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NOTE ON CERTAIN PARTITIONS OF POINTS IN R^d

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Let $\mathscr{X}$ be a finite set of points in the d-dimensional Euclidean space R^d. In [4] Radon partitions of types $\{r, s\}$ (i.e. $\mathscr{X}$ admits a partition into non-empty subsets $\mathscr{X}_1$ and $\mathscr{X}_2$ such that card $\mathscr{X}_1 = r$, card $\mathscr{X}_2 = s$ and $\text{conv}\,\mathscr{X}_1 \cap \text{conv}\,\mathscr{X}_2 \neq \emptyset$) are studied. In this note a similar question for the cone hulls (with certain singularity) is solved.

Let o be the fixed origin and $\mathscr{X} = (x_1, \ldots, x_f)$ be an f-tuple of (not necessarily different) points in the d-dimensional Euclidean space R^d, $f \geqq d + 2$, for which $o \notin \mathscr{X}$ and $\dim \mathscr{X} = d$. We say that $\mathscr{X}$ has the *property* (r), r being a natural number, $1 \leqq r \leqq f - 1$, if there exists $J \subset F = \{1, 2, \ldots, f\}$ such that card $J = r$ and either $\text{conv}\,\mathscr{X}(J) \cap \text{conv}\,\mathscr{X}(F - J) = \{o\}$ or $\text{cone}\,\mathscr{X}(J) \cap \text{cone}\,\mathscr{X}(F - J)$ contains a ray. (By $\mathscr{X}(J)$ we denote the n-tuple $(x_{i_1}, \ldots, x_{i_n})$ with indices $J = \{i_1, \ldots, i_n\} \subset F$.)

Let $x_i = (x_{i1}, \ldots, x_{id})$ for $i = 1, \ldots, f$ in a basis $\mathscr{K}$. We shall consider the f by d matrix

$$X = \begin{pmatrix} x_{11} \ldots x_{1d} \\ \ldots\ldots\ldots \\ x_{f1} \ldots x_{fd} \end{pmatrix}$$

and we put $L(X) = \text{lin}\,(x^{(1)}, \ldots, x^{(d)})$, where $x^{(i)} \in R^f$ is the ith column in X, $D(X)$ its orthogonal complement in R^f. It is $\dim L(X) = d$, $\dim D(X) = f - d$.

Forming the matrix

$$\bar{X} = \begin{pmatrix} \alpha_{11} \ldots \alpha_{1\,f-d} \\ \ldots\ldots\ldots\ldots \\ \alpha_{f1} \ldots \alpha_{f\,f-d} \end{pmatrix}$$

whose columns $\alpha^{(j)}$, $j = 1, \ldots, f - d$ form a basis of $D(X)$, we shall assign the ith row $\bar{x}_i \in R^{f-d}$ of $\bar{X}$ to each x_i, $i \in F$; the f-tuple $\bar{\mathscr{X}} = (\bar{x}_1, \ldots, \bar{x}_f)$ of these points in R^{f-d} is called a *linear representation of* $\mathscr{X}$ (see [5]).

By an *affine representation* (or *Gale transform*) we understand an f-tuple $\tilde{\mathscr{X}} = (\tilde{x}_1, \ldots, \tilde{x}_f)$, $\tilde{x}_i = (\beta_{i1}, \ldots, \beta_{i\,f-d-1})$ $i = 1, \ldots, f$ of points in R^{f-d-1}, where the columns of the matrix

$$\begin{pmatrix} \beta_{11} \ldots \beta_{1\,f-d-1} \\ \ldots\ldots\ldots\ldots \\ \beta_{f1} \ldots \beta_{f\,f-d-1} \end{pmatrix}$$

form a basis of the orthogonal complement of the $(d + 1)$-space $\text{lin}\,(x^{(1)}, \ldots, x^{(d)}, 1)$ in R^f (see [3], 5.4, $o \in \mathscr{X}$ possibly.)

