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A CHARACTERIZATION OF OSCULATING MAPS

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In this paper we characterize osculating maps of higher order of a differentiable map $V_n \to \tilde{V}_m$, V_n being a simply connected manifold and $\tilde{V}_m$ an affine space or a Lie group.

In the following, all manifolds, maps, vector bundles and their sections, respectively are supposed to be differentiable of class C^∞.

Let V_n be a manifold of dimension n. For $p \in V_n$ let f be a local function at p such that $f(p) = 0$. Then a k-jet $j_p^k(f)$ is called *a covelocity of order k on V_n at the point p*. Let $T^{k*}(V_n)_p$ be a vector space of all covelocities of order k at p. Each linear form $X_p^{(k)}$ on $T^{k*}(V_n)_p$ is called *a vector of order k at p*. The set of all $X_p^{(k)}$ is a vector space $T_k(V_n)_p$. We put $T_k(V_n) = \bigcup_{p \in V_n} T_k(V_n)_p$.

For any k, $T_k(V_n)$ is naturally a vector bundle over V_n and $T_1(V_n) = T(V_n)$ is the tangent bundle of V_n. (See [1], [3].)

Each vector $X_p^{(k)} \in T_k(V_n)$ is a linear differential operator on V_n and, with respect to a local coordinate system $(u_1, \ldots, u_n)$ at p, it is represented uniquely in the form

$$X_p^{(k)} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial u^i} + \sum_{1 \leqq i \leqq j \leqq n} a_{ij} \frac{\partial^2}{\partial u^i \partial u^j} + \ldots + \sum_{1 \leqq i_1 \leqq \ldots \leqq i_k \leqq n} a_{i_1 \ldots i_k} \frac{\partial^k}{\partial u^{i_1} \ldots \partial u^{i_k}}. \tag{1}$$

For any sequence of indices $0 \leqq i_1 \leqq \ldots \leqq i_k \leqq n$, $i_k > 0$, we can introduce an operator $\frac{\partial^k}{\partial u^{i_1} \ldots \partial u^{i_k}}$ putting inductively: $\frac{\partial^{l+1}}{\partial u^0 \partial u^{j_1} \ldots \partial u^{j_l}} = \frac{\partial^l}{\partial u^{j_1} \ldots \partial u^{j_l}}$ for each $l < k$, $0 \leqq j_1 \leqq \ldots \leqq j_l \leqq n$, $j_l > 0$. Then (1) takes a simple form

$$X_p^{(k)} = \sum_{\substack{0 \leqq i_1 \leqq \ldots \leqq i_k \leqq n \\ i_k > 0}} a_{i_1 \ldots i_k} \frac{\partial^k}{\partial u^{i_1} \ldots \partial u^{i_k}}. \tag{1'}$$

In a coordinate neighbourhood $U \subset V_n$, the operators $\frac{\partial^k}{\partial u^{i_1} \ldots \partial u^{i_k}}$, $0 \leqq i_1 \leqq \ldots \leqq i_k \leqq n$, $i_k > 0$, form a basis of $T_k(V_n)_q$ for each $q \in U$.

On the other hand, any vector $X_p^{(k+1)} \in T_{k+1}(V_n)$ may be written in the form

$$X_p^{(k+1)} = \sum_{i=1}^{r} X_{i,p} X_i^{(k)}, \tag{2}$$

where $X_i^{(k)}$ are suitable local sections of $T_k(V_n)$ defined at p, and $X_{i,p} \in T(V_n)_p$, $i = 1, 2, \ldots, r$. For any $l \leqq k$ we have a canonical injection $I_{l,k}\colon T_l(V_n) \to T_k(V_n)$.

Following P. Libermann, *a symmetric surconnection* S_k *of order* k *on* V_n is a bundle homomorphism $S_k\colon T_{k+1}(V_n) \to T(V_n)$ such that $S_k \circ I_{1,k+1}\colon T(V_n) \to T(V_n) =$ the identity map. (See [2].) It is easy to check that a symmetric surconnection $S_1\colon T_2(V_n) \to T(V_n)$ is an ordinary linear connection ∇ on V_n the torsion form of which vanishes. (See [3], p. 158.) Further, the successive interations ∇^k of ∇ $(k = 1, 2, \ldots)$ determine a sequence of symmetric surconnections S_k of orders $k = 1, 2, \ldots$, if, and only if, the curvature form of ∇ vanishes, too. In this case, we can define the sequence $\{S_k\}$ by induction: for $X_p^{(k+1)} \in T_{k+1}(V_n)$, $X_p^{(k+1)} = \sum_{i=1}^{r} X_{i,p} X_i^{(k)}$, we put

$$S_k X_p^{(k+1)} = \sum_{i=1}^{r} \nabla_{X_{i,p}}(S_{k-1} X_i^{(k)}), \tag{3}$$

It must be shown that (3) does not depend on a representation of $X_p^{(k+1)}$ in the form (2). But this is just guaranteed by vanishing of both torsion and curvature forms of ∇. (The proof is routine nad will be omitted.) For $l \leqq k$, S_k is a prolongation of S_l, i.e., $S_l = S_k \circ I_{l+1,k+1}$ on $T_{l+1}(V_n)$.

