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TWO ELEMENTARY REMARKS ON NON-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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To Professor O. Borůvka, on his 70th birthday

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Here two remarks are made concerning the differential equation

$$(1) \qquad (p(x)\, y')' + q(x) f(y)\, h(p(x)\, y') = 0, \qquad a < x < b,$$

where $p(x) > 0$, $f(y)$ is of the same sign as y, $h(x) > 0$, and for which a uniqueness theorem for the initial value problem holds ((i.e., the conditions $y(\eta) = y'(\eta) = 0$, $a < \eta < b$, η a fixed value, imply that $y(x) \equiv 0$, $a < x < b$). Theorem 1′ of [1] provides such a uniqueness theorem.

The equation (1) is a generalization of the classic self-adjoint linear differential equation

$$(2) \qquad (py')' + q(x)\, y = 0, \qquad p(x) > 0,$$

and, *a fortiori*, of

$$(3) \qquad y'' + q(x)\, y = 0.$$

To the study of equation (3), Professor O. Borůvka and his many pupils have made numerous contributions. His recent book [2] describes this work. This note, however, deals with a different type of problem.

Recently [3], there was occasion to make two remarks concerning equation (3) in order to establish some minor properties of Bessel functions, Jacobi polynomials and Laguerre polynomials. These are the remarks which are extended here to equation (1).

(I) *If* $q(x) > 0$, *then the zeros and extremum points of any non-trivial solution of* (1) *are interlaced.*

Proof. Clearly, no zero can be an extremum point, from the uniqueness condition. It remains to show only that each positive extremum is a maximum, each negative extremum a minimum.

Let $((\xi, y(\xi))$ be a positive extremum. Then $y(x) > 0$, $\xi \leqq x \leqq \xi + \delta$, for some $\delta > 0$, since $y(x)$ is continuous, being differentiable. From equation (1) it follows that $(p(x)\, y')' < 0$ when $y(x) > 0$, in particular for $\xi \leqq x \leqq \xi + \delta$. Hence $p(x)\, y'(x)$ decreases in the closed interval

$[\xi, \xi + \delta]$. But $p(\xi)\, y'(\xi) = 0$ so that $p(x)\, y'(x) < 0$ for $\xi < x \leqq \xi + \delta$. Consequently, $y'(x) < 0$ in $[\xi, \xi + \delta]$, since $p(x) > 0$, and so $(\xi, y(\xi))$ is a maximum, as asserted.

The corresponding observation for negative extrema can be established similarly, and so (I) is proved. The other remark is a partial converse of (I).

(II) *If $y(x)$ is a solution of* (1) *with the property that its zeros and extremum points are interlaced, and if $q(x)$ is monotonic (either nondecreasing or non-increasing), then $q(x) > 0$ throughout an open interval containing all extremum points of $y(x)$.*

Proof. Under these hypotheses, all relative maxima are positive, all relative minima are negative. Let $(\xi, y(\xi))$ be an extremum (taken to be positive with no loss of generality).

The case in which $q(x)$ is non-decreasing is considered first. If there exists an $x_0 < \xi$ for which $q(x_0) > 0$, then the non-decreasing character of $q(x)$ implies that $q(x) > 0$ for all $x > x_0$, i.e., in an open interval including ξ and all subsequent extremum points, as is to be proved.

Suppose now that $q(x) \leqq 0$, $a < x < \xi$. From equation (1) it follows that $(py')' \geqq 0$, $a < x < \xi$, so that $p(x)\, y'(x)$ is non-decreasing in (a, ξ). But $p(\xi)\, y'(\xi) = 0$, and so $p(x)\, y'(x) \leqq 0$, $a < x < \xi$, whence $y'(x) \leqq 0$, $a < x < \xi$, since $p(x) > 0$.

If $y'(x) \equiv 0$ in $(\xi - \delta, \xi)$ for some $\delta > 0$, $y(x)$ would be constant in $(\xi - \delta, \xi)$ and, consequently, the zeros and extremum points of $y(x)$ would not interlace there, contrary to the hypothesis.

Thus, $y'(x) < 0$ for at least one point in the interval $(\xi - \delta, \xi)$ for all $\delta > 0$. Hence $y(x)$ has points of decrease in each such interval.

But this is impossible, since $(\xi, y(\xi))$ is a maximum. Hence $q(x)$ must become positive for some $x_0 < \xi$ and, being non-decreasing, must remain positive for all $x > x_0$.

This completes the proof for $q(x)$ non-decreasing. If $q(x)$ is non-increasing, the argument is similar, with the interval $(\xi, \xi + \delta)$ playing the role assumed above by the interval $(\xi - \delta, \xi)$, and the interval (ξ, b) replacing the interval (a, ξ).

Comments

(a) It is not necessary to assume the presence of a uniqueness theorem for the purposes of Remark (II). There the interlacing hypothesis, alone, implies that a zero cannot coincide with an extremum point, so that the uniqueness condition stipulated holds vacuously.

(b) The remarks in [3] were related to Sonin's theorem which infers the monotonicity of the magnitude of the extrema of solutions of (**3**). The corresponding result for (2) is the Sonin-Pólya theorem. An extension to equation (1) was obtained by I. Bihari (1, Theorem 1′).

REFERENCES

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