(*) Under the assumption of $o \in \text{conv}\,\mathscr{X}$ we denote by K the k-face of the polytope $\text{conv}\,\mathscr{X}$, $1 \leqq k \leqq d$, for which $o \in \text{relint}\,K$ and put $G = \{i \in F \mid x_i \in K\}$, $g = \text{card}\,G$.

By $h(X)$ we shall denote the dimension of the projection of $D(X)$ on the coordinate $(f - g)$-space in R^f determined by the axes with indices $F - G$ in the direction of the complementary coordinate g-space; according to the definition we put $h(X) = -1$ if $G = F$.

1. *Under the situation* (*) *it is* $h(X) = h(X')$, *where* X *or* X' *are the matrices belonging to* $\mathcal{X}$ *in two arbitrary Cartesian systems* $\mathcal{K}$ *or* $\mathcal{K}'$ *in* R^d, *respectively, with the same origin* o.

Proof. We simplify the denotation as follows: We denote by G or $H = G^{\perp}$ the coordinate g-space or its orthogonal complement, resp. and let $1, 2, \ldots, g$ be the indices of the coordinate axes of G. We shall write briefly D, D', h, h' instead of $D(X)$, $D(X')$, $h(X)$, $h(X')$.

Putting $\beta = D \cap H$, $\beta' = D' \cap H$ we shall prove that $\dim \beta = \dim \beta'$. To this purpose we denote the g by $(f - d)$ matrix formed from the first g rows of $\bar{X}$ or $\bar{X}'$ by X^* or X'^*, resp. Then it is $\beta = \{x \in R^f | x = \bar{X}\bar{\lambda}$, where $\bar{\lambda} \in \Lambda = \{\lambda \in R^{f-d} \mid X^*\lambda = o\}\}$, $\beta' = \{x \in R^f \mid x = \bar{X}'\bar{\lambda}$, where $\bar{\lambda} \in \Lambda' = \{\lambda \in R^{f-d} \mid X'^*\lambda = o\}\}$. Since the columns of $\bar{X}$ and $\bar{X}'$ are linearly independent, it is $\dim \beta = \dim \Lambda$, $\dim \beta' = \dim \Lambda'$. Considering that $X'^* = X^*R$ for a suitable regular matrix R (As R we can take a regular matrix such that $\bar{X}' = \bar{X}R$ which exists because the column vectors of both matrices form the spaces of the same dimension $f - d$), it is $\dim \Lambda = \dim \Lambda'$ and hence $\dim \beta = \dim \beta'$.

Replacing H by G we shall prove that $\dim \gamma = \dim \gamma'$, where $\gamma = G \cap D$, $\gamma' = G \cap D'$. If we put $\delta = \beta_D^{\perp}$ (the orthogonal complement of β in D), $\varepsilon = \gamma_\delta^{\perp}$, $\delta' = \beta'^{\perp}_{D'}$, $\varepsilon' = \gamma'^{\perp}_{\delta'}$, we have $\dim \delta = \dim \delta'$, $\dim \varepsilon = \dim \varepsilon'$ and since $h = \dim \beta + \dim \varepsilon$, $h' = \dim \beta' + \dim \varepsilon'$, it follows $h = h'$.

Remark. Thus the number $h(\mathcal{X})$ can be defined by the relation $h(\mathcal{X}) = h(X)$, where X corresponds to arbitrary basis in R^d with the origin o (under conditions (*)).

2. $\mathcal{X}$ *has the property* (r) *if and only if there exists* $J \subset F$, caxd $J = r$ *and a hyperplane* H *in* R^{f-d}, $o \in H$ *such that* $\bar{\mathcal{X}}(J) \subset H_1$, $\bar{\mathcal{X}}(F - J) \subset H_2$, *where* H_1, H_2 *are closed halfspaces determined by* H *and* $\operatorname{int} H_1 \cap \bar{\mathcal{X}}(J) \neq \emptyset \neq \operatorname{int} H_2 \cap \bar{\mathcal{X}}(F - J)$.

Such a separation is called the *semiseparation of* points.