Note 1. If the curvature of ∇ is non-zero, the successive iterations ∇^k define a sequence $\{\bar{S}_k\}$ of *semiholonomic surconnections*; see [2].

Note 2. On a paracompact V_n, there are symmetric surconnections of any order k. In fact, we can construct such a surconnection on each coordinate neighbourhood of V_n and then use a C^∞-partition of unity subjected to a locally finite atlas of V_n.

A differentiable map $\varphi\colon V_n \to \tilde{V}_m$ induces canonically a sequence $\{T_k(\varphi) : T_k(V_n) \to T_k(\tilde{V}_m)\}$ of bundle morphisms such that all the diagrams

$$\begin{array}{ccc} T_k(V_n) & \xrightarrow{T_k(\varphi)} & T_k(\tilde{V}_m) \\ \pi_k \downarrow & \varphi & \tilde{\pi}_k \downarrow \\ V_n & \longrightarrow & \tilde{V}_m \end{array}$$

are commutative, $k = 1, 2, \ldots$ Let be now $\tilde{V}_m = A^m$, an affine space of dimension m. Let us denote by W^m the associated vector space of A^m and for each $x \in A^m$ let $\omega_x : T(A^m)_x \to$

$\to W^m$ be the canonical isomorphism. The maps ω_x, $x \in A^m$, determine a vector form ω on A^m, $\omega : T(A^m) \to W^m$.

In A^m, there is a canonical flat connection ∇. Its successive iterations determine a canonical sequence $\{S_k\}$ of symmetric surconnections on A^m. In a linear coordinate system $(x_1, \ldots, x_m)$ of A^m, each S_k, $k \geqq 1$, may be reprezented as follows: for $X_p^{(k+1)} \in T_{k+1}(A^m)$, $X_p^{(k+1)} = \sum a_{i_1 \ldots i_{k+1}} \cdot \cdot \frac{\partial^{k+1}}{\partial x^{i_1} \ldots \partial x^{i_{k+1}}}$, $0 \leqq i_1 \leqq \ldots \leqq i_{k+1} \leqq m$, $i_{k+1} > 0$ we have $S_k(X_p^{(k+1)}) = = \sum_{i=1}^{m} a_{0 \ldots 0i} \frac{\partial}{\partial x^i} =$ the first order part of $X_p^{(k+1)}$. Fork $k = 0$ we put S_0: $T(A^m) \to T(A^m) =$ the identity map. Let φ: $V_n \to A^m$ be a smooth map. For any $k \geqq 1$, we shall denote by φ_k^*: $T_k(V_n) \to W^m$ the compositions of maps of the sequence

$$T_k(V_n) \xrightarrow{T_k(\varphi)} T_k(A^m) \xrightarrow{S_{k-1}} T(A^m) \xrightarrow{\omega} W^m. \tag{4}$$

We can see that any φ_k^* is a composition of a bundle morphism $\tilde{\varphi}_k$: $T_k(V_n) \to V_n \times W^m$ and a canonical projection pr_2: $V_n \times W^m \to \to W^m$. In the regular case there is an index s such that $\tilde{\varphi}_s$ is a bundle epimorphism. If $(f_1, \ldots, f_m)$ is a basis of W_m corresponding to a linear coordinate system $(x^1, \ldots, x^m)$, we have

$$\varphi_k^*(X_p^{(k)}) = \sum_{i=1}^{m} [X_p^{(k)}(x^i \circ \varphi)] \cdot f_i . \tag{5}$$

For any $l > k$, $\varphi_l^* = \varphi_k^*$ holds on the bundle $T_k(V_n)$ and hence it is possible to omit k. From (5) we obtain immediately

$$\varphi^*(X_p X^{(k)}) = X_p \varphi^*(X^{(k)}). \qquad (k = 1, 2, \ldots) \tag{6}$$

(Here $\varphi^*(X^{(k)})$ is to be considered as a local vector function on V_n with values in W^m.) Therefore, if $\varphi^*(X^{(k)}) =$ const. for a local section $X^{(k)}$ of the bundle $T_k(V_n)$, we have $\varphi^*(X_p X^{(k)}) = 0$.

Our task is to prove the converse: *in the regular case, the last property is characteristic for the maps* φ_k^*.

