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ONE CHARACTERIZATION OF SPECIAL TRANSLATION PLANES

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§ **1** M. Hall gives in [1] a description of near-field planes using some permutation groups. This concept will be now adapted also for affine planes over Veblen-Wedderburn systems with the right inverse property.

Let S be a set with at least two distinct elements. Let Σ and Σ^* be sets of permutations of S satisfying the following conditions:

(a) Let x_1, y_1, x_2, y_2 be elements of S such that $x_1 \neq x_2$ and $y_1 \neq y_2$. Then there is precisely one $\sigma \in \Sigma$ such that $x_1^\sigma = y_1$ and $x_2^\sigma = y_2$.

(b) Let $\alpha \in \Sigma$ and $x_0, y_0 \in S$ with $x_0^\alpha \neq y_0$. Then there is at most one $\beta \in \Sigma$ such that $x_0^\beta = y_0$ and that $\beta^{-1} \bigcirc \alpha$ changes all elements of S.

(c) If $\sigma \in \Sigma$ then $\sigma^{-1} \in \Sigma$.

(d) The identity permutation id_S belongs to Σ^*, Σ^* is properly contained in Σ and each $\sigma \in \Sigma \setminus \Sigma^*$ keeps one element of S fixed.

(e) If $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$ then $\sigma^* \bigcirc \sigma \in \Sigma$.

(f) The sets $\lambda_{\sigma^*} = \{(x, y) \in S \times S \mid y = x^{\sigma^*}\}$, $\sigma^* \in \Sigma^*$ form a decomposition of $S \times S$.

Now we deduce some conclusions.

1. $(\Sigma^*, \bigcirc)$ is a group. In fact, by (e) we have $\beta \bigcirc \alpha \in \Sigma$ for any $\alpha, \beta \in \Sigma^*$. From (f) it follows that $\beta \bigcirc \alpha$ changes each element of S so that $\beta \bigcirc \alpha \in \Sigma^*$ by (d). If $\alpha \in \Sigma^*$ then $\alpha^{-1} \in \Sigma^*$ by (b). Following (f), α^{-1} must change all elements of S. Consequently by (c) it is $\alpha^{-1} \in \Sigma^*$. But $id_S \in \Sigma^*$ by (d) so that $(\Sigma^*, \bigcirc)$ is a group. Q.E.D.

2. Let $\alpha \in \Sigma$ and $x_0, y_0 \in S$ with $x_0^\alpha \neq y_0$. Then there is at least one $\beta \in \Sigma$ such that $x_0^\beta = y_0$ and that $\beta_0^{-1} \bigcirc \alpha$ displaces all elements of S. In fact, the sets $\lambda_{\sigma^* \circ \alpha} = \{(x, y) \mid y = x^{\sigma^* \circ \alpha}\}$, $\sigma^* \in \Sigma^*$ form a decomposition of $S \times S$ by (e) and (f). So there is a $\beta^* \in \Sigma^*$ such that $x_0^{\beta^* \circ \alpha} = y_0$ and that $\lambda_{\beta^* \circ \alpha}, \lambda_\alpha$ are disjoint. But this means that $(\beta^* \bigcirc \alpha)^{-1} \bigcirc \alpha$ displaces each element of S. Q.E.D.

3. $\mathbf{A} = (S \times S, L, \in)$ is an affine plane where $L = \{\{(x, y) \mid x = a\} \mid \alpha \in S\} \cup \{\{(x, y) \mid y = b\} \mid b \in S\} \cup \{\{(x, y) \mid y = x^\sigma\} \mid \sigma \in \Sigma\}$. In fact using (a), (b), Proposition 2 and the assumption card $S \geqq 2$ we easily verify the axioms of affine planes: Any two points are contained in exactly one common line. Through each point it goes precisely one line parallelly to a given line (i.e. disjoint to a given line or coinciding with it). There are three points which are not contained in the same line. Q.E.D.

