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HIGHER ORDER TORSIONS OF MANIFOLDS WITH CONNECTION

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Investigating a so-called generalized space with connection $\mathscr{S}$, we define some projections of its curvature form and we deduce that these projections represent some natural "obstructions" for holonomy of the successive developments of $\mathscr{S}$. Since the first of them is the usual torsion form of $\mathscr{S}$, [9], these projections are called the higher order torsion forms of $\mathscr{S}$. (For a space with Cartan connection, we have studied this question in [4].) In the general case, the torsion form of a sufficiently higher order coincides with the curvature form. Thus, if the torsion form of this order vanishes, then the connection of $\mathscr{S}$ is integrable and a development of an arbitrary order of $\mathscr{S}$ is holonomic. Moreover, if $\mathscr{S}$ has some additional properties, we also introduce the "weak" torsion forms of $\mathscr{S}$, which represent some natural obstructions for holonomy of the successive weak developments of $\mathscr{S}$ (see § 3). As an example, we treat in details a surface of a 3-dimensional space with affine connection. We also find a remarkable difference between the methods of investigation used in the "strong" and in the "weak" case, cf. § 6 and § 7.

Our considerations are in the category C^∞.

1. In [5], we have defined a generalized space with connection as a quadruple $\mathscr{S} = \mathscr{S}(P(B, G), F, C, \sigma)$, where $P(B, G, \pi)$ is a principal fibre bundle, C is a connection (of the first order) on the groupoid PP^{-1} associated with P and σ is a cross section of the associated fibre bundle $E = E(B, F, G, P)$. Let $G_x \subset PP^{-1}$ be the group of all isomorphisms of E_x, $x \in B$, and let $H_x \subset G_x$ be the stability group of $\sigma(x)$. The curvature form $\Omega(x)$ of C at x is an element of $\mathfrak{g}_x \otimes \bigwedge^2 T_x^*(B)$ and the torsion form (or the strong torsion form) of order zero $\tau^\circ(x)$ of $\mathscr{S}$ at x is introduced as the canonical projection of $\Omega(x)$ into $(\mathfrak{g}_x/\mathfrak{h}_x) \otimes \bigwedge^2 T_x^*(B)$, [4]. Further, the development of order r of $\mathscr{S}$ (or the absolute differential of order r of σ with respect to C, [7]) is the cross section $\sigma^r : B \to \bigcup_{x \in B} \bar{J}^r(B, E_x) = (B, \bar{J}^r(B, F), G, P)$, $\sigma^r(x) = [C^{(r-1)}]^{-1}(x)(\sigma)$, [3]. Sometimes, we shall also say that σ^r is the strong development of order r of $\mathscr{S}$. In [4], Proposition 1, we have proved that $\sigma^2(x)$ is a holonomic 2-jet of B into E_x if and only if $\tau^\circ(x) = 0$. Let μ be the canonical mapping $\mu : \mathfrak{g}_x \to T_{\sigma(x)}(E_x)$. In the course of the proof of Proposition 1 of [4], we have established the relation

$$\Delta([C']^{-1}(x)(\sigma)) = -\mu(\Omega(x)), \tag{1}$$

where Δ means the mapping assigning to every semi-holonomic 2-jet its difference tensor, [3], [4].

A generalized space with connection $\mathscr{S}$ will be said to be a generalized manifold with connection, if it satisfies

a) $m = \dim B < \dim F = n$,

b) $C^{-1}(x)(\sigma)$ **is regular for every** $x \in B$.

If it further holds

c) G **acts on** F **transitively,**

then $\mathcal{S}$ will be called a manifold with connection. Such a manifold with connection is locally equivalent to a submanifold of a space with Cartan connection.

