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# INEQUALITIES FOR SOME WHITTAKER FUNCTIONS

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**1. Introduction and Statement of Results.** By a Whittaker function is meant a non-trivial solution of E. T. Whittaker's self-adjoint form of the differential equation for confluent hypergeometric functions [2; 3, Vol. 1, Ch. 6; 4 § 1.6]:

$$y'' + \left\{-\tfrac{1}{4} + \frac{\varkappa}{x} + \frac{\tfrac{1}{4} - \nu^2}{x^2}\right\} y = 0,\ 0 < x < \infty. \tag{1}$$

This has been studied extensively for complex values of the parameters $\varkappa$, $\nu$ and of the variables $x$, $y$, as well as in the real case. Here, however, the parameters and variables will be restricted to be real. (Reference [2] replaces $\nu$ by $\frac{1}{2}\mu$ in (1), with corresponding changes in other notations.)

Standard forms of the Whittaker functions are

$$M_{\varkappa,\nu}(x) = e^{-\frac{1}{2}x}\, x^{\frac{1}{2}+\nu}\, \frac{\Gamma(1+2\nu)}{\Gamma(\frac{1}{2}+2\nu-\varkappa)} \sum_{r=0}^{\infty} \frac{\Gamma(\nu-\varkappa+\frac{1}{2}+r)}{r!\,\Gamma(1+2\nu+r)}\, x^r \tag{2}$$

and

$$W_{\varkappa,\nu}(x) = \frac{\Gamma(-2\nu)}{\Gamma(-\nu-\varkappa+\frac{1}{2})} M_{\varkappa,\nu}(x) + \frac{\Gamma(2\nu)}{\Gamma(\nu-\varkappa+\frac{1}{2})} M_{\varkappa,-\nu}(x), \tag{3}$$

where, in case $2\nu$ is an integer, we follow the usual convention of taking the limit of the right member of (3) as $2\nu$ approaches the integer involved. When $2\nu$ is an odd negative integer, the function $M_{\varkappa,\nu}(x)$ is not defined, but the function $M_{\varkappa,\nu}(x)/\Gamma(1+2\nu)$ is.

Important special cases of (2) and (3) include

$$M_{0,\nu}(2x) = \Gamma(1+\nu)\, 2^{2\nu+\frac{1}{2}}\, x^{\frac{1}{2}}\, I_\nu(x),\ \nu > -1, \tag{4}$$

and

$$W_{0,\nu}(x) = (x/\pi)^{\frac{1}{2}}\, K_\nu(\tfrac{1}{2}x), \tag{5}$$

where $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions of order $\nu$.

Concerning these special cases, R. P. Soni [5] has shown that

(A) $$I_\nu(x) - I_{\nu+1}(x) > 0 \text{ for } \nu > -\tfrac{1}{2},\ x > 0$$

and

(B) $$K_{\nu+\varepsilon}(x) - K_\nu(x) > 0 \text{ for } \nu > 0,\ \varepsilon > 0,\ x > 0.$$

(Inequality (A) is true also for $\nu = -\frac{1}{2}$; the definitions of $I_{\frac{1}{2}}(x)$ and $I_{-\frac{1}{2}}(x)$ make this obvious. Inequality (B) holds also for $\nu = 0$, as Soni's own proof shows.) His proofs were based on integral representations of $I_\nu(x)$ and $K_\nu(x)$.

Here these inequalities will be generalized to Whittaker functions and, at the same time, made more precise. The function $W_{\varkappa,\nu}(x)$ will be considered as it stands, but it is convenient to replace $M_{\varkappa,\nu}(x)$ by another solution of (1), $m_{\varkappa,\nu}(x)$, defined as follows:

$$\begin{aligned} m_{\varkappa,\nu}(x) &= \frac{\Gamma(\frac{1}{2} + \nu - \varkappa)}{\Gamma(\frac{1}{2} + \nu)} \frac{\sqrt{2\pi}\, M_{\varkappa,\nu}(x)}{2^{2\nu+\frac{1}{2}}\Gamma(1+\nu)} \\ &= \frac{\Gamma(\frac{1}{2} + \nu - \varkappa)}{\Gamma(1 + 2\nu)} M_{\varkappa,\nu}(x) \qquad (6) \\ &= e^{-\frac{1}{2}x} x^{\nu+\frac{1}{2}} \sum_{r=0}^{\infty} \frac{\Gamma(\nu - \varkappa + r + \frac{1}{2})}{r!\,\Gamma(2\nu + 1 + r)} x^r, \end{aligned}$$