Now choose in **A** a *coordinatizing frame* F consisting of the points O, J_x, J, J_y forming a parallelogram such that $O = (0, 0)$, $J_x = (1, 0)$, $J = (1, 1)$, $J_y = (0,1)$ where 0, 1 are some distinct elements of S.[1] The Hall's coordinatization principle ([2]) gives the ternary operation τ: $S \times S \times S \to S$ where $\tau(x, 0, v) = v$ for all $x, v \in S$ and $\tau(x, u, v) = y$ for $u \neq 0$ should mean that the point (x, y) lies on the line going through $(0, v)$ parallelly to the line $O(1, u)$. The *derived addition* and *multiplication* on S are defined by $x \overset{+}{_\tau} v = \tau(x, 1, v)$ and $x \overset{\cdot}{_\tau} u = \tau(x, u, 0)$.

4. The *linearity property* $\tau(x, u, v) = x \overset{\cdot}{_\tau} u \overset{+}{_\tau} v$ is valid. In fact, denote $\alpha^*: t \to t \overset{+}{_\tau} v$ and $\alpha: x \to x \overset{\cdot}{_\tau} u$. Then $\alpha^* \in \Sigma^*$, $\alpha \in \Sigma$ and $\sigma^* \bigcirc \sigma: x \to x \overset{\cdot}{_\tau} u \overset{+}{_\tau} v$ belongs also to Σ, by (e). Since $0^{\sigma^* \circ \sigma} = v$ and $1^{\sigma^* \circ \sigma} = u \overset{+}{_\tau} v$ (by $0 \overset{\cdot}{_\tau} u = 0$ and $1 \overset{\cdot}{_\tau} u = u$), it follows $\sigma^* \bigcirc \sigma : x \to \tau(x, u, v)$. Thus $\tau(x, u, v) = x \overset{\cdot}{_\tau} u \overset{+}{_\tau} v$. Q.E.D.

5. The *right inverse property* $(b \overset{\cdot}{_\tau} a) \overset{\cdot}{_\tau} a'' = b$ holds for all $a \in S \setminus \{0\}$, $b \in S$ where $a'' \in S$ is determined by $a \overset{\cdot}{_\tau} a'' = 1$. Consequently $a'' \overset{\cdot}{_\tau} a = 1$ (so that a'' can by denoted us usually by a^{-1}). In fact, let $a \neq 0$. Consider the inverse mapping $\alpha^{-1} : y \to y \overset{\cdot}{_\tau} c \overset{+}{_\tau} d$ of the mapping $\alpha : x \to x \overset{\cdot}{_\tau} a$. Here α must belong to Σ so that by (c) we have $\alpha^{-1} \in \Sigma$. From $0^\alpha = 0$ and $1^\alpha = a$ it follows $d = 0$ and $c = a''$. Thus $\alpha^{-1} \bigcirc \alpha : x \to (x \overset{\cdot}{_\tau} a) \overset{\cdot}{_\tau} a''$. Let $a' \in S$ be determined by $a' \overset{\cdot}{_\tau} a = 1$. Then for $x = a'$ we obtain $(a')^{\alpha^{-1} \circ \alpha} = a''$, i.e. $a' = a''$. Q.E.D.

6. $(S, \overset{+}{_\tau})$ is a group. This follows at once from the definition of $\overset{+}{_\tau}$ and from Proposition 1. The element 0 is neutral for the investigated group. Q.E.D.