We define the weak development of order r of a generalized manifold with connection $\mathcal{S}$ as the cross section $\lambda^r : B \to \bigcup_{x \in B} \bar{K}^r_m(E_x) = (B, \bar{K}^r_m(F), G, P)$, $\lambda^r(x) = k(\sigma^r(x))$, where $k(\sigma^r(x))$ means the contact element determined by $\sigma^r(x)$. (For definition of contact elements and for the corresponding notations, we refer the reader to [1].) Further, let K_x be the m-dimensional subspace of $T_{\sigma(x)}(E_x)$ determined by $C^{-1}(x)(\sigma)$, let $\varepsilon : T_{\sigma(x)}(E_x) \to T_{\sigma(x)}(E_x)/K_x$ be the natural projection and let $\mathfrak{k}_x \subset \mathfrak{g}_x$ be the kernel of the projection $\varepsilon\mu$. The canonical projection of $\Omega(x)$ into $(\mathfrak{g}_x/\mathfrak{k}_x) \otimes \bigwedge^2 T^*_x(B)$ will be called the weak torsion form of order 0 of $\mathcal{S}$ at x and will be denoted by $\nu^\circ(x)$. (If $\mathcal{S}$ is a manifold with connection, then $\nu^\circ(x)$ coincides with the reduced torsion form of $\mathcal{S}$ introduced in [3].) The following assertion extends Theorem 2 of [3] to the case of an arbitrary action of G on F.

Proposition 1. *Let $\mathcal{S}$ be a generalized manifold with connection. The second weak development of $\mathcal{S}$ at x is holonomic (i.e. $\lambda^2(x) \in K^2_m(E_x)$) if and only if $\nu^\circ(x) = 0$.*

Proof. This is a direct consequence of (1) and of Proposition 11 of [3].

2. Consider an arbitrary generalized space with connection $\mathcal{S}$ and denote by $H^r_x \subset G_x$ the stability group of $\sigma^r(x) \in \bar{J}^r(B, E_x)$. The canonical projection of $\Omega(x)$ into $(\mathfrak{g}_x/\mathfrak{h}^r_x) \otimes \bigwedge^2 T^*_x(B)$ will be called the (strong) torsion form of order r of $\mathcal{S}$ at x and will be denoted by $\tau^r(x)$. On the other hand, $\mathcal{S}$ will be said to be (strongly) r-holonomic at x, if $\sigma^r(x) \in J^r(B, E_x)$. By Proposition 2 of [4], we obtain immediately.

Proposition 2. *Assume that the strong torsion form of order r-1 of $\mathcal{S}$ vanishes in a neighbourhood of $x \in B$. Then $\mathcal{S}$ is strongly $(r+2)$-holonomic at x if and only if $\tau^r(x) = 0$.*

Let $s_r(F, m)$ be the minimum of the dimensions of the stability groups of the elements of $T^r_m(F)$. The smallest r satisfying $s_r(F, m) = 0$ will be denoted by $i_s(F, m)$ and will be said to be the index of isotropy of F with respect to m-dimensional velocities or the strong m-index of isotropy of F. Let $D_1 \subset T^{r_1}_m(F)$, $r_1 = i_s(F, m)$, be the set of all elements whose stability group has dimension zero; one sees easily that D_1 is an open subset. We shall say that $\mathcal{S}$ is of general type of the first kind at x, if for some (and hence for every) $u \in P_x$ it holds $u^{-1}(\sigma^{r_1}(x)) \in D_1$.

Proposition 3. *Assume that a generalized space with connection $\mathcal{S}$ is of general type of the first kind at every point. If $\mathcal{S}$ is strongly $(i_s(F, m) + 2)$-holonomic at every point, then its connection is integrable.*

Proof is obvious.

Remark 1. One can easily see that the strong n-index of isotropy of a homogeneous space $F(n = \dim F)$ coincides with the order of isotropy of F as introduced in [4] and that a space with Cartan connection is of general type of the first kind at every point. Hence Corollary 2 of [4] is a special case of Proposition 3.

3. Consider a generalized manifold with connection $\mathcal{S}$. Let N^r_x be the stability group of $\lambda^r(x) = z$ and let μ_r be the canonical projection of $\mathfrak{g}_x$ into $T_z(\bar{K}^r_m(E_x))$. Let $K^r_x \subset T_z(\bar{K}^r_m(E_x))$ be the m-dimensional subspace determined by $C^{-1}(x)(\lambda^r)$, let ε_r be the canonical projection $T_z(\bar{K}^r_m(E_x)) \to T_z(\bar{K}^r_m(E_x))/K^r_x$ and let $\mathfrak{k}_r \subset \mathfrak{g}_x$ be the kernel of the projection $\varepsilon_r\mu_r$. The canonical projection of $\Omega(x)$ into $(\mathfrak{g}_x/\mathfrak{k}^r_x) \otimes \otimes \bigwedge^2 T^*_x(B)$ will be called the weak torsion form of order r of $\mathscr{S}$ at x and will be denoted by $\nu^r(x)$. On the other hand, $\mathscr{S}$ will be said to be weakly r-holonomic at x, if $\lambda^r(x) \in \in K^r_m(E_x)$. (We introduce a more precise terminology than in [1] or [2]. For comparison, "holonomic" in the sense of [1] or [2] means "weakly holonomic" according to our present terminology.)