Proof. I. Let $\mathcal{X}$ have the property (r), i.e. there exists $J \subset F$, card $J = r$ and a point $b \in R^f$, $b = (\beta_1, \ldots, \beta_f)$ such that $\sum_{i \in F} \beta_i x_i = o$, $\beta_i \geqq 0$ for $i \in J$, $\beta_i \leqq 0$ for $i \in F - J$ and at least in one case there holds the inequality. Since $b \in D(X)$, it is $b = \sum_{j=1}^{f-d} \gamma_j \alpha^{(j)}$. Put $c = (\gamma_1, \ldots, \gamma_{f-d})$. It holds $(c, \bar{x}_i) = \beta_i$ for each $i \in F$. Thus the hyperplane in R^{f-d} whose normal is determined by c semiseparates the $\bar{\mathcal{X}}(J)$ and $\bar{\mathcal{X}}(F - J)$.

II. On the contrary, let $\bar{\mathcal{X}}(J)$, $\bar{\mathcal{X}}(F - J)$ be semiseparated by the hyperplane with $c = (\gamma_1, \ldots, \gamma_{f-d})$ as its normal. Put $\beta_i = (c, \bar{x}_i)$ for $i \in F$, $b = (\beta_1, \ldots, \beta_f)$. Then $\beta_i \geqq 0$ for $i \in J$, $\beta_i \leqq 0$ for $i \in F - J$ and in both cases at leest one inequality appears. It holds $b = \sum_{j=1}^{f-d} \gamma_j \alpha^{(j)}$ and hence $\sum_{i=1}^{f} \beta_i x_i = o$. From this it follows that $\sum_{i \in J} \beta_i x_i$ is the common point of cone $\mathcal{X}(J)$ and cone $\mathcal{X}(F - J)$. If $\sum_{i \in J} \beta_i x_i = o$, it is $o \in \operatorname{conv} \mathcal{X}(J) \cap \operatorname{conv} \mathcal{X}(F - J)$ and if $\sum_{i \in J} \beta_i x_i \neq o$, then cone $\mathcal{X}(J)$ and cone $\mathcal{X}(F - J)$ have the common ray.

3. (see [2], 358). *If the points* $x_1, \ldots, x_f \in R^d$ *satisfy the condition* $o \in \operatorname{int} \operatorname{conv} \{x_1, \ldots, x_f\}$, *then there exist positive numbers* λ_i, $i = 1, \ldots, f$ *such that* $o = \sum_{i=1}^{f} \lambda_i x_i$.

4. *Under the situation (*) a hyperplane H of R^{f-d}, $o \in H$ exists for which* $\bar{\mathscr{X}}(F - G) \subset H$, $\bar{\mathscr{X}}(G)$ *lies in one of open halfspaces determined by* H, cone $\bar{\mathscr{X}}(F - G) = \operatorname{lin} \bar{\mathscr{X}}(F - G)$ *and* $h(\mathscr{X}) = \dim \operatorname{cone} \bar{\mathscr{X}}(F - G)$.

Proof. Since $o \in \operatorname{relint} K$, it is (according to 3) $o = \sum_{i=1}^{f} \beta_i \bar{x}_i$ for a suitable $(\beta_1, \ldots, \beta_f)$ where $\beta_i > 0$ for $i \in G$, $\beta_i = 0$ for $i \in F - G$. From this it follows that $b = (\beta_1, \ldots, \ldots, \beta_f) \in D(X)$ and thus $b = \sum_{j=1}^{f-d} \gamma_j \alpha^{(j)}$ for some $c = (\gamma_1, \ldots, \gamma_{f-d})$; c is the normal vector of the required hyperplane H because of $(c, \bar{x}_i) = \beta_i$ for $i \in F$. Further on it holds that, for no supporting hyperplane of conv $\bar{\mathscr{X}}$ through o, more than g points from $\bar{\mathscr{X}}$ lie in the corresponding open halfspace. (In fact, if more than g points from $\bar{\mathscr{X}}$ lay in the open halfspace determined by such hyperplane, then there would exist more than g points of $\mathscr{X}$ lying in K.) From this it follows cone $\bar{\mathscr{X}}(F - G) = \operatorname{lin} \bar{\mathscr{X}}(F - G)$. We put $h^*(X) = \dim \operatorname{lin} \bar{\mathscr{X}}(F - G)$. Then the equality $h^*(X) = h(\mathscr{X})$ holds. ($h^*(X)$ equals the rank of the $f - g$ by $f - d$ matrix formed from the rows of $\bar{X}$ with indices $F - G$, which is also equal to the rank of the f by $(f - d)$ matrix if we replace the rows with indices G by the zero rows and hence it equals the dimension of the projection of $D(X)$ on the coordinate $(f - g)$-space.); q.e.d. Note that evidently $f - d > h(\mathscr{X}) \geqq -1$.