7. The *distributivity law* $(a \overset{+}{_\tau} b) \overset{\cdot}{_\tau} c = a \overset{\cdot}{_\tau} c \overset{+}{_\tau} b \overset{\cdot}{_\tau} c$ holds. In fact, choose $\beta : x \to x \overset{\cdot}{_\tau} a \overset{+}{_\tau} b$ where $a, b \in S \setminus \{0\}$. Then β^{-1} has the form $x \to x \overset{\cdot}{_\tau} a^{-1} \overset{+}{_\tau} c$ because for $\alpha : x \to x \overset{\cdot}{_\tau} a$ we have $\alpha^{-1} : x \to x \overset{\cdot}{_\tau} a^{-1}$ and consequently $\alpha^{-1} \bigcirc \beta$, $\alpha \bigcirc \beta^{-1}$ change all elements of S. Thus $\beta^{-1} \bigcirc \beta : x \to (x \overset{\cdot}{_\tau} a^{-1} \overset{+}{_\tau} c) \overset{\cdot}{_\tau} a \overset{+}{_\tau} b$. For $x = 0$ we obtain $0 = c \overset{\cdot}{_\tau} a \overset{+}{_\tau} b$ and for $x = b \overset{\cdot}{_\tau} a$ we have $b \overset{\cdot}{_\tau} a \overset{+}{_\tau} x \overset{\cdot}{_\tau} a = (b \overset{\cdot}{_\tau} a \overset{\cdot}{_\tau} a^{-1} \overset{+}{_\tau} c) \overset{\cdot}{_\tau} a$, i.e. $b \overset{\cdot}{_\tau} a \overset{+}{_\tau} c \overset{\cdot}{_\tau} a = (b \overset{+}{_\tau} c) \overset{\cdot}{_\tau} a$. The cases $a = 0$, $b = 0$ are easily to consider. Q.E.D.

The arguments of [3] imply now the commutativity of $\overset{+}{_\tau}$ and as a final result, *$(S, \overset{+}{_\tau}, \overset{\cdot}{_\tau})$ is shown to be a Veblen-Wedderburn system* (in the sense of [4]) *with the right inverse property*. Conversely, *if $(S, +, .)$ is a Veblen-Wedderburn system with the right inverse property then the set Σ of all mappings $x \to x \,.\, u + v (u \in S \setminus \{0\}, v \in S)$ and the set Σ^* of all mappings*

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[1]) If **A** is an affine line with a general coordinatizing frame $\mathfrak{F} = OJ_xJJ_y$ then we shall use the denotation $OJ_x = \xi_{\mathfrak{F}}$, $OJ_y = \eta_{\mathfrak{F}}$, $OJ = \xi_{\mathfrak{F}}$. A parallelogram $ABCD$ will be called $\mathfrak{F}$-*distinguished* if $AB \parallel CD \parallel \eta_{\mathfrak{F}}$, $DA \parallel BC \parallel \xi_{\mathfrak{F}}$ and $BD = \zeta_{\mathfrak{F}}$.

$x \rightarrow x + w (w \in S)$ *satisfy conditions* (a) to (f). The evident proof may be omitted. Of course, the alternative field is a particular case of the above Veblen-Wedderburn system. But no example of a proper (i.e. non-distributive and non-associative) Veblen-Wedderburn system with the right inverse properry is known to the author.

As a complement we shall formulate still two assertions:

8. For the plane **A** of Proposition 3, $\sigma \in \Sigma$, $\sigma^* \in \Sigma^* \Rightarrow \sigma^{-1} \bigcirc \sigma^* \bigcirc \bigcirc \sigma \in \Sigma$. In fact, let $\sigma : x \rightarrow x \dot{\tau} u_\tau v$ and $\sigma^*: x \rightarrow x \overset{+}{\tau} w (u \in S \setminus \{0\}, v \in S, w \in S)$. Then $\sigma^{-1}: x \rightarrow x \dot{\tau} u^{-1} \overset{+}{\tau} (\bar{\tau} v) \dot{\tau} u^{-1}$, $\sigma^{-1} \bigcirc \sigma^* \bigcirc \sigma : x \rightarrow (x_\tau u^{-1} \overset{+}{\tau} (\bar{\tau} v) \dot{\tau} u^{-1} \overset{+}{\tau} w) \dot{\tau} u \overset{+}{\tau} v = x \overset{+}{\tau} ((\bar{\tau} v \overset{+}{\tau} w) \dot{\tau} u \overset{+}{\tau} v)$. Thus $\sigma^{-1} \bigcirc \sigma^* \bigcirc \sigma \in \Sigma^*$. Q.E.D.

9. Let **A** be an arbitraty affine plane with a coordinatizing frame $\mathfrak{F}$. Let Σ be the set of all mappings $x \rightarrow \tau(x, u, v)$ with $u \in S \setminus \{0\}$, $v \in S$ where τ is the corresponding ternary operation. Then (c) $\Leftrightarrow$ (I) where

(I) If *ABCD* is a variable $\mathfrak{F}$-distinguished parallelogram with the point *A* ranging over a line *m* non $\parallel \xi_\mathfrak{F}, \eta_\mathfrak{F}$, then *C* ranges also over a line.