Proposition 4. *Assume that ν^{r-1} vanishes in a neighbourhood of $x \in B$. Then $\mathscr{S}$ is weakly $(r + 2)$-holonomic at x if and only if $\nu^r(x) = 0$.*

Proof is based on some properties of contact elements which will be deduced in the following two lemmas. Let M, V be two manifolds, $n = \dim M > \dim V = m$. Let $\tau : \tilde{K}^r_m(M) \to M$ be the jet projection.

Lemma 1. *Let X be a 1-jet of V into $\tilde{K}^r_m(M)$ such that $\tau X \in J^1(V, M)$ is regular. Then X is canonically identified with an element $\varkappa(X) \in \tilde{K}^{r+1}_m(M)$.*

Proof of Lemma 1. Consider first the case $V = \mathbf{R}^m$, $\alpha X = 0$. Let $X = j^1_0\varrho$ and let $t_x : \mathbf{R}^m \to \mathbf{R}^m$ be the translation $y \to y + x$. Take a local mapping ψ of $\mathbf{R}^m$ into $\tilde{T}^r_m(M)$ such that $k(\psi(x)) = \varrho(x)$. Then $x \to \psi(x)\, t_x^{-1}$ is a local cross section of $\tilde{J}^r(\mathbf{R}^m, M)$ so that $Y = j^1_0(\psi(x)\, t_x^{-1})$ is an element of $\tilde{T}^{r+1}_m(M)$ and one finds easily that Y is regular. Put $\varkappa(X) = k(Y)$; we have to show that this definition is correct. If we take another local mapping $\hat{\psi}$ of $\mathbf{R}^m$ into $\tilde{T}^r_m(M)$ such that $k(\hat{\psi}(x)) = \varrho(x)$, then $\hat{\psi}(x) = \psi(x)\, \varphi(x)$, where φ is a local mapping of $\mathbf{R}^m$ into $\tilde{L}^r_m$. Set $Y = j^1_0[\hat{\psi}(x) t_x^{-1}]$; then we have $\hat{Y} = j^1_0[\psi(x)\, t_x^{-1}\, t_x\, \varphi(x)\, t_x^{-1}] = Y j^1_0[t_x\, \varphi(x)\, t_x^{-1}]$. But $j^1_0[t_x\varphi(x)\, t_x^{-1}] \in \tilde{L}^{r+1}_m$ which implies $k(Y) = k(\hat{Y})$. — If V is arbitrary, we take an element $h \in H^1(V)$ such that $\beta h = \alpha X$ and we define $\varkappa(X) = \varkappa(Xh)$.

In the course of such a consideration, it is often convenient to use an auxiliary fibering on M. As well known, for any fibered manifold (W, π, W_1), $\dim W_1 = m$, the elements of $\tilde{J}^r(W, \pi, W_1)$ are identified with those elements of $\tilde{K}^r_m(W)$ which are transversal with respect to the projection $\pi : W \to W_1$. Let $X \in J^1(V, \tilde{K}^r_m(M))$, $X = j^1_x\varrho$. On a neighbourhood $U \subset M$ of the point $\tau(\varrho(x))$, take an auxiliary fibering $\mu : U \to U_1$, $\dim U_1 = m$, such that βX and τX are transversal with respect to μ. Then ϱ is locally identified with a local cross section χ of $\tilde{J}^r(U, \mu, U_1)$ and, by the preceding construction, $\varkappa(X) \in \tilde{K}^{r+1}_m(M)$ is identified with $j^1_t\chi \in \tilde{J}^{r+1}(U, \pi, U_1)$, $t = \mu(\tau(\varrho(x)))$.