5. (see [3], 5.4. iii)

If $Z = (z_1, \ldots, z_f)$ *is an f-tuple of points in* R^{f-d-1} *for which* $\sum_{i=1}^{f} z_i = o$ *and* $\dim \operatorname{lin} Z = f - d - 1$, then there exists an *f-tuple* $\mathscr{X} \subset R^d$ *such that* $\dim \operatorname{aff} \mathscr{X} = d$ *and* Z *is its affine representation.*

6. (see [3], 7.1.4)

If P is a k-neighbourly d-polytope (i.e. each k-membered subset $K \subset$ vert *P forms a face S of P for which $K =$* vert *S) and* $k > \left[\frac{d}{2}\right]$, *then P is a d-simplex.*

Note. Corollary. If $\mathscr{X}$ is the set of all vertices of some d-polytope (with f vertices) and $f \geqq d + 2$, then there exists $k \leqq \left[\frac{d}{2}\right]$ such that conv $\mathscr{X}$ is an l-neighbourly polytope for each $1 \leqq l \leqq k$ and for $l > k$ it is not l-neighbourly.

7. (see [4], lemma 2)

For each affine representation $\bar{X} \subset R^{f-d-1}$, $f \geqq d + 2$ *of an f*-tuple $\mathscr{X} \subset R^d$, $\dim \mathscr{X} = d$ *it holds:*

Every open halfspace of R^{f-d-1} determined by a hyperplane H, $o \in H$ contains,

(i) *at least one point of* $\bar{X}$; *and some of them contains exactly one point if* $\mathscr{X} \neq$ vert P *for every convex d-polytope P with f vertices,*

(ii) *at least $k + 1$ points of* $\bar{X}$ *if* $\mathscr{X} =$ vert *P for some k-neighbourly convex d-polytope P with f vertices; and some of such halfspaces contains exactly $k + 1$ points of* $\bar{X}$ *if P is k- but not $(k + 1)$-neighbourly convex d-polytope.*

8. *The range of the value r for which the given f-tuple $\mathscr{X} \subset R^d$ has the property (r) forms the integer interval.*

Proof. Let H, $o \in H$ be a hyperplane of R^{f-d} that semiseparates r points of $\bar{\mathscr{X}}$. There exists a point $x \neq o$ such that $x \in H \cap \operatorname{int} \operatorname{cone} \bar{\mathscr{X}}$. Let λ a be any $(f-d-2)$-space going through o, x and lying in H. If H rotates around λ from 0° to 180°, then for every r', $r \leqq r' \leqq f - r$ there exists the position of H such that r' points from $\overline{\mathscr{X}}$ are semiseparated.