Theorem 1. *Let V_n be a simply connected manifold and $s \geqq 1$ an integer. Let be given a map Φ: $T_{s+1}(V_n) \to W^m$ of the form $\Phi = pr_2 \circ \tilde{\Phi}$, where $\tilde{\Phi}$: $T_{s+1}(V_n) \to V_n \times W^m$ is a bundle morphism and pr_2: $V_n \times W^m \to W^m$ is a canonical projection. Suppose that*

a) *the restriction of $\tilde{\Phi}$ to the subbundle $T_s(V_n)$ is a bundle epimorphism,*

b) *if $X_p \in T(V_n)$ and $X^{(s)}$ is a local section of $T_s(V_n)$ defined at p such that $\Phi(X^{(s)}) =$ const., then $\Phi(X_p X^{(s)}) = 0$. Under these assumptions* *there is exactly one map φ: $V_n \to A^m$ satisfying initial condition $\varphi(p) = x$ and such that $\varphi^* = \Phi$ on $T(V_n)$. Moreover, we have $\varphi^* = \Phi$ on the whole bundle $T_{s+1}(V_n)$.*

Proof. Let be given $p \in V_n$ and a basis $(f_1, \dots, f_m)$ of W^m. Denote by ν the dimension of a fibre of $T_s(V_n)$. As $\tilde{\Phi}$ induces a bundle epimorphism $T_s(V_n) \to V_n \times W^m$, the following assertion may be easily verified: there is a coordinate neighbourhood $U(u_1, \dots, u_m)$ at p and local sections $X_1^{(s)}, \dots, X_\nu^{(s)}$ of $T_s(V_n)$ over U such that (i) the vectors $X_{1,p}^{(s)}, \dots, X_{\nu,p}^{(s)}$ are linearly independent, (ii) we have

$$\Phi(X_i^{(s)}) = f_i, \qquad i = 1, 2, \dots, m$$
$$\Phi(X_i^{(s)}) = 0, \qquad i = m+1, \dots, \nu$$

identically on U.

Put

$$X_{i,q}^{(s)} = \sum_{\substack{0 \leq i_1 \leq \dots \leq i_s \leq n \\ i_s > 0}} a_{i\,(q)}^{i_1 \dots i_s} \left(\frac{\partial^s}{\partial u^{i_1} \dots \partial u^{i_s}} \right), \qquad i = 1, \dots, \nu,$$

then the determinant $|a_{i\,(p)}^{i_1 \dots i_s}| \neq 0$. Now

$$\frac{\partial}{\partial u^k} X_i^{(s)} = \sum \left\{ \frac{\partial a_i^{i_1 \dots i_s}}{\partial u^k} \left(\frac{\partial^s}{\partial u^{i_1} \dots \partial u^{i_s}} \right) + a_i^{i_1 \dots i_s} \left(\frac{\partial^{s+1}}{\partial u^{i_1} \dots \partial u^k \dots \partial u^{i_s}} \right) \right\},$$

$$\frac{\partial}{\partial u^k} \Phi(X_i^{(s)}) = \sum \left\{ \frac{\partial a_i^{i_1 \dots i_s}}{\partial u^k} \Phi \left(\frac{\partial^s}{\partial u^{i_1} \dots \partial u^{i_s}} \right) + \right.$$
$$\left. + a_i^{i_1 \dots i_s} \frac{\partial}{\partial u^k} \Phi \left(\frac{\partial^s}{\partial u^{i_1} \dots \partial u^{i_s}} \right) \right\} = 0,$$

and according to the assumption **b** of the Theorem,

$$\Phi \left(\frac{\partial}{\partial u^k} X_i^{(s)} \right) = \sum \left\{ \frac{\partial a_i^{i_1 \dots i_s}}{\partial u^k} \Phi \left(\frac{\partial^s}{\partial u^{i_1} \dots \partial u^{i_s}} \right) + \right.$$
$$\left. + a_i^{i_1 \dots i_s} \Phi \left(\frac{\partial^{s+1}}{\partial u^{i_1} \dots \partial u^k \dots \partial u^{i_s}} \right) \right\} = 0.$$

Thus we have, for any $k = 1, 2, \dots, n$ and $i = 1, 2, \dots, \nu$,

$$\sum a_{i\,(p)}^{i_1 \dots i_s} \left\{ \Phi_p \left(\frac{\partial^{s+1}}{\partial u^{i_1} \dots \partial u^k \dots \partial u^{i_s}} \right) - \left(\frac{\partial}{\partial u^k} \right)_p \Phi \left(\frac{\partial^s}{\partial u^{i_1} \dots \partial u^{i_s}} \right) \right\} = 0.$$

In view of $|a_{i\,(p)}^{i_1 \dots i_s}| \neq 0$,

$$\Phi \left(\frac{\partial^{s+1}}{\partial u^{i_1} \dots \partial u^k \dots \partial u^{i_s}} \right) = \frac{\partial}{\partial u^k} \Phi \left(\frac{\partial^s}{\partial u^{i_1} \dots \partial u^{i_s}} \right)$$

for any sequence $0 \leqq i_1 \leqq i_2 \leqq \ldots \leqq k \leqq \ldots \leqq i_s \leqq n$, $k > 0$. Hence we obtain easily

$$\Phi(X_p X^{(s)}) = X_p \Phi(X^{(s)}) \tag{7}$$

for any vector $X_p \in T(V_n)$ and any local section $X^{(s)}$ of $T_s(V_n)$ defined at p.