A cross section ϱ of V into $J^1(V, \tilde{K}^r_m(M))$ will be called **admissible**, if the values of the mapping $y \to \tau\varrho(y)$ of V into $J^1(V, M)$ are regular 1-jets. If ϱ is admissible, then $k(\varrho(y)) \in \tilde{K}^1_m(K^r_m(M))$ and $\varkappa(j^1_x[k(\varrho(y))]) \in \tilde{K}^2_m(\tilde{K}^r_m(M))$, $x \in V$. On the other hand, $\varkappa(\varrho(y))$ is a mapping of V into $\tilde{K}^{r+1}_m(M)$, so that $\varkappa(j^1_x[\varkappa(\varrho(y))]) \in \tilde{K}^{r+2}(M)$.

Lemma 2. *Let ϱ be an admissible cross section of $J^1(V, K^r_m(M))$ such that the values of $\varkappa(\varrho(y))$ lie in $K^{r+1}_m(M)$. Then $\varkappa(j^1_x[\varkappa(\varrho(y))]) \in K^{r+2}_m(M)$ if and only if $\varkappa(j^1_x[\varkappa[\varrho(y)]]) \in \in \bar{K}^{r+2}_m(M)$ and $\varkappa(j^1_x[k(\varrho(y))]) \in K^2_m(K^r_m(M))$.*

Proof of Lemma 2 is based on the same idea as the proof of Proposition 2 of [4]. On a neighbourhood U of $\tau(\beta\varrho(x))$, choose an auxiliary fibering $\mu : U \to U_1$, $\dim U_1 = m$, such that $\tau\varrho(x)$ is transversal with respect to μ. Then $\varkappa(\varrho(y))$ is locally identified with a local cross section χ of $J^{r+1}(U, \mu, U_1)$. Let $a^\alpha(y)$, $a^\alpha_i(y)$, $\ldots$, $a^\alpha_{i_1} \ldots$

$\dots i_{r+1}\,(y)$ (symmetric in all subscripts) be the coordinates of χ in a local coordinate system. Then the coordinates of $j^1_t\ \chi$ are $a^\alpha(t)$, $a^\alpha_i(t)$, ..., $a^\alpha_{i_1} \dots i_{r+1}\,(t)$ and $\partial_{r+2}\, a^\alpha_{i_1} \dots$ $\dots i_{r+1}\,(t) = b^\alpha_{i_1} \dots i_{r+1} i_{r+2}$. Hence $b^\alpha_{i_1} \dots i_{r+1} i_{r+2}$ are symmetric in the first $r+1$ subscripts. Further, the elements of $K^2_m(K^r_m(U))$ transversal with respect to the corresponding fibering are identified with the elements of $J^2(J^r U)$, which implies $b^\alpha_{i_1} \dots i_{r+1} i_{r+2} = b^\alpha_{i_1} \dots i_{r+2} i_{r+1}$. Then $j^1_t\ \chi \in J^{r+2}U$ and $\varkappa(j^1_x[\varkappa[\varrho(y))]) \in K^{r+2}_m(M)$. The converse assertion is obvious.

We are now in a position to prove Proposition 4. According to the proof of Lemma 1, it is $\varkappa(C^{-1}(x)\,(\lambda^r)) = \lambda^{r+1}(x)$. Then Proposition 4 follows easily from Lemma 2 and Proposition 1, QED.

One can say that a generalized manifold with connection $\mathscr{S}$ is (s, r)-holonomic at x, $s \leqq r$, if it is strongly s-holonomic and weakly r-holonomic at x. We can also pose a natural question: What is the geometrical meaning of vanishing of the canonical projection of $\Omega(x)$ into $(\mathfrak{g}_x/\mathfrak{n}^r_x) \otimes \bigwedge^2 T^*_x(B)$ (where $\mathfrak{n}^r_x$ means the Lie algebra of N^r_x)? Since $\mathfrak{n}^r_x = \mathfrak{k}^r_x \cap \mathfrak{h}_x$, we deduce from Proposition 1 of [4] and from Proposition 4 the following

