

$$y(x) \, y''(x) - y'^2(x) \leqq y(x) \, y''(x) - \frac{1}{2} y'^2(x) \leqq -\frac{1}{2} q(x) \, y^2(x)$$

thus for $x \geqq x_1$

$$\frac{d}{dx}\left(\frac{y'(x)}{y(x)}\right) \leqq -\frac{1}{2} q(x)$$

and therefore

$$\frac{y'(x)}{y(x)} \leqq \frac{y'(x_1)}{y(x_1)} - \frac{1}{2} \int_{x_1}^{x} q(t) \, dt. \tag{4}$$

Since (2) holds, there exists $x_2 \geqq x_1$ such that for $x \geqq x_2$ from (4) we have

$$\frac{y'(x)}{y(x)} \leqq -k, \text{ where } k > 0. \tag{5}$$

Suppose that $y(x) > 0$ for $x \geqq x_1$. From (5) we have $y'(x) < 0$ for $x \geqq x_2$. Therefore it is necessary that $y(x) \geqq C = \lim_{x \to \infty} y(x) \geqq 0$ for any $x \geqq x_2$. Let $C > 0$. Then for $x \geqq x_2$:

$$\frac{y'(x)}{C} \leqq \frac{y'(x)}{y(x)} \leqq -k,$$

so that $y(x) \to -\infty$ for $x \to \infty$ which is a contradiction. Hence $C = 0$ and $\lim_{x \to \infty} y(x) = 0$.

The following part of this paper is concerned with the oscillatory behaviour of solutions of the differential equation

$$y^{(4)} + p(x)y'' + q(x) \, y' + r(x) \, y = f(x), \tag{6}$$

with $p(x) \in C_0 \langle x_0, \infty)$, $q(x) \in C_1 \langle x_0, \infty)$, $r(x) \in C_0 \langle x_0, \infty)$, $f(x) \in C_0 \langle x_0, \infty)$ and $\int_{x_0}^{\infty} | f(t) | \, dt < \infty$, where $x_0 \in (-\infty, \infty)$. Suppose further that

$$F_1(y(x)) = y(x) \, y'''(x) - y'(x) \, y''(x) + \frac{1}{2} q(x) \, y^2(x),$$

where $y(x)$ is a solution of (6). Then we have

Theorem 3. *Suppose that for all $x \in \langle x_0, \infty)$*

$$q(x) \geqq 0, \; | p(x) | \leqq 2, \; 2r(x) - | p(x) | - q'(x) - | f(x) | \geqq 0.$$

If (2) holds, then a solution y(x) of (6) which satisfies

$$F_1(y(x_0)) + \frac{1}{2} \int_{x_0}^{\infty} | f(t) | \, dt \leqq 0 \tag{7}$$

is oscillatory on $\langle x_0, \infty)$.

Proof. Suppose that a solution $y(x)$ of (6) satisfies (7) and that $y(x) \neq 0$ for $x \in \langle x_1, \infty)$, $x_1 \geqq x_0$. From (6) we get

$$F_1(y(x)) + \int_{x_0}^{x} \left[1 - \frac{1}{2} | p(t) | \right] y''^2(t) \, dt + \int_{x_0}^{x} \left[r(t) - \frac{1}{2} | p(t) | - \frac{1}{2} q'(t) - \right.$$

$$\left. - \frac{1}{2} | f(t) | \right] y^2(t) \, dt \leqq F_1(y(x_0)) + \frac{1}{2} \int_{x_0}^{x} | f(t) | \, dt$$

and therefore

$$y(x) \, y'''(x) - y'(x) \, y''(x) \leqq -\frac{1}{2} q(x) \, y^2(x)$$

thus for $x \geqq x_1$

$$\frac{d}{dx} \left(\frac{y''(x)}{y(x)} \right) \leqq -\frac{1}{2} q(x)$$

and hence

$$\frac{y''(x)}{y(x)} \to -\infty \text{ for } x \to \infty. \tag{8}$$

Suppose that $y(x) > 0$ for $x \geqq x_1$. From (8) we can see that $y''(x) < 0$ for $x \in \langle x_2, \infty)$ with suitable $x_2 \geqq x_1$. Since $y'(x)$ is monotonous, only the following two cases are possible:

1) $y'(x) > 0$ for all $x > x_2$

2) there exists $x_3 \geqq x_2$ such that $y'(x_3) < 0$.

Evidently in the second case there exists $\xi \geqq x_2$ such that $y(\xi) = 0$ which contradicts the hypothesis. Therefore let $y'(x) > 0$ for all $x \geqq x_2$; thus $y(x)$ is an increasing function on $\langle x_2, \infty)$ and therefore

$$\frac{y''(x)}{y(x_2)} \leqq \frac{y''(x)}{y(x)} \text{ for } x \in \langle x_2, \infty),$$

so that, owing to (8), $\lim_{x \to \infty} y''(x) = -\infty$, which is agai contradictory to the assumption that $y(x) > 0$ for $x \geqq x_1$.

Analogously we prove that (6) has no solution $y(x)$ satisfying (7) such that $y(x) < 0$ for all $x \geqq x_1 \geqq x_0$.

This completes the proof.

Remark. *Theorem 3 is a generalization of certain theorems in* [1] *and* [2] *and of Theorem* 6 *in* [4] *which deal with equations*

$$y^{(4)} + 2A(x)\,y' + [b(x) + A'(x)]\,y = 0,$$

or

$$y^{(4)} + q(x)\,y' + r(x)\,y = f(x).$$

REFERENCES

[1] MAMRILLA J.: *O niektorých vlastnostiach riešení lineárnej dif. rovnice* $y^{(4)} + 2Ay' + [A' + b]\,y = 0$, Acta F.R.N. Univ. Comen., VII, 11, Mathem. 1963

[2] MAMRILLA J.: *O niektorých vlastnostiach riešení lineárnej dif. rovnice* $y^{(4)} + 2Ay' + [A' + b]\,y = 0$, Acta F.R.N. Univ. Comen., X, 3, Mathem. 1965

[3] ŠOLTÉS P.: *O niektorých vlastnostiach riešení dif. rovnice tvaru* $y''' + p(x)y'' + q(x)y' + r(x)y = f(x)$, Mat. časopis 22 (1972), No. 2, 123–130

[4] ŠOLTÉS P.: *O niektorých vlastnostiach riešení dif. rovnice* 4. *pásu*, Spisy přírod. fak. Univ. J. E. Purkyně v Brně, č. 518 (1970), 429–444

P. Šoltés
Department of Mathematics
P. J. Šafárik University
Košice, nám. Febr. víť. 9
Czechoslovakia