To complete our proof we shall use the Frobenius Theorem. Put $D = V_n \times A^m$. At each point $\alpha \in D$, $\alpha = (p, x)$ we have $T(D)_\alpha = T(V_n)_p + T(A^m)_x$. We shall construct on D a differentiable distribution Δ_α of dimension n as follows: for any $\alpha \in D, \alpha = (p, x)$, let $\Delta_\alpha \subset T(D)_\alpha$ be a linear subspace of all vectors of the form $X_p + \omega_x^{-1} \Phi(X_p)$, $X_p \in T(V_n)_p$. *The distribution Δ_α is involutive.* In fact, let $\pi: D \to V_n$ be a canonical projection. For any $\alpha = (p, x) \in D$, there are linearly independent vector fields $X_1, \ldots, X_n$ defined on a neighbourhood $U \ni p$. Then the vector fields $\tilde{X}_{i,\beta} = X_{i,q} + \omega_y^{-1} \Phi(X_{i,q})$, $i = 1, 2, \ldots, n$, generate the distribution Δ_β, $\beta = (q, y)$, on a neighbourhood $\pi^{-1}(U)$ of α. It suffices to prove that $[\tilde{X}_i, \tilde{X}_j]_\alpha$ belongs to Δ_α. But since $\tilde{X}_i$, $\tilde{X}_j$ do not depend essentially on y, we have $\omega_x^{-1} \Phi(X_{i,p}) \tilde{X}_j = 0$, $\omega_x^{-1} \Phi(X_{j,p}) \tilde{X}_i = 0$, and hence $[\tilde{X}_i, \tilde{X}_j]_\alpha = [X_i, X_j]_p + X_{j,p}\omega_y^{-1}\Phi(X_{i,q}) - X_{i,p}\omega_y^{-1} \Phi(X_{j,p}) = [X_i, X_j]_p + \omega_x^{-1}\{X_{i,p} \Phi(X_j) - X_{j,p} \Phi(X_i)\} = [X_i, X_j]_p + \omega_x^{-1}\Phi([X_i, X_j]_p)$, according to (7).

There is only one maximal integral manifold $\tilde{V}_n$ of the distribution Δ_α, passing through a prescribed point $\alpha_0 \in D$. Then for any $\alpha \in \tilde{V}_n$, $\alpha = (p, x)$, we have $d\pi[T(\tilde{V}_n)_\alpha] = d\pi(\Delta_\alpha) = T(V_n)_p$. Hence π is a local diffeomorphism. Since Δ is invariant with respect to all transformations of D of the form $(q, y) \to (q, y + a)$, *$\tilde{V}_n$ is a covering space of V_n.* As V_n is simply connected, π is a diffeomorphism. *If $\varrho: D \to A^m$ is a canonical projection, we obtain a map $\varphi: V_n \to A^m$, $\varphi = \varrho \circ \pi^{-1}$.* Here $d\varphi(X_p) = \omega_{\varphi(p)}^{-1} \Phi(X_p)$ for any $X_p \in T(V_n)_p$ *and consequently, in view of* (4), $\Phi = \omega \circ d\varphi = \omega \circ T_1(\varphi) = \varphi^*$ *on* $T(V_n)$. *Finally, from* (6), (7), we see, step by step, that $\varphi^* = \Phi$ on $T_2(V_n)$, $T_3(V_n)$, $\ldots$, $T_{s+1}(V_n)$, q.e.d.

*

As an application of Theorem 1, we can re-prove a result of KOČANDRLE (see [6]). First we shall present some concepts of [6]. Let be given a covariant tensor $t(x_1, \ldots, x_r)$ of degree r on W^m. We shall denote by ${}^iS$ the set of all vectors $y \in W^m$ such that $t(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_r) = 0$ for arbitrary vectors $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r$ from W^m. The intersection $S = \bigcap_{i=1}^{r} {}^iS$ is called the *singular space of t*. The *automorphism group of the tensor t* is the group of all transformations $g \in GL(m)$ such that $t(x_1, \ldots, x_r) = t(x_1 g, \ldots, x_r g)$ for each $x_1, \ldots, x_r \in W^m$. It is a Lie subgroup $G^0 \subset GL(m)$. *Let be given a fixed regular tensor* $t(x_1, \ldots, x_r)$ on W^m, i.e., such that its singular space $S = \{0\}$. If $\varphi\colon V_n \to A_m$ is a map, $\varphi_k^*\colon T_k(V_n) \to W^m$ the induced maps given by (4), $k = 1, 2, \ldots$, we can define an r-linear form $t_k^* = t \circ (\overset{r}{\otimes} \varphi_k^*)$ on $T_k(V_n)$ for each $k = 1, 2, \ldots$ For each $k > l$, t_k^* coincides with t_l^* on $T_l(V_n)$. The sequence $\{t_k^*\}$ of multilinear forms is called the *fundamental tensor of the manifold* V_n. Now, the main result of [6] is a characterization of the fundamental tensor.

Let us consider the following conditions:

I. On V_n, there is given a differentiable tensor t^*, covariant of degree r, acting at each point $p \in V_n$ on the $(k_0 + 1)$-vectors from $T_{k_0+1}(V_n)_p$, k_0 is a given number.