where the equality of the middle two members of (6) follows from Legendre's duplication formula for the gamma function [3, Vol. 1, p. 5 (15)]. When $\varkappa = 0$, $m_{\varkappa,\nu}(x)$ is defined for all $\nu$, and when $\varkappa \neq 0$, $m_{\varkappa,\nu}(x)$ is meaningful when $\nu + \frac{1}{2} - \varkappa$ is other than a non-positive integer.

A non-trivial solution of (1) can have only a finite number of positive zeros [6, p. 20 (Theorem 1.82.3)], since the coefficient of $y(x)$ in (1) is negative for $x > 2\varkappa + (4\varkappa^2 - 4\nu^2 + 1)^{\frac{1}{2}}$, if this last quantity is real, and for all $x > 0$ otherwise. In some instances (e.g., $\varkappa = 0, \nu > -1$, when the modified Bessel functions $I_\nu(x)$, $K_\nu(x)$ arise), there are no positive zeros at all. In other cases (cf. [2, pp. 208—216] and [4, Chapter 5]), positive zeros do occur, although in a finite number.

Accordingly, it is appropriate to *define the symbol* $z(\varkappa, \nu)$ *to be the largest positive zero of* $m_{\varkappa,\nu}(x)$, *if there are any positive zeros, and to be* 0 *otherwise, and the symbol* $\zeta(\varkappa, \nu)$ *similarly for* $W_{\varkappa,\nu}(x)$.

In particular, $\zeta(0, \nu) = 0$, and, when $\nu \geqq -1$, $z(0, \nu) = 0$. Moreover, $z(\varkappa, \nu) = 0$ if $\varkappa < \frac{1}{2} + \nu$ and $\nu > -\frac{1}{2}$, since the coefficients in the power series in (6) are all positive under these conditions, while $\zeta(\varkappa, \nu) < 2\varkappa +$

$+ (4\varkappa^2 - 4\nu^2 + 1)^{\frac{1}{2}}$ [6, p. 20 (Theorem 1.82.3)], since the coefficient of $y(x)$ in (1) is negative for larger values of $x$ and (cf. § 3) $W_{\varkappa,\nu}(x) = o(1)$ as $x \to \infty$. More detailed information on the zeros of Whittaker functions can be gleaned from [3, Vol. 1, pp. 288—289; 4, Chapter 6].

In formulating the results to be established here, an obvious notation is helpful: the expression "$g(x) \uparrow 1, a < x \uparrow b$" means that $g(x)$ increases in the interval $a < x < b$ and has limit 1 as $x$ increases to $b$. Similarly for "$g(x) \downarrow 1, a < x \uparrow b$."

In this symbolism, the results are:

**Theorem 1.** *If* $z(\varkappa, \nu) \leqq z(\varkappa, \nu + \varepsilon)$, *then*

$$0 < \frac{m_{\varkappa,\nu+\varepsilon}(x)}{m_{\varkappa,\nu}(x)} \uparrow 1,\ z(\varkappa, \nu + \varepsilon) < x \uparrow \infty, \tag{7}$$

*where* $\varepsilon > 0, \nu > -\frac{1}{2}\varepsilon$ *and, when* $\varkappa \neq 0$, *neither* $\frac{1}{2} + \nu - \varkappa$ *nor* $\frac{1}{2} + \nu + \varepsilon - \varkappa$ *equals* 0, —1, —2, .... *The conclusion holds also when* $\nu = -\frac{1}{2}\varepsilon$ (here $\nu < 0$), *if* $\Gamma(\frac{1}{2} - \frac{1}{2}\varepsilon - \varkappa)\, \Gamma(\frac{1}{2} + \frac{1}{2}\varepsilon - \varkappa) \sin(\pi\varepsilon) > 0$.

*When* $\varkappa = 0$ *and* $\nu > -1$, *we have*

$$0 < \frac{I_{\nu+\varepsilon}(x)}{I_\nu(x)} \uparrow 1, \quad 0 < x \uparrow \infty, \tag{8}$$

*for* $\varepsilon > 0$, $\nu > -\frac{1}{2}\varepsilon$, *and if* $\sin(\frac{1}{2}\pi\varepsilon) > 0$, *also for* $\nu = -\frac{1}{2}\varepsilon$.