Corollary 1. *Assume that ν^{r-1} vanishes in a neighbourhood of $x \in B$. Then the canonical projection of $\Omega(x)$ into $(\mathfrak{g}_x/\mathfrak{n}^r_x) \otimes \bigwedge^2 T^*_x(B)$ vanishes if and only if $\mathscr{S}$ is $(2, r+2)$-holonomic at x.*

4. Let $w_r(F, m)$ be the minimum of the dimensions of the stability groups of the elements of $K^r_m(F)$. The smallest r satisfying $w_r(F, m) = 0$ will be denoted by $i_w(F, m)$ and will be called the index of isotropy of F with respect to m-dimensional contact elements or the weak m-index of isotropy of F. (It should be underlined that $i_s(F, m)$ is defined for arbitrary m, while $i_w(F, m)$ has a non-trivial meaning only for $m <$ $<\dim F$.) Let $D_2 \subset K^{r_2}_m(F)$, $r_2 = i_w(F, m)$, be the set of all elements whose stability group has dimension zero; one can easily see that D_2 is an open subset. We shall say that $\mathscr{S}$ is of general type of the second kind at x, if for some (and hence for every) $u \in P_x$ it is $u^{-1}(\lambda^{r_2}(x)) \in D_2$. By Corollary 1, we deduce

Proposition 5. *Let r_2 be the weak m-index of isotropy of F. If a generalized manifold with connection $\mathscr{S}$ of general type of the second kind is $(2, r_2+2)$-holonomic at every point, then its connection is integrable.*

5. As an example, we shall treat the case $F = A_3$(= the 3-dimensional affine space) and $\dim B = 2$. Such a manifold with connection $\mathscr{S}$ is locally equivalent to a surface of a 3-dimensional space with affine connection. To simplify the evaluations, we shall apply some convenient specializations of frames. It will be sufficient to use the simpliest case of such a specialization, which is based on the following well known fact. Let $P(B, G, \pi)$ be a principal fibre bundle, let G acts transitively on the left on a manifold M, let ϱ be a cross section of the associated fibre bundle (B, M, G, P) and let $p \in M$ be a point. Then

$$Q = \{u \in P;\ u^{-1}(\varrho(\pi(u))) = p\} \tag{2}$$

is a reduction of P to the stability group H of p. We shall say that Q is the reduction determined by the pair (ϱ, p). Let x^b be some local coordinates on M, let a^b be the coordinate functions of ϱ, see [6], and let x^b_0 be the coordinates of p. Then $Q \subset P$ is characterized by

$$a^b = x^b_0. \tag{3}$$

Further, let $X_\alpha = \xi^b_\alpha(x)\dfrac{\partial}{\partial x^b}$ be the coordinate expressions of the vector fields on M corresponding to a basis $\tilde{\omega}^\alpha$ of $\mathfrak{g}^*$, see [6]. Then the differential equations of the stability group H of p are

$$\xi^b_\alpha(x_0)\,\tilde{\omega}^\alpha = 0. \tag{4}$$

6. For the sake of simplicity, we shall first discuss the "weak" case and we shall use the method of investigation explained in [8] combined with some specializations of frames. Fix an affine coordinate system on A_3; this implies an identification of A_3 and $\mathbf{R}^3$. The fundamental section σ of $\mathscr{S}$ and the point $p_0 = (0, 0, 0)$ determine a reduction Q of P to a subgroup H of the fundamental group G of A_3. The differential equations of H are

$$\tilde{\omega}^i = 0, \qquad i, j, \ldots = 1, 2, 3. \tag{5}$$

where $\tilde{\omega}^i, \tilde{\omega}^i_j$ is the natural basis of $\mathfrak{g}^*$. The first weak development λ^1 of $\mathscr{S}$ can be considered as a cross section of $(B, K^1_{3,2}, H, Q)$. On $K^1_{3,2}$, there are natural local coordinates

$$y_p = y^3_p, \qquad p, q, \ldots = 1, 2. \tag{6}$$

Let ω be the restriction of the connection form to $\hat{Q}$, see [8], and let ω^i, ω^i_j be the components of ω. According to [8], the coordinate functions $a_p : \hat{Q} \to \mathbf{R}$ of λ^1 are determined by

$$\omega^3 = a_1\omega^1 + a_2\omega^2. \tag{7}$$

On $\hat{Q}$, it further holds

$$\begin{aligned} d\omega^i &= \omega^j \wedge \omega^i_j + R^i\omega^1 \wedge \omega^2, \\ d\omega^i_j &= \omega^k_j \wedge \omega^i_k + R^i_j\omega^1 \wedge \omega^2. \end{aligned} \tag{8}$$