Let us denote by $S_{k_0,p}$, $S_{k_0+1,p}$ the singular spaces of t^* on $T_{k_0}(V_n)_p$ and $T_{k_0+1}(V_n)_p$, respectively.

II. For any differentiable fields of k_0-vectors $X_1^{(k_0)}, \ldots, X_r^{(k_0)}$ and any vector $Y_p \in T(V_n)_p$, we have $Y_p t^*(X_1^{(k_0)} \ldots, X_r^{(k_0)}) = \sum_{i=1}^{r} t^*(X_{1,p}^{(k_0)}, \ldots, X_{i-1,p}^{(k_0)}, Y_p X_i^{(k_0)}, X_{i+1,p}^{(k_0)}, \ldots, X_{r,p}^{(k_0)})$.

III. $\dim T_{k_0}(V_n)_p/S_{k_0,p} = \dim T_{k_0+1}(V_n)_p/S_{k_0+1,p} = m$ for each point $p \in V_n$; $S_{k_0+1,p} \cap T(V_n)_p = \{0\}$ for each $p \in V_n$.

Let P denote the principal fibre bundle of all bases of the spaces $T_{k_0}(V_n)_p/S_{k_0,p}$, $p \in V_n$.

IV. To each point $p \in V_n$ there is a neighbourhood $U \subset V_n$ of p and a local section s of the fibre bundle P over U such that the components of t^* with respect to the basis s_q are constant functions of q on U.

V. There is a point $p \in V_n$ such that the vector space W^m with the given tensor t is isomorphic to the space $T_{k_0}(V_n)_p/S_{k_0,p}$ with the tensor t^*.

Let us introduce the abbreviations $T_{k_0} = T_{k_0}(V_n)$, $S_{k_0} = \bigcup_{q \in V_n} S_{k_0,q}$, and similarly for the index $k_0 + 1$. From III we obtain easily a commutative diagram

$$\begin{array}{ccc} S_{k_0} & \longrightarrow & S_{k_0+1} \\ \downarrow & & \downarrow \\ T_{k_0} & \longrightarrow & T_{k_0+1} \end{array}$$

over V_n, and a canonical isomorphism $\sigma\colon T_{k_0}/S_{k_0} \to T_{k_0+1}/S_{k_0+1}$ of factor bundles. Let

$$\pi_{k_0}\colon T_{k_0} \to T_{k_0}/S_{k_0}, \quad \pi_{k_0+1}\colon T_{k_0+1} \to T_{k_0+1}/S_{k_0+1}$$

be canonical projections. We have a commutative diagram

$$\begin{array}{ccc} T_{k_0} & \xrightarrow{I_{k_0,k_0+1}} & T_{k_0+1} \\ \big\downarrow \pi_{k_0} & & \big\downarrow \pi_{k_0+1} \\ T_{k_0}/S_{k_0} & \xrightarrow{\sigma} & T_{k_0+1}/S_{k_0+1}. \end{array}$$

Let G_0 be the automorphism group of the tensor t on W^m. We can prove that V is satisfied at each point $q \in V_n$. *Let P_q^0 be the set of all isomorphisms χ_q: $T_{k_0, q}/S_{k_0, q} \to W^m$ with the property* V, i.e., *such that* $t^* = t \circ \overset{r}{\otimes} (\chi_q \circ \circ \pi_{k_0, q})$. Then $P_0 = \bigcup_{q \in V_n} P_q^0$ is a principal fibre bundle over V_n with the structural group G_0. If we choose a fixed basis ϱ^w of W^m, we obtain *a canonical injection* $P^0 \to P$. Let be given $p \in V_n$. To each section s of P^0 over a neighbourhood $U \ni p$, we can join a matrix form ω_p^s on $T(V_n)_p$ as follows: put $s = (\xi_1^{(k_0)}, \ldots, \xi_m^{(k_0)})$ over U, $\xi_i^{(k_0)}$ being local sections of T_{k_0}/S_{k_0}. Let $X_1^{(k_0)}, \ldots, X_m^{(k_0)}$ be sections of T_{k_0} over U such that $\pi_{k_0} X_i^{(k_0)} = \xi_i^{(k_0)}$, $i = 1, 2, \ldots, m$. For any $X_p \in T(V_n)_p$ the elements $\eta_{i,p}^{(k_0+1)} = \pi_{k_0+1}(X_p X_i^{(k_0)})$, $\eta_{i,p}^{(k_0+1)} \in T_{k_0+1, p}/S_{k_0+1, p}$, do not depend on the representation of $\xi_i^{(k_0)}$ by $X_i^{(k_0)}$ and we can write

$$X_p s = \{X \xi_1^{(k_0)}, \ldots, X_p \xi_m^{(k_0)}\} = \{\eta_{1,p}^{(k_0+1)}, \ldots, \eta_{m,p}^{(k_0+1)}\} = \sigma[s_p \cdot \omega_p^s(X_p)],$$

where $\omega_p^s(X_p)$ is a matrix of type $m \times m$.