9. *Let* $\tilde{\mathscr{X}} = (\tilde{x}_1, \ldots, \tilde{x}_f)$ *be an f-tuple of points in* R^l, $l \geqq 1, f \geqq l + 1, \dim \tilde{\mathscr{X}} = l$, $o \in \operatorname{int} \operatorname{conv} \tilde{\mathscr{X}}$. *Then for every natural number r for which* $\frac{f - l - 1}{2} < r < \frac{f + l + 1}{2}$ *there exists a hyperplane containing o that semiseparates r points from* $\tilde{\mathscr{X}}$; *this interval cannot be enlarged.*

Proof. In 8 it is shown that the range of r is an interval. Since $o \in \operatorname{int} \operatorname{conv} \tilde{\mathscr{X}}$, there exist (see 3) numbers $\lambda_1, \ldots, \lambda_f > 0$ such that $o = \sum_{i=1}^{f} \lambda_i \tilde{x}_i$. According to 5 there exists an f-tuple $\mathscr{X} \subset R^{f-l-1}$ such that the f-tuple $\lambda_1\tilde{x}_1, \ldots, \lambda_f\tilde{x}_f$ is its affine representation and $\dim \operatorname{aff} \mathscr{X} = f - l - 1$. The semiseparation of $(\tilde{x}_1, \ldots, \tilde{x}_f)$ is equivalent to the semiseparation of $(\lambda_1\tilde{x}_1, \ldots, \lambda_f\tilde{x}_f)$. If $f = l + 1$, r points can be semiseparated for arbitrary r, $1 \leqq r \leqq l$ because $\tilde{\mathscr{X}}$ is the set of vertices of an l-simplex and $o \in \operatorname{int} \operatorname{conv} \tilde{\mathscr{X}}$. Thus the assertion holds.

Let $f \geqq l + 2$. A) If $\mathscr{X}$ is the set of vertices of some convex $(f - l - 1)$-polytope P ($\operatorname{card} \operatorname{vert} P = f$), then there exists exactly one k, $1 \leqq k \leqq \left[\frac{f - l - 1}{2}\right]$ such that P is a k-neighbourly polytope and not m-neighbourly for every $m > k$ (see 6). According to 7 (put $l = f - d - 1$) every open halfspace in R^l determined by a hyperplane going through o contains at least $k + 1$ points and some of them contains exactly $k + 1$ points from $\tilde{\mathscr{X}}$. In general, the semiseparation of $\left[\frac{f - l - 1}{2}\right] + 1$ points from $\tilde{\mathscr{X}}$ is guaranteed and no less. B) If A) does not work, then $\mathscr{X}$ is not the set of vertices of the convex $(f - l - 1)$-polytope with f vertices and by 7 one point of $\tilde{\mathscr{X}}$ can be semiseparated by a suitable hyperplane; q.e.d.

10. (see [4], theorem)

Let $\mathscr{X}$ *be an f-tuple of points in* R^d, $\operatorname{card} \mathscr{X} \geqq d + 3$. *Then*

(i) *if* $\mathscr{X}$ *is not the set of vertices of a convex polytope with f vertices,* $\mathscr{X}$ *has a Radon partition of the type* $\{r, f - r\}$ *for arbitrary* $r = 1, \ldots, f - 1$.

(ii) *If* $\mathscr{X}$ *is the set of a k-neighbourly convex polytope P, then there is no partition of the type* $\{r, f - r\}$ *for* $r \leqq k$, *and if P is exactly k-neighbourly, then it admits Radon partitions for every* r, $f - k - 1 \geqq r \geqq k + 1$.

11. (see [1], 3.2.)

If $y \in \operatorname{int} \operatorname{conv} X$, $X \subset R^d$, *then* $y \in \operatorname{int} \operatorname{conv} Y$ *where* $Y \subset X$, $\operatorname{card} Y \leqq 2d$.