Hence $\mathscr{S}$ is weakly 2-holonomic if and only if

$$R^3 = a_1R^1 + a_2R^2. \tag{9}$$

Let π^i_j be the restriction of $\tilde{\omega}^i_j$ to $H \subset G$. By [8], the equations of the fundamental distribution on $H \times K^1_{3,2}$ are

$$dy_p - y_q\pi^q_p - y_qy_p\pi^q_3 + y_p\pi^3_3 + \pi^3_p = 0. \tag{10}$$

Let $p_1 \in K^1_{3,2}$ be the point with the coordinates $y_1 = y_2 = 0$. The pair (λ^1, p_1) determines a reduction Q_1 of Q to a subgroup H_1 of H with the differential equations

$$\pi^3_p = 0. \tag{11}$$

Let ϖ^i, ϖ^i_j be the restriction of ω^i, ω^i_j to Q_1. By (3) and (7), it is

$$\varpi^3 = 0. \tag{12}$$

From now on, we shall suppose $\mathscr{S}$ is weakly 2-holonomic. Let ${}^2K \subset K^2_{3,2}$ be the subspace of all elements lying over p_1. Then λ^2 can be considered as a cross section of $(B, {}^2K, H_1, Q_1)$. On ${}^2K$, there are natural coordinates

$$y_{11}, y_{12}, y_{22}. \tag{13}$$

According to [8], the corresponding coordinate functions of λ^2 satisfy

(14) $$\bar{\omega}_p^3 = a_{pq}\bar{\omega}^q.$$

Hence $\mathscr{S}$ is weakly 3-holonomic if and only if it holds (on Q_1)

(15) $$R^3 = 0,\ R_p^3 = a_{pq}R^q.$$

By the standard procedure, cf. [8], we find the equations of the fundamental distribution on $H_1 \times {}^2K$ in the form

(16) $$\begin{aligned} &\mathrm{d}y_{11} - y_{11}(2\pi_1^1 - \pi_3^3) - 2y_{12}\pi_1^2 = 0,\\ &\mathrm{d}y_{12} - y_{12}(\pi_1^1 + \pi_2^2 - \pi_3^3) - y_{11}\pi_2^1 - y_{22}\pi_1^2 = 0,\\ &\mathrm{d}y_{22} - y_{22}(2\pi_2^2 - \pi_3^3) - 2y_{12}\pi_2^1 = 0,\end{aligned}$$

where π_i^p, π_3^3 are the restrictions of the corresponding π's to $H_1 \subset H$.

Denote by ${}^2K_h \subset {}^2K$ the subspace of all "hyperbolic" contact 2^2-elements on A_3 and assume that the values of λ^2 are hyperbolic. Choose the point $p_2 \in K_h$ with the coordinates $y_{11} = y_{22} = 0$, $y_{12} = 1$. According to § 5, the pair (λ^2, p_2) determines a reduction $Q_2 \subset Q_1$ to a subgroup $H_2 \subset H_1$ with the differential equations

(17) $$\pi_1^2 = 0, \qquad \pi_2^1 = 0, \qquad \pi_1^1 + \pi_2^2 - \pi_3^3 = 0.$$

Let $\bar{\bar{\omega}}^i$, $\bar{\bar{\omega}}_j^i$ be the restriction of $\bar{\omega}^i$, $\bar{\omega}_j^i$ to Q_2. By (3) and (14), we obtain

(18) $$\bar{\bar{\omega}}^3 = 0, \qquad \bar{\bar{\omega}}_1^3 = \bar{\bar{\omega}}^2, \qquad \bar{\bar{\omega}}_2^3 = \bar{\bar{\omega}}^1.$$