It may be deduced from II that $\omega_p^s(X_p)$ belongs to the Lie algebra $\mathfrak{g}_0$ of G^0. Further, the forms $\omega_p^s(p \in V_n$, s being a local section of P^0 defined at p) determine a *connection* ω in P^0. (See [6] and, for instance [7].) Now, the last condition of the Paper [6] is the following:

VI. *The curvature form of the connection ω is equal to* 0.

The main result of [6] is the following: *if the conditions* I—VI *are satisfied then there is a covering manifold V_n' with the covering map π: $V_n' \to V_n$ and a regular map Ψ: $V_n' \to A^m$ such that we have locally $t^* = t \circ (\overset{r}{\otimes} \varphi_{k_0+1}^*)$; here $\varphi = \Psi \circ \pi^{-1}$ is a local embedding $V_n \to A^m$.*

Proof. First let us suppose that the manifold V_n is simply connected. Because the curvature form of ω vanishes, there are local horizontal sections in P_0, and from the monodromy theorem (see [8]) follows that there is a global horizontal section $\tilde{s}$: $V_n \to P^0$. We have global horizontal sections in the associated fibre bundle $E = T_{k_0}/S_{k_0} \cong \cong T_{k_0+1}/S_{k_0+1}$, too. Let $p \in V_n$ be a fixed point, χ: $E_p \to W^m$ a fixed isomorphism such that $t^* = t \circ \overset{r}{\otimes} (\chi \circ \pi_{k_0+1, p})$ (Condition V). Let h_q: $E_q \to E_p$ be the parallel translation with respect to the connection ω. Put $\Phi_q = \chi \circ h_q \circ \pi_{k_0+1, q}$, Φ_q: $T_{k_0+1}(V_n)_q \to W^m$, for $q \in V_n$. Then the restriction of Φ_q to $T_{k_0}(V_n)_q$ is an epimorphism. Let $X^{(k_0)}$ be a local section of $T_{k_0}(V_n)$ over a neighbourhood $U \ni q$, and $X_q \in T(V_n)_q$ a vector. Suppose that $\Phi(X^{(k_0)}) = \text{const}$. Then $\pi_{k_0} X^{(k_0)}$ is a horizontal section of E and there is a constant matrix $a = (a_1, \ldots, a_m)$ such that $\pi_{k_0} X^{(k_0)} = = \tilde{s} \cdot a$. We have $\omega_q^{\tilde{s}} = 0$ because the section $\tilde{s}$ is horizontal. Now, $\pi_{k_0+1, q}(X_q X^{(k_0)}) = X_q(\pi_{k_0} X^{(k_0)}) = X_q \tilde{s} \cdot a = \sigma[\tilde{s}_q \cdot \omega_q^{\tilde{s}}(X_q) \cdot a] = 0$; hence $\Phi(X_q X^{(k_0)}) = 0$.

The conditions of Theorem 1 are satisfied and consequently, there is a map $\varphi\colon V_n \to A^m$ such that $\Phi = \varphi^*_{k_0+1}$ on $T_{k_0+1}(V_n)$. Since the restriction of Φ_q to $T(V_n)_q$ is a monomorphism (the second condition of III), we can see easily that φ is an immersion. Now from the construction of the principal bundle P^0 we see that, on each $T_{k_0+1}(V_n)_q$,

$$t^*_q = t \circ \overset{r}{\otimes} (\chi \circ h_q \circ \pi_{k_0+1,\,q}) = t \circ (\overset{r}{\otimes} \Phi_q) = t \circ (\overset{r}{\otimes} \varphi^*_{k_0+1,\,q}),$$

which proves our assertion for V_n simply connected.

In case V_n to be not simply connected, let us consider the universal covering manifold $\tilde{V}_n$ of V_n (see [8]). Then the proof will be easily traced back to the preceding case.

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In the second part of this Paper we shall try to generalize Theorem 1, at least in a weakened form, to the case when A^m is replaced by an arbitrary Lie group. So, let G be a Lie group, $\mathbf{g}$ its Lie algebra. For $X_g \in T(G)$ let us denote by $\omega(X_g)$ the left invariant vector field on G determined by X_g. Then $\omega\colon T(G) \to \mathbf{g}$ is a vector form on G, each partial map $\omega_g\colon T(G)_g \to \mathbf{g}$ being an isomorphism. Let $S_k\colon T_{k+1}(G) \to T(G)$ be a surconnection on G and $\varphi\colon V_n \to G$ a differentiable map. Then we have a sequence of maps, analogous to (4):

$$T_{k+1}(V_n) \xrightarrow{T_{k+1}(\varphi)} T_{k+1}(G) \xrightarrow{S_k} T(G) \xrightarrow{\omega} \mathbf{g}. \tag{8}$$

Let $\varphi^*\colon T_{k+1}(V_n) \to \mathbf{g}$ denote the composed map of the sequence. Obviously φ^* may be written as a composition $T_{k+1}(V_n) \xrightarrow{\tilde{\varphi}} V_n \times \mathbf{g} \xrightarrow{pr_2}$ $\xrightarrow{pr_2} \mathbf{g}$, of a bundle morphism and a canonical projection.

