Archivum Mathematicum

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Archivum Mathematicum, Vol. 5 (1969), No. 2, 85--92

Persistent URL: http://dml.cz/dmlcz/104685

OPTIMAL PROBLEMS IN DISTRIBUTED PARAMETER CONTROL SYSTEMS I

by

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To Professor OTAKAR BORŮVKA, whose research work and character are worth to be admired everywhere, on his 70th birthday

(Received January 31, 1969)

INTRODUCTION

Two recent sets of authors' papers, partially elaborated in collaboration with L. E. Krivoshein [3], [4], devoted respectively to boundary problems for integro-differential equations with caloric operators and retarded arguments and to polylocal ordinary integro-differential systems, are to be related with O. Borůvka's research work, partially published in the ,,Bulletin of the Polytechnic Institute of Jassy" and subsequently continued there by some of his disciples, as well as with Minoru Urabe's and Setsuzo Yosida's papers published in the Professor Masuo Hukuhara's valuable journal [5], [6]. While our own research work, related with ,,polyvibrating" or draftsman's equations[1]) with the prototype the boundary value problem

$$[A(x)\,u' + \lambda B(x)\,u]' + \lambda[B(x)\,u' + C(x)\,u] = 0,$$
$$\text{boundary of } R = 0, \quad R = [a_i \leqq x_i \leqq b_i]\ (i = 1, 2, \ldots, m),$$
$$x = (x_1, x_2, \ldots, x_m),$$

or the corresponding variational problem of the minimum value of the functional

$$D(f) = \int_R A(x)\,f'^2(x)\,\mathrm{d}x,$$

subject to the condition that

$$H(f) = \int_R (2B(x)\,f(x)\,f'(x) + C(x)\,f^2(x))\,\mathrm{d}x = +1 \text{ (or } -1)$$

and $f = 0$ on the boundary of R, where we have considered various problems pertaining to functional equations in the theory of polyvibrating systems [7]—[10], is strongly related with the aims and the

[1]) The polyvibrating equations were called by various scientists ,,Mangeron's equations" [15]—[18].

targets of the „Funkcialaj Ekvaciaj" and the „Equationes Mathematicae" journals.

The novel aspects of our theorems lies in the interpretation of R as a m-dimensional rectangular domain and the symbol $'$ as designating „total differentiation" in the sense that $u' = \partial^m u/(\partial x_1 \dots \partial x_m)$ (Picone's sense [11]). Completely new are to be considered also various results pertaining to majoration formulas, to generalized Green's, Hadamard's, Janet's a.o. inequalities [12], [13] as well as to eigenexpansion theory [14].

In what follows an optimization problem concerning a distributed parameter system is discussed while in a subsequent one a basic optimization problem concerning a polyvibrating distributed parameter system will be solved, and the results of the paper [2] are extended.

I. DESCRIPTION OF THE CONTROL SYSTEM

Let S be a given 1-dimensional control system defined on the interval $J \equiv \{x \mid a \leqq x \leqq b\}$. Let $I = [t_0, t_1]$ be a given time interval. We assume that the state of the system S at the time $t \in I$ is described by $u = u(t, x; v)$, an ordinary function in (t, x) and a functional in $v \in V$, where V is a set in a given set of functions continuous on $I \times J$. Specifically, we assume that

$$
\text{(I.1)} \qquad \begin{aligned} u(t, x; v) = {} & \varphi(t, x) + \int_{t_0}^{t_1}\int_a^b K_1(t, x; \sigma, \zeta)\, d\sigma\, d\zeta + \\ & + \frac{\lambda}{2!}\int_{t_0}^{t_1}\int_a^b\int_{t_0}^{t_1}\int_a^b K_2(t, x; \sigma_1, \zeta_1; \sigma_2, \zeta_2) \times \\ & \times v(\sigma_1, \zeta_1)\, v(\sigma_2, \zeta_2)\, d\sigma_1\, d\zeta_1\, d\sigma_2\, d\zeta_2 + \\ & + \dots + \\ & + \frac{\lambda^{n-1}}{n!}\int_{t_0}^{t_1}\int_a^b \dots \int_{t_0}^{t_1}\int_a^b K_n(t, x; \sigma_1, \zeta_1; \dots; \sigma_n, \zeta_n) \times \\ & \times v(\sigma_1, \zeta_1) \dots v(\sigma_n, \zeta_n)\, d\sigma_1\, d\zeta_1 \dots d\sigma_n\, d\zeta_n + \dots, \\ & + \dots, \end{aligned}
$$

where λ is a real parameter, $\varphi(t, x)$ is a given function continuous and continuously differentiable in $(t, x) \in I \times J$, $K_n(t, x; \sigma_1, \zeta_1; \dots; \sigma_n, \zeta_n)$ $(n = 1, 2, 3, \dots)$ are given functions continuous in $t, x, \sigma_1, \zeta_1, \dots, \sigma_n, \zeta_n$, and continuously differentiable in t and x for (t, x), (σ_1, ζ_1), $\dots$, (σ_n, ζ_n).