The proof may be omitted. This configuration theorem (I) can be used for a geometrical proof of Proposition 5.

§ 2 Now we shall investigate the independence of the „inversing" operation and express the corresponding situations by some algebraic or geometric conditions.

Let **A** be the plane of Proposition 3. We shall suppose that the coordinatizing frame $\mathfrak{F} = OJ_xJJ_y$ introduced after Proposition 3 is now fixed. If $\mathfrak{F}' = OJ_xJ'J_y'$ is a coordinatizing frame then define the *inversing operation* $\imath_{\mathfrak{F}'}$ as the mapping $S \setminus \{0\} \rightarrow S \setminus \{0\}$ which sends each element $a \in S \setminus \{0\}$ onto the element $a' \in S \setminus \{0\}$ such that $1 = \tau'(a', a, 0)$ where τ' is the ternary operation determined by $\mathfrak{F}'$.

10. The left inverse property $a^{-1} \dot{\tau} (a \dot{\tau} b) = b$ is valid iff $\imath_{\mathfrak{F}'}$ is independent on the change of $\zeta_{\mathfrak{F}'}$, by fixed $\eta_{\mathfrak{F}'}$. In fact, take $a \in S \setminus \{0\}$, $b \in S \setminus \{0, 1\}$ and construct the line $\{(x, y) \mid y = x \dot{\tau} b\}$, the points $(a, a_\tau b)$, $(1, a \dot{\tau} b)$, $(1, b)$, the line $\{(x, y) \mid y = x \dot{\tau} (a_\tau b)\}$ and finally the point $(a^{-1}, a^{-1} \dot{\tau} (a_\tau b))$. The mentioned independence of $\imath_{\mathfrak{F}'}$ is now expressed by $(a^{-1}, a^{-1} \dot{\tau} (a \dot{\tau} b)) \in \{x, y) \mid y = b\}$. Q.E.D.

11. The independence of $\imath_{\mathfrak{F}'}$ on the choice of $\eta_{\mathfrak{F}'}$ by fixed $\zeta_{\mathfrak{F}'}$ is equivalent to

(II) If $A_0A_1A_2A_3A_4A_5$, $B_0B_1B_2B_3B_4B_5$ are polygonal lines with $A_0 = B_0 = (a, 0) \neq 0$; $A_1 = (a, a)$; $A_0A_1 \parallel A_2A_3 \parallel A_4A_5$; $A_1A_2 \parallel \parallel A_3A_4 \parallel \xi_\mathfrak{F}$; $J \in A_2A_3$; $A_3 \in \zeta_\mathfrak{F}$; $A_4 \in OA_2$, $A_5 \in \xi_\mathfrak{F}$; $B_1 = (b, b) \neq 0$; $B_0B_1 \parallel B_2B_3 \parallel B_4B_5$; $B_1B_2 \parallel B_3B_4 \parallel \xi_\mathfrak{F}$; $B_3 \in \zeta_\mathfrak{F}$; $B_4 \in OB_2$; $B_5 \in \xi_\mathfrak{F}$ then $A_5 = B_5$.

The proof is obvious.

12. Let **A** satisfy the harmonic point axiom ([5]). Then $\imath_{\mathfrak{F}'}$ is independent on the change of $\eta_{\mathfrak{F}'}$ by fixed $\zeta_{\mathfrak{F}'}$.