Proposition 1. *There is a map $\Psi^*\colon T(V_n) \otimes T_k(V_n) \to \mathbf{g}$, a composition of a bundle morphism $T(V_n) \otimes T_k(V_n) \to V_n \times \mathbf{g}$ and the canonical projection* pr_2: *$V_n \times \mathbf{g} \to \mathbf{g}$, with the following property*:

$$\varphi^*(X_p X^{(k)}) = X_p \varphi^*(X^{(k)}) + \Psi^*(X_p \otimes X^{(k)}_p) \tag{9}$$

for any vector $X_p \in T(V_n)$ and any local section $X^{(k)}$ of $T_k(V_n)$ defined at p.

Proof. Let be given $X_p \in T(V_n)_p$, $X^{(k)}_p \in T_k(V_n)_p$. Let $X^{(k)}$ be a local section of $T_k(V_n)$ passing through $X^{(k)}_p$. It suffices to prove that the expression $\varphi^*(X_p X^{(k)}) - X_p \varphi^*(X^{(k)})$ depends on X_p, $X^{(k)}_p$ only and that it is linear in each argument. Choose a local coordinate system $(u_1, \ldots, u_n)$ at p and put

$$X_p = \sum_{i=1}^{n} a^i \frac{\partial}{\partial u^i}, \quad X^{(k)} = \sum_{\substack{0 \le i_1 \le \ldots \le i_k \le n \\ i_k > 0}} a^{i_1, \ldots, i_k}(q) \frac{\partial^k}{\partial u^{i_1} \ldots \partial u^{i_k}}$$

Then

$$X_p X^{(k)} = \sum \left\{ a^i \frac{\partial a^{i_1 \dots i_k}}{\partial u^i} \cdot \frac{\partial^k}{\partial u^{i_1} \dots \partial u^{i_k}} + \right.$$
$$\left. + a^i a^{i_1 \dots i_k}(p) \frac{\partial^{k+1}}{\partial u^{i_1} \dots \partial u^i \dots \partial u^{i_k}} \right\},$$

$$\varphi^*(X_p X^{(k)}) - X_p \varphi^*(X^{(k)}) = \sum a^i a^{i_1, \dots, i_k}(p) \left\{ \varphi_p^* \left(\frac{\partial^{k+1}}{\partial u^{i_1} \dots \partial u^i \dots \partial u^{i_k}} \right) - \right.$$
$$\left. - \left(\frac{\partial}{\partial u^i} \right)_p \varphi^* \left(\frac{\partial^k}{\partial u^{i_1} \dots \partial u^{i_k}} \right) \right\}.$$

This proves our assertion.

Proposition 2. *For any* X_p, $Y_p \in T(V_n)_p$ *we have* $\Psi^*(X_p \otimes Y_p - - Y_p \otimes X_p) = [\varphi^*(X_p), \varphi^*(Y_p)]$, $[\ \ ,\ \ ]$ *being the bracket operation in the algebra* **g**.

Proof. Let us remind the equations $d\omega = -1/2\,[\omega, \omega]$, $d\omega(X, Y) = = X\omega(Y) - Y\omega(X) - \omega([X, Y])$, where ω: $T(G) \to$ **g** is the canonical form. (See [5], for instance). These equations are still valid if we substitute the form ω by the form $\omega' = \omega \circ d\varphi$, $d\varphi$: $T(V_n) \to T(G)$ being the tangent map of φ. According to (8), we have $\omega' = \varphi^*$ on $T(V_n)$. Let X, Y be local tangent fields at p passing through X_p, Y_p, respectively. Then

$$\varphi^*(X_p Y) - \varphi^*(Y_p X) = \varphi^*([X, Y]_p) = X_p \omega'(Y) - Y_p \omega'(X) - - d\omega'(X_p, Y_p) = X_p \omega'(Y) - Y_p \omega'(X) + [\omega'(X_p), \omega'(Y_p)] = = X_p \varphi^*(Y) - Y_p \varphi^*(X) + [\varphi^*(X_p), \varphi^*(Y_p)].$$

On the other hand

$$\varphi^*(X_p Y) - \varphi^*(Y_p X) = X_p \varphi^*(Y) + \Psi^*(X_p \otimes Y_p) - Y_p \varphi^*(X) - - \Psi^*(Y_p \otimes X_p).$$

This proves our assertion.