The series in (I.1) as well as the series corresponding to $u_t = \frac{\partial u}{\partial t}$ and $u_x = \frac{\partial u}{\partial x}$ are supposed to be uniformly convergent for $(t, x) \in I \times J$ and $|\lambda| \leqq R$, R being a positive number. We also assume that the kernels $K_n(t, x; \sigma_1, \zeta_1; \ldots; \sigma_n, \zeta_n)$ are symmetric with respect to $(\sigma_1, \zeta_1), \ldots, (\sigma_n, \zeta_n)$. Any function $v = v(t, x)$ will be called an admissible control.

II. OPTIMAL PROBLEM

We assume that the performance of the system S under the control $v \in V$ is measured by a functional of the form

$$\text{(II.1)} \qquad F(u, v, u_t, u_x) = \int_{t_0}^{t_1}\int_{a}^{b} Q(u(t, x), v(t, x), u_t(t, x), u_x(t, x), t, x)\, \mathrm{d}t\, \mathrm{d}x,$$

where $Q = Q(u, v, u_t, u_x, t, x)$ is a given non-negative function, continuous in t and x, continuously differentiable in u, v, u_t and u_x for $(t, x) \in I \times J$, $v \in V$, $u = u(t, x; v)$ being given by Eq. (I.1). Then, our optimization problem can be formulated as follows:

Optimization Problem. Under the above assumptions, find an admissible control $v^0 \in V$ for which the functional $F(u, v, u_t, u_x)$ assumes its minimum; i.e.,

$$\text{(II.2)} \qquad F(u^0, v^0, u_t^0, u_x^0) = \min_{v \in V} F(u, v, u_t, u_x),$$

where $u \equiv u(t, x; v)$ and $u^0 \equiv u(t, x; v^0)$.

Any $v^0 \in V$ for which Eq. (II.2) holds, will be called an **optimal control.** The existence of an optimal control can be demonstrated under very general conditions on the set V. Here we shall deal only with the necessary conditions for optimality of an admissible control v, using a dynamic programming approach.

III. APPLICATION OF THE PRINCIPLE OF OPTIMALITY

To emphasize the fact that the minimization operation in (II.2) with respect to $v \in V$ is performed on the interval $I = [t_0, t_1]$, we introduce the notation

$$f(t_0, t_1) = \min_{v \in V} F(u, v, u_t, u_x). \tag{III.1}$$

More generally, let τ be any point in I, put $I_\tau = [\tau, t_1]$, and define

$$f(\tau, t_1) = \min_{v \in V} \left\{ \int_\tau^{t_1} \int_a^b Q(u(t, x), v(t, x), u_t(t, x), u_x(t, x), t, x) \, \mathrm{d}t \, \mathrm{d}x \right\}. \tag{III.2}$$

Let Δ be a small discrete t-interval and let us divide the interval I_τ into two parts, namely $I''_\tau \equiv [\tau, \tau + \Delta]$ and $I_{\tau+\Delta} = [\tau + \Delta, t_1]$. Making use the hypotheses assumed in § I—II, we find

$$\begin{aligned} f(\tau, t_1) &= \min_{v \in V} \left\{ \int_\tau^{\tau+\Delta} \int_a^b Q(u, v, u_t, u_x, t, x) \, \mathrm{d}t \, \mathrm{d}x + \right. \\ &\quad \left. + \int_{\tau+\Delta}^{t_1} \int_a^b Q(u, v, u_t, u_x, t, x) \, \mathrm{d}t \, \mathrm{d}x \right\} = \\ &= \min_{v \in V} \left\{ \Delta \int_a^b Q(u(\Theta, x), v(\Theta, x), u_t(\Theta, x), u_x(\Theta, x), \Theta, x) \, \mathrm{d}x + \right. \\ &\quad \left. + \int_{\tau+\Delta}^{t_1} \int_a^b Q(u(t, x), v(t, x), u_t(t, x), u_x(t, x), t, x) \, \mathrm{d}t \, \mathrm{d}x \right\}, \end{aligned} \tag{III.3}$$