Further, let $\bar{\bar{\pi}}_1^1$, $\bar{\bar{\pi}}_2^2$, $\bar{\bar{\pi}}_3^1$, $\bar{\bar{\pi}}_3^2$ be the restrictions of the corresponding forms to $H_2 \subset H_1$. Let ${}^3K \subset K_{3,2}^3$ be the subspace of all elements lying over p_2. Suppose that $\mathscr{S}$ is weakly 3-holonomic. Then λ^3 can be considered as a cross section of $(B, {}^3K, H_2, Q_2)$. On ${}^3K$, there are natural coordinates

(19) $$y_{111},\ y_{112},\ y_{122},\ y_{222}.$$

By [8], the corresponding coordinate functions of λ^3 satisfy

(20) $$\begin{aligned} &-2\bar{\bar{\omega}}_1^2 = a_{111}\bar{\bar{\omega}}^1 + a_{112}\bar{\bar{\omega}}^2,\\ &\bar{\bar{\omega}}_3^3 - \bar{\bar{\omega}}_1^1 - \bar{\bar{\omega}}_2^2 = a_{112}\bar{\bar{\omega}}^1 + a_{122}\bar{\bar{\omega}}^2,\\ &-2\bar{\bar{\omega}}_2^1 = a_{122}\bar{\bar{\omega}}^1 + a_{222}\bar{\bar{\omega}}^2.\end{aligned}$$

Thus, $\mathscr{S}$ is weakly 4-holonomic if and only if it holds (on Q_2)

(21) $$\begin{aligned} &R^3 = 0,\ R_p^3 = 0,\ -2R_1^2 = a_{111}R^1 + a_{112}R^2,\\ &R_3^3 - R_1^1 - R_2^2 = a_{112}R^1 + a_{122}R^2,\ -2R_2^1 = a_{122}R^1 + a_{222}R^2.\end{aligned}$$

According to [8], we deduce the equations of the fundamental distribution on $H_2 \times$ $\times\ {}^3K$ in the form

(22) $$\begin{aligned} &\mathrm{d}y_{111} - y_{111}(2\bar{\bar{\pi}}_1^1 - \bar{\bar{\pi}}_2^2) = 0,\ \mathrm{d}y_{122} - y_{122}\bar{\bar{\pi}}_2^2 - 2\bar{\bar{\pi}}_3^1 = 0,\\ &\mathrm{d}y_{112} - y_{112}\bar{\bar{\pi}}_1^1 - 2\bar{\bar{\pi}}_3^2 = 0,\ \mathrm{d}y_{222} - y_{222}(2\bar{\bar{\pi}}_2^2 - \bar{\bar{\pi}}_1^1) = 0.\end{aligned}$$

One finds easily that the coordinates of an element of ${}^3K$ satisfy $y_{111} \neq 0 \neq y_{222}$ if and only if it is a contact element of the third order of a non-ruled surface. The stability group of such an element is characterized by

$$\overline{\overline{\pi}}{}_1^1 = \overline{\overline{\pi}}{}_2^2 = \overline{\overline{\pi}}{}_3^1 = \overline{\overline{\pi}}{}_3^2 = 0, \tag{23}$$

i.e. its dimension is zero. Hence the weak 2-index of isotropy of A_3 is equal to 3 and a hyperbolic contact 2^3-element is of general type if and only if it is a contact element of a non-ruled surface. By Proposition 5, we obtain.

Proposition 6. *If a hyperbolic non-ruled manifold $\mathcal{S}$ without the strong torsion of order 0 is weakly 5-holonomic at every point, then its connection is integrable.*

Remark 2. An analogous treatment ot the case $F = P_3$ and dim $B = 2$ is, in fact, carried out in [2]. However, our investigations in [2] were based only on some computational analogies with the "flat" case. On the contrary, in the present paragraph, we intended to justify every step of our evaluations in all details.