Let $\mathscr{X}$ be an f-tuple of points in R^d, $f \geqq d + 2$, $o \notin \mathscr{X}$ $\dim \mathscr{X} = d$. Let us define for it the number $s(\mathscr{X})$ as follows:

1. In the case of $o \notin \operatorname{conv} \mathscr{X}$ put $s(\mathscr{X}) = \frac{d - 1}{2}$

2. In the case of $o \in \operatorname{conv} \mathscr{X}$, i.e. if (*) is fulfilled, we put

2.1. $s(\mathscr{X}) = 0$ for $g > 2k$

and for $g \leqq 2k$ we define

2.2.1. $s(\mathscr{X}) = \frac{g}{f - d} - 1$ if $h(\mathscr{X}) = 0$ or $= -1$

2.2.2. $s(\mathscr{X}) = \frac{d - g}{2}$ if $h(\mathscr{X}) = f - d - 1$

2.2.3. $s(\mathscr{X}) = \min\left\{\frac{g}{f-d-h(\mathscr{X})} - 1\ ,\ \frac{f-g-h(\mathscr{X})-1}{2}\right\}$
if $1 \leqq h(\mathscr{X}) \leqq f-d-2$.

Theorem. *Let $\mathscr{X}$ be an f-tuple of points in R^d, $f \geqq d+2$, $o \notin \mathscr{X}$, $\dim \mathscr{X} = d$. Then for every natural number r for which $s(\mathscr{X}) < r < f - s(\mathscr{X})$ $\mathscr{X}$ has the property (r) an this interval is the maximal one.*

Proof. According to 8 the range of admissible value of r is an interval.

Case 1. We choose a hyperplane H in R^d in order that it may strictly separate o and $\operatorname{conv} \mathscr{X}$ and project $\mathscr{X}$ from o on H; we denote the projection by $\mathscr{X}'$. It holds dim conv $\mathscr{X}' = d-1$ (because of dim conv $\mathscr{X} = d$) and $\mathscr{X}$ has the property (r) if and only if there exists $J \subset F$, card $J = r$ such that $\operatorname{conv} \mathscr{X}'(J) \cap \operatorname{conv} \mathscr{X}'(F-J) \neq \varnothing$. If for some $i, j \in F$, $i \neq j$, it is $x_i' = x_j'$, $\mathscr{X}$ has the property (r) for every $r = 1, 2, \ldots, f-1$. In other case the assumptions of 10 are satisfied because of card $\mathscr{X}' \geqq$ $\geqq (d-1)+3$. From 10 and 6 it follows that $\mathscr{X}$ has the property (r) for every r, $\left[\frac{d-1}{2}\right] + 1 \leqq r \leqq f - \left[\frac{d-1}{2}\right] - 1$ or, equivalently, for $\frac{d-1}{2} < r < f - \frac{d-1}{2}$. The remaining part of the assertion follows from the existence of the $\left[\frac{d-1}{2}\right]$ — neighbourly polytope (see [3]).

Case 2.1. Under the situation (*) it is $o \in \operatorname{relint}_k \operatorname{conv} \mathscr{X}(G)$ and card $G = g > 2k$. Hence (by 11) there exists $j \in G$ such that $o \in \operatorname{relint}_k \operatorname{conv} \mathscr{X}(G - \{j\})$ and the property (r) for $r = 1$ can be achieved by the choice of $J = \{j\}$.

Case 2.2.1. If $h(\mathscr{X}) = -1$ (i.e. $G = F$), all the points of $\bar{\mathscr{X}}$ lie in the open half-space ϱ determined by the hyperplane H from 4. If $h(\mathscr{X}) = 0$, it is $\bar{x}_i = o$ for at least one $i \in F$ and for the remaining $j \in F$ it is $\bar{x}_j = o$ or $\bar{x}_j$ lies in ϱ. In both cases, consequently, cone$\bar{\mathscr{X}}$ is the sharp cone of dimension $f-d$. Let E be the set of all $i \in F$ for which cone $\{\bar{x}_i\}$ is an extreme ray in cone $\bar{\mathscr{X}}$ and k_i be the multiplicity of that ray. Put $k = \min_{t \in E} k_i$. Then there exists a hyperplane in R^{f-d} going through o which semiseparates k points of $\bar{\mathscr{X}}$. Since $\max k = \left[\frac{g}{f-d}\right]$ when max operates on all the f-tuples $\mathscr{X}$ for which $h(\mathscr{X}) = 0$ or $= -1$ (the inequality $\leqq$ is evident, the equality is proved by the following example), it is (by 2) $s(\mathscr{X}) = \frac{g}{f-d} - 1$; q.e.d.