where $\tau \leqq \Theta \leqq \tau + \Delta$; and, applying here the principle of optimality of R. E. Bellman [1], we obtain

$$\begin{aligned} f(\tau, t_1) = f(\tau + \Delta, t_1) + \Delta \min_{v \in V} \left\{ \int_a^b Q(u(\Theta, x), v(\Theta, x), \right. \\ \left. u_t(\Theta, x), u_x(\Theta, x), \Theta, x) \, \mathrm{d}x \right\}, \end{aligned} \tag{III.4}$$

from which the following functional equation is obtained:

$$\begin{aligned} -\frac{\partial}{\partial \tau} f(\tau, t_1) = \min_{v \in V} \left\{ \int_a^b Q(u(\tau, x), v(\tau, x), u_t(\tau, x), \right. \\ \left. u_x(\tau, x), \tau, x) \, \mathrm{d}x \right\}. \end{aligned} \tag{III.5}$$

Thus, optimal controls are among the controls $v \in V$ for which the functional

$$G(v; \tau) = -\int_a^b Q(u(\tau, x), v(\tau, x), u_t(\tau, x), u_x(\tau, x), \tau, x) \, \mathrm{d}x \tag{III.6}$$

assumes its minimum for each $\tau \in I$.

IV. NECESSARY CONDITIONS FOR OPTIMALITY

Let $v \in V$ be an extreme point for the functional $G(v;\, \tau)$. $\tau \in I$. Then, we have

(IV.1) $$\delta_v G(v;\, \tau) = 0,$$

where $\delta_v G$ denotes the first variation of the functional G with respect to v. Further,

$$\delta_v G(v;\, \tau) = -\int_a^b \left\{ \frac{\partial Q}{\partial u}\, \delta_v u + \frac{\partial Q}{\partial v}\, \delta_v + \frac{\partial Q}{\partial u_t}\, \delta_v u_t + \frac{\partial Q}{\partial u_x}\, \delta_v u_x \right\} \mathrm{d}x$$

for any $\tau \in I$, where $Q \equiv Q(u(\tau, x),\ v(\tau, x),\ u_t(\tau, x),\ u_x(\tau, x),\ \tau,\ x)$ and $\delta v \equiv \delta v(t, x)$ is an arbitrary function continuous for $(t, x) \in I \times J$ such that $v + \delta v \in V$. Thus, according to Eq. (IV.1), optimal controls are among the functions $v \in V$ for which the equality

(IV.2) $$\int_a^b \left\{ \frac{\partial Q}{\partial u}\, \delta_v u + \frac{\partial Q}{\partial v}\, \delta_v + \frac{\partial Q}{\partial u_t}\, \delta_v u_t + \frac{\partial Q}{\partial u_x}\, \delta_v u_x \right\} \mathrm{d}x = 0$$

holds for any $\tau \in I$ and for any arbitrary admissible increment δ_v. Clearly, Eq. (IV.1) is only a necessary condition for an extreme point $v \in V$ for $G(v;\, \tau)$.

Now, since $u = u(t, x;\, v)$ is given by Eq. (I.1), we have

(IV.3) $$\delta_v u(t, x;\, v) = \int_{t_0}^{t_1}\int_a^b K(t, x;\, \sigma, \zeta)\, \delta_v(\sigma, \zeta),$$

where

(IV.4) $$K(t, x;\, \sigma, \zeta) = K_1(t, x;\, \sigma, \zeta) +$$

$$+ \frac{\lambda}{1!} \int_{t_0}^{t_1}\int_a^b K_2(t, x;\, \sigma_1, \zeta_1;\, \sigma, \zeta)\, v(\sigma_1, \zeta_1)\, \mathrm{d}\sigma_1\, \mathrm{d}\zeta_1 +$$

$$+ \frac{\lambda^2}{2!} \int_{t_0}^{t_1}\int_a^b\int_{t_0}^{t_1}\int_a^b K_3(t, x;\, \sigma_1, \zeta_1;\, \sigma_2, \zeta_2;\, \sigma, \zeta)$$

$$\times\ v(\sigma_1, \zeta_1)\, v(\sigma_2, \zeta_2)\, \mathrm{d}\sigma_1\, \mathrm{d}\zeta_1\, \mathrm{d}\sigma_2\, \mathrm{d}\zeta_2 +$$