Proof. If suffices to verify the configuration theorem (I). Let all assumptions of (I) be satisfied; we have to show that $A_5 = B_5$. Both polygonal lines $A_0A_1A_2A_3A_4A_5$, $B_0B_1B_2B_3B_4B_5$ can be „shortened" onto $A_0A_1JA_3A_5$, $A_0B_1JB_3B_5$ where $A_1J \parallel A_3A_5$, $BJ \parallel B_3B_5$.[2]) Now $A_5 = B_5$ follows by the further lemma: Let Q be constructed using the polygonal line $ABJCQ$ with $B = (b, b) \neq 0$, $JC \parallel AB$, $C \in \zeta_{\mathfrak{F}}$, $CQ \parallel JB$, $Q \in \xi_{\mathfrak{F}}$. Then $Q = (a^{-1}, 0)$. In fact, let h be the line through $(a^{-1}, 0)$ parallelly to BJ. The equations of JC and h are $y = (x \underset{\tau}{-} 1) \underset{\tau}{\cdot} ((b \underset{\tau}{-} a)^{-1}b]$ and $y = (x \underset{\tau}{-} a)^{-1} \underset{\tau}{\cdot} ((b \underset{\tau}{-} 1)^{-1} \underset{\tau}{\cdot} b$ respectively. The point of instersection C' of these lines lies on $\zeta_{\mathfrak{F}}$ iff $x = (x \underset{\tau}{-} 1) \underset{\tau}{\cdot} ((b \underset{\tau}{-} a)^{-1} \underset{\tau}{\cdot} b) = (x \underset{\tau}{-} a^{-1}) \underset{\tau}{\cdot} ((b \underset{\tau}{-} 1)^{-1} \underset{\tau}{\cdot} b)$, i.e. iff $x \underset{\tau}{\cdot} (1 \underset{\tau}{-} b^{-1} \underset{\tau}{\cdot} a) = x \underset{\tau}{-} 1$, $x \underset{\tau}{\cdot} (1 \underset{\tau}{-} b^{-1}) = x \underset{\tau}{-} a^{-1}$. By the distributivity law $x \underset{\tau}{\cdot} (y \underset{\tau}{+} z) = x \underset{\tau}{\cdot} y \underset{\tau}{+} x \underset{\tau}{\cdot} z$ we obtain in both cases $x = a^{-1} \underset{\tau}{\cdot} b$ so that $C = C'$. Q.E.D.

Corollary. If **A** satisfies the harmonic point axiom then $\iota_{\mathfrak{F}'}$ is independent on the choice of $\eta_{\mathfrak{F}'}$ and $\zeta_{\mathfrak{F}'}$.

For the case $1 + 1 \neq 0$ this is proved by other methods in [6].

13. Let **A** be such that the left inverse property (cf. Proposition 10) be valid. Construct the polygonal line $A_0A_1A_2A_3A_4$ with $A_0 = (a, 0) \neq 0$, $A_1 = (1, a)$, $A_2 = (1, 0)$, $A_3 = (a^{-1}, 1)$, $A_4 = (a^{-1}, 0)$. Then $A_0A_1 \parallel A_2A_3$

Proof. The lines A_0A_1, A_2A_3 have the slopes $(1 \underset{\tau}{-} a)^{-1} \underset{\tau}{\cdot} a$, $(a^{-1} \underset{\tau}{-} 1)^{-1}$ respectively. By the left and right inverse properties it follows that $(1 \underset{\tau}{-} a)^{-1} \underset{\tau}{\cdot} a^{-1} = (a^{-1} \underset{\tau}{-} 1)^{-1}$ is equivalent to $a^{-1} \underset{\tau}{-} 1 = a^{-1} \underset{\tau}{-} 1$. Q.E.D.

It is an open problem whether the independence of $\iota_{\mathfrak{F}'}$ on the choice of $\eta_{\mathfrak{F}'}$ and $\zeta_{\mathfrak{F}'}$ implies the remaining distributivity law.

REFERENCES

[1] M. Hall, The theory of groups, N. York 1959, § 20.7.
[2] The same book, § 20.3.
[3] G. Pickert, Projektive Ebenen, Berlin-Göttingen-Heidelberg 1955, p. 91.
[4] Cf. the cited Hall's book, § 20.4.
[5] N. S. Mendelsohn, Non-Desarguesian plane geometries which satisfy the harmonic point axiom, Canad. Journ. Math. 8 (1956), 532—562; cf. p. 540.
[6] Cf. Mendelsohn's article, pp. 550—551.

[2]) This can be verified by the direct computation.