Example. Let $\mathscr{X}$ be an f-tuple of points in R^d its matrix $\bar{X}$ of which is the f by $f-d$ matrix whose rows $\bar{x}_i$, $i = 1, \ldots, f$ are the vectors

$$\bar{x}_1 = \ldots = \bar{x}_k = (1, 0, 0, \ldots, 0)$$
$$\bar{x}_{k+1} = \ldots = \bar{x}_{2k} = (0, 1, 0, \ldots, 0)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\bar{x}_{(f-d-1)\,k+1} = \ldots = \bar{x}_g = (0, 0, \ldots, 0, 1)$$
$$\bar{x}_{g+1} = \ldots = \bar{x}_f = (1, 1, \ldots, 1)$$

where $k = \left[\frac{g}{f-d}\right]$, Such an f-tuple $\mathscr{X}$, $o \notin \mathscr{X}$ exists and has not the property (r) for every $r < \left[\frac{g}{f-d}\right]$ because $\left[\frac{g}{f-d}\right]$ point sin $\bar{\mathscr{X}}$ are the least number of points which may be semiseparated.

Case 2.2.2. Let H be the hyperplane in R^{f-d} from 4. Since for every hyperplane $H' \neq H$ in R^{f-d} going through o $H' \cap H$ semiseparates in H at least one point of $\overline{\mathscr{X}}(F - G)$ (because of cone $\overline{\mathscr{X}}(F - G) = H$), the least number of points in $\overline{\mathscr{X}}$ that can be semiseparated equals the minimal number of points from $\overline{\mathscr{X}}(F - G)$ which can be semiseparated by a hyperplane in H going through o. According to 9, for every r where $\frac{d - g}{2} < r < f - \frac{d - g}{2}$, r points of $\overline{\mathscr{X}}$ can be semiseparated and this interval cannot be enlarged. By 2 it is $s(\mathscr{X}) = \frac{d - g}{2}$.

Case 2.2.3. First of all it holds $f - d \geqq \dim \text{cone } \overline{\mathscr{X}}(G) \geqq f - d - h(\mathscr{X})$ and cone $\overline{\mathscr{X}}(G)$ is a sharp cone. Denote by τ $(f - d - h)$-dimensional orthogonal complement to h-space cone $\overline{\mathscr{X}}(F - G)$ and project the g-tuple $\overline{\mathscr{X}}(G)$ on τ in the direction of this h-space; denote by $\overline{\mathscr{X}}_\tau(G)$ the projected g-tuple. The semiseparation of some points from $\overline{\mathscr{X}}(G)$ by a hyperplane in R^{f-d} going through the h-space cone $\overline{\mathscr{X}}(F - G)$ is equivalent to the semiseparation of points from $\overline{\mathscr{X}}_\tau(G)$ by a hyperplane in R^{f-d-h}. According to the case 2.2.1 $\left[\frac{g}{f - d - h}\right]$ points from $\overline{\mathscr{X}}_\tau(G)$ can be semiseparated and this number is generally the minimal one. At the same time it equals the least number of points which can be semiseparated in $\overline{\mathscr{X}}$ if the separating hyperplane contains cone $\overline{\mathscr{X}}(F - G)$. If the separating hyperplane (note it by H') is not of this kind, then $H \cap H'$ is such a hyperplane that in each of its open halfspaces there lies at least one point of $\overline{\mathscr{X}}(F - G)$. According to 9 the semiseparation of $\leqq \left[\frac{f - g - h - 1}{2}\right]$ points from $\overline{\mathscr{X}}(F - G)$ by $H \cap H'$ cannot be guaranteed and this estimation is the best one. This number is the same even for the semiseparation of points from $\overline{\mathscr{X}}$ Since every separating hyperplane in R^{f-d} is one of the above types and the estimations in 9 and 2.2.1 are the best ones, our assertion follows from 2.

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