$$+ \ldots$$

$$+ \frac{\lambda^n}{n!} \int_{t_0}^{t_1}\int_a^b \cdots \int_{t_0}^{t_1}\int_a^b K_{n+1}(t, x;\, \sigma_1, \zeta_1;\, \ldots;\, \sigma_n, \zeta_n;\, \sigma, \zeta) \times$$

$$\times\ v(\sigma_1, \zeta_1) \ldots v(\sigma_n, \zeta_n)\, \mathrm{d}\sigma_1\, \mathrm{d}\zeta_1 \ldots \mathrm{d}\sigma_n\, \mathrm{d}\zeta_n + \ldots$$

which, by the hypotheses of § I, is continuous in t, x, σ, ζ and continuously differentiable in t and x for (t, x), $(\sigma, \zeta) \in I \times J$. Hence

(IV.5) $$\delta_v u_t(t, x) = \int_{t_0}^{t_1}\int_a^b K_t(t, x; \sigma, \zeta)\, \delta v(\sigma, \zeta)\, d\sigma\, d\zeta,$$

and

(IV.6) $$\delta_v u_x(t, x) = \int_{t_0}^{t_1}\int_a^b K_x(t, x; \sigma, \zeta)\, \delta v(\sigma, \zeta)\, d\sigma\, d\zeta,$$

for $(t, x) \in I \times J$. Then, combining Eqs. (IV.2), (IV.3), (IV.5) and (IV.6), and changing the order of integration suitably, we find

(IV.7) $$\int_a^b \frac{\partial Q}{\partial v}\, \delta v(\tau, x)\, dx + \int_{t_0}^{t_1}\int_a^b H(\tau; \sigma, \zeta)\, \delta v(\sigma, \zeta)\, d\sigma\, d\zeta = 0$$

for $\tau \in I$, where

(IV.8) $$H(\tau; \sigma, \zeta) = \int_a^b \left\{ \frac{\partial Q}{\partial u} K(\tau, x; \sigma, \zeta) + \frac{\partial Q}{\partial u_t} K_t(\tau, x; \sigma, \zeta) + \frac{\partial Q}{\partial u_x} K_x(\tau, x; \sigma, \zeta) \right\} dx.$$

It can be shown that Eq. (IV.7) can hold for any continuous $\delta v(t, x)$ if and only if

(IV.9) $$\frac{\partial Q}{\partial v} = 0$$

and

$$H(\tau; \sigma, \zeta) = 0$$

for $\tau \in I$, $(\sigma, \zeta) \in I \times J$ (cf. [2]). Thus,

Optimal controls are among the admissible controls v satisfying the equations in (IV.9), simultaneously.

V. AN EXAMPLE

Suppose that the state of the system S at (t, x) is described by

(V.1) $$u(t, x) = \int_{t_0}^{t_1}\int_a^b K(t, x; \sigma, \zeta)\, v(\sigma, \zeta)\, d\sigma\, d\zeta,$$

where $K(t, x; \sigma, \zeta)$ is symmetric with respect to (t, x) and (σ, ζ), and satisfies the general conditions of § I. Let the cost functional F associated with the system S be the following quadratic functional

$$(V.2)\qquad F(u, v, u_t, u_x) = \int_{t_0}^{t_1}\int_a^b \{u^2 - 2uv + \mu v^2 + u_t^2 + u_x^2\}\, dt\, dx,$$

where μ is a real number greater than 1. We assume that $v(t, x) \equiv 0$ is not admissible. In this case we have

$$(V.3)\qquad \int_{t_0}^{t_1}\int_a^b K(t, x; \sigma, \zeta)\, v(\sigma, \zeta)\, d\sigma\, d\zeta = \mu v(t, x),$$

by the first equation of (IV.9), and

$$(V.4)\qquad \int_a^b \{(\mu - 1)\, v(\tau, x)\, K(\tau, x; \sigma, \zeta) + u_t(\tau, x)\, K_t(\tau, x; \sigma, \zeta) + u_x(\tau, x)\, K_x(\tau, x; \sigma, \zeta)\}\, dx = 0,$$

for $\tau \in I$, $(\sigma, \zeta) \in I \times J$, by the second equation of (IV.9). Hence, optimal controls are among the eigenfunctions of the integral operator T

$$(V.5)\qquad T(v)\,(t, x) = \int_{t_0}^{t_1}\int_a^b K(t, x; \sigma, \zeta)\, v(\sigma, \zeta)\, d\sigma\, d\zeta,$$

for which Eq. (V.4) helds.

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