7. To the "strong" case, we shall apply some results of [7]. The natural equations of the fundamental distribution on $G \times A_3$ are

$$dx^i + x^j\tilde{\omega}_j^i + \tilde{\omega}^i = 0. \tag{24}$$

Let $\bar{\omega} : T(P) \to \mathfrak{g}$ be the connection form and let $\bar{\omega}^i$, $\bar{\omega}_j^i$ be the components of $\bar{\omega}$. It holds

$$d\bar{\omega}^i = \bar{\omega}^j \wedge \bar{\omega}_j^i + \mathbf{D}\bar{\omega}^i, \tag{25}$$

$$d\bar{\omega}_j^i = \bar{\omega}_j^k \wedge \bar{\omega}_k^i + \mathbf{D}\bar{\omega}_j^i.$$

Let $\bar{a}^i$ be the coordinate functions of the fundamental section σ of $\mathcal{S}$. By [4], $\mathcal{S}$ is strongly 2-holonomic if and only if

$$\bar{a}^j\mathbf{D}\bar{\omega}_j^i + \mathbf{D}\bar{\omega}^i = 0. \tag{26}$$

Using an algorithm of [7], we deduce the equations of the fundamental distribution on $(G \times L_2^1) \times T_2^1(A_3)$ in the form (24) and

$$dx_p^i - x_q^i\hat{\omega}_p^q + x_p^j\tilde{\omega}_j^i = 0, \tag{27}$$

where $\hat{\omega}_q^p$ is the natural basis of $\mathfrak{l}_2^{1*}$. In particular, the equations of the fundamental distribution on $G \times T_2^1(A_3)$ are (24) and

$$dx_p^i + x_p^j\tilde{\omega}_j^i = 0. \tag{28}$$

Let φ be the canonical form of $H^1(B)$, let p_1, p_2 be the product projections of $H^1(B) \oplus \oplus P$, let $\varphi = p_1^*\bar{\varphi}$, $\omega = p_2^*\bar{\omega}$ and let φ^p, ω^i, ω_j^i be the components of these forms. Then we have

$$d\omega^i = \omega^j \wedge \omega_j^i + R^i\varphi^1 \wedge \varphi^2, \tag{29}$$

$$d\omega_j^i = \omega_j^k \wedge \omega_k^i + R_j^i\varphi^1 \wedge \varphi^2.$$

Let a^i, $a^i : H^1(B) \oplus P \to \mathbf{R}$ be the coordinate functions of σ^1. (According to [7], they are determined by $a^i = \beta^*\bar{a}^i$ and by

$$da^i + a^j\omega_j^i + \omega^i = a_p^i\varphi^p.) \tag{30}$$

By Proposition 2, $\mathcal{S}$ is strongly 3-holonomic if and only if

$$a^jR_j^i + R^i = 0, \qquad a_p^jR_j^i = 0. \tag{31}$$

In the same way as before, we deduce the equations of the fundamental distribution on $G \times T^2_2(A_3)$ in the form (24), (28) and

$$\mathrm{d}x^i_{pq} + x^j_{pq}\tilde{\omega}^i_j = 0. \tag{32}$$

The group G acts transitively on the set of all regular 2^1-velocities on A_3. Let $p_1 \in T^1_2(A_3)$ be the element with the coordinates $x^i = 0$, $x^p_q = \delta^p_q$, $x^3_p = 0$. The differential equations of the stability group $H_1 \subset G$ of p_1 are

$$\tilde{\omega}^i = 0,\ \tilde{\omega}^p_q = 0,\ \tilde{\omega}^3_p = 0. \tag{33}$$

Denote by ${}^2T \subset T^2_2(A_3)$ the subspace of all elements lying over p_1. By (4) and (33), the equations of the stability group $H_2 \subset H_1$ of an element of ${}^2T$ are (33) and

$$x^3_{pq}\tilde{\omega}^i_3 = 0. \tag{34}$$

If at least one of the coordinates x^3_{pq} is different from zero, then $\tilde{\omega}^i_3 = 0$ and dim $H_2 = 0$; such a 2^2-velocity will be said to be non-planar. Analogously, a 2-jet X of B into A_3 with source x will be said to be non-planar, if the 2^2-velocity XY is non-planar for some (and hence for every) $Y \in H^2_x(B)$. Thus, we have deduced that the strong 2-index of isotropy of A_3 is equal to 2 and that a 2-jet of B into A_3 is of general type if and only if it is non-planar. By Proposition 3, we obtain.

Proposition 7. *If the second development of $\mathcal{S}$ is non-planar and if $\mathcal{S}$ is strongly 4-holonomic at every point, then its connection is integrable.*

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