Theorem 2. *Let* V_n *be a simply connected manifold,* G *a Lie group with the algebra* **g**. *Let be given differentiable maps* Φ: $T_{s+1}(V_n) \to$ **g**; ψ: $T(V_n) \otimes T_s(V_n) \to$ **g**, *which are compositions of bundle morphisms* $\tilde{\Phi}$: $T_{s+1}(V_n) \to V_n \times$ **g**, $\tilde{\psi}$: $T(V_n) \otimes T_s(V_n) \to V_n \times$ **g**, *respectively and of the canonical projection* pr_2: $V_n \times$ **g** $\to$ **g**. *Suppose that*

a) *the restriction of* $\tilde{\Phi}$ *to the subbundle* $T_s(V_n)$ *is a bundle epimorphism,*

b) *if* $X_p \in T(V_n)$ *and* $X^{(s)}$ *is a local section of* $T_s(V_n)$ *defined at* p *such that* $\Phi(X^{(s)}) = const.$, *then* $\Phi(X_p X^{(s)}) = \psi(X_p \otimes X_p^{(s)})$,

c) *for any two vectors* X_p, $Y_p \in T(V_n)$ *we have*

$$\psi(X_p \otimes Y_p - Y_p \otimes X_p) = [\Phi(X_p), \Phi(Y_p)].$$

Then there is exactly one map φ: $V_n \to G$ satisfying initial condition $\varphi(p) = g$ and such that $\Phi = \omega \circ d\varphi$ on $T(V_n)$.

Proof. An argument like that in the proof of Theorem 1 shows that

$$\Phi(X_p X^{(s)}) = X_p \Phi(X^{(s)}) + \psi(X_p \otimes X_p^{(s)}) \tag{10}$$

for any vector $X_p \in T(V_n)$ and any local section $X^{(s)}$ of $T_s(V_n)$ defined at p. Let us define a distribution Δ_α on $D = V_n \times G$ as follows: if $\alpha = (p, g) \in D$, then Δ_α consists of all vectors of the form $X_p + \omega_g^{-1}\Phi(X_p)$, $X_p \in T(V_n)_p$. For two vector fields $\tilde{X}_\beta = X_q + \omega_h^{-1}\Phi(X_q)$, $\tilde{Y}_\beta = Y_q + \omega_h^{-1}\Phi(Y_q)$ belonging to Δ in a neighbourhood of $\alpha = (p, g)$ we obtain $[\tilde{X}, \tilde{Y}]_\alpha = [X, Y]_p + \omega_g^{-1}X_p\Phi(Y) - \omega_g^{-1}Y_p\Phi(X) + \omega_g^{-1}[\Phi(X_p), \Phi(Y_p)] = [X, Y]_p + \omega_g^{-1}\Phi([X, Y]_p)$. It is a consequence of (10) and of the assumption c) of the Theorem. Thus the distribution Δ is involutive. The rest of the proof is the same as at Theorem 1.

Note 3. If the restriction of $\tilde{\Phi}$ to the subbundle $T(V_n)$ is a monomorphism, then φ is an immersion.

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There is an interesting special case when we may prove a stronger result, an exact analog of Theorem 1. It is the case when $[[X, Y], Z] = 0$ for any $X, Y, Z \in \mathfrak{g}$. As known, on any Lie group G there is exactly one connection ∇ with the following properties:

a) The geodesics of ∇ are the integral curves of left invariant vector fields on G,

b) the torsion form $T(X, Y) = 0$. (See [5], Chapter 6.) We have

$$\nabla_{X_g} Y = \omega_g^{-1}\left\{X_g\omega(Y) + \frac{1}{2}[\omega(X_g), \omega(Y_g)]\right\} \tag{11}$$

for any vector $X_g \in T(G)_g$ and any vector field Y at g. Finally, the curvature form is given by $R(X, Y, Z) = 1/4[[X, Y], Z]$. In our special case we have $R(X, Y, Z) = 0$ and consequently, the iterations ∇^k generate a canonical sequence $\{S_k\}$ of symmetric surconnections on G. According to (3), (11) we have

$$S_k(X_g X^{(k)}) = \nabla_{X_g} S_{k-1}(X^{(k)}) = \omega_g^{-1}\left\{X_g[\omega \circ S_{k-1}](X^{(k)}) + \frac{1}{2}[\omega(X_g), [\omega \circ S_{k-1}](X_g^{(k)})]\right\} \tag{12}$$

for $k \geq 2$.

Now, for any differentiable map $\varphi\colon V_n \to G$ and any $k \geqq 0$ we can define a map $\varphi^*_{k+1}\colon T_{k+1}(V_n) \to \mathfrak{g}$ by $\varphi^*_{k+1} = \omega \circ S_k \circ T_{k+1}(\varphi)$. (Here $\varphi^*_1 = \omega \circ T_1(\varphi)$.) From (11), (12) we get a formula

$$\varphi^*(X_p X^{(k)}) = X_p \varphi^*(X^{(k)}) + \frac{1}{2}[\varphi^*(X_p), \varphi^*(X_p^{(k)})]. \tag{13}$$

If the mapping $\psi\colon T(V_n) \otimes T_s(V_n) \to \mathfrak{g}$ introduced in Theorem 2 is given by $\psi(X_p \otimes X_p^{(s)}) = 1/2[\Phi(X_p), \Phi(X_p^{(s)})]$, the condition **c** is fulfilled. From Theorem 2 and (10), (13), we obtain the following result: *there is exactly one map $\varphi\colon V_n \to G$ satisfying an initial condition and such that $\varphi^*_s = \Phi$ on $T(V_n)$. Moreover, we have $\varphi^* = \Phi$ on the whole $T_{s+1}(V_n)$.*

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