**Theorem 2.** *If* $z(\varkappa, \nu) > z(\varkappa, \nu + \varepsilon)$, *then either*

$$\frac{m_{\varkappa,\nu+\varepsilon}(x)}{m_{\varkappa,\nu}(x)} \downarrow 1, \qquad z(\varkappa, \nu) < x \uparrow \infty, \tag{9}$$

*or there exist unique* $\alpha_1, \alpha_2$, *with* $z(\varkappa, \nu) < \alpha_2 < \alpha_1 < \infty$, *such that*

$$\frac{m_{\varkappa,\nu+\varepsilon}(x)}{m_{\varkappa,\nu}(x)} \downarrow \frac{m_{\varkappa,\nu+\varepsilon}(\alpha_1)}{m_{\varkappa,\nu}(\alpha_1)} < 1,\ z(\varkappa, \nu) < x \uparrow \alpha_1, \tag{10}$$

*while*

$$0 < \frac{m_{\varkappa,\nu+\varepsilon}(x)}{m_{\varkappa,\nu}(x)} \uparrow 1,\ \alpha_1 < x \uparrow \infty, \tag{11}$$

*and*

$$m_{\varkappa,\nu+\varepsilon}(\alpha_2) = m_{\varkappa,\nu}(\alpha_2). \tag{12}$$

*Again, it is assumed throughout that* $\nu > -\frac{1}{2}\varepsilon$, $\varepsilon > 0$ *and that neither* $\frac{1}{2} + \nu - \varkappa$ *nor* $\frac{1}{2} + \nu + \varepsilon - \varkappa$ *equals* 0, —1, —2, ....

The relation (8) shows that the hypotheses of Theorem 1 can be realized, with $z(\varkappa, \nu + \varepsilon) = z(\varkappa, \nu) = 0$. It is also possible to have $z(\varkappa, \nu) > z(\varkappa, \nu + \varepsilon)$, as supposed in Theorem 2. One may take $\varkappa = 0{,}3$, $\nu = -1{,}1$, $\varepsilon = 3$ (note that $\nu > -\frac{1}{2}\varepsilon$) and observe from [3, Vol. 1, p. 289] that $z(\varkappa, \nu) > 0$, while $z(\varkappa, \nu + \varepsilon) = 0$.

The inequality (8) both generalizes (A) and makes it more precise. A corresponding extension of (B) is

**Theorem 3.**

$$\frac{W_{\varkappa,\nu+\varepsilon}(x)}{W_{\varkappa,\nu}(x)} \downarrow 1, \qquad \zeta(\varkappa,\nu) < x \uparrow \infty, \tag{13}$$

*when* $\varepsilon > 0$, $\nu \geqq 0$.

Putting $\varkappa = 0$ yields the following more precise version of (B):

$$\frac{K_{\nu+\varepsilon}(x)}{K_{\nu}(x)} \downarrow 1, \qquad 0 < x \uparrow \infty \tag{14}$$

*for* $\varepsilon > 0$, $\nu \geqq 0$.

The proofs of (7) and (13), and of their respective corollaries (8) and (14), will be based on the Whittaker differential equation (1). The essential tools are Sturm-type comparison theorems incorporating side conditions introduced by G. Szegö (cf. [6, pp. 18—21]).

**2. The Comparison Theorems.** The theorem appropriate to the proof of (7) and (8) is

(I) *Let* $y(x)$ *and* $Y(x)$ *be positive solutions of the differential equations*

$$y'' + f(x)\, y = 0, \qquad Y'' + F(x)\, Y = 0, \qquad a < x < b,$$

*respectively, with* $f(x) < F(x)$, $a < x < b$, *and such that*

$$\lim_{x\to a+} \{y'(x)\, Y(x) - y(x)\, Y'(x)\} \geqq 0 \tag{15}$$

*and*

$$\lim_{x\to b-} \frac{y(x)}{Y(x)} \leqq 1. \tag{16}$$

*Then* $Y(x) > y(x)$, $a < x < b$, *and, if equality holds in* (16), $y(x)/Y(x) \uparrow 1$, $a < x \uparrow b$.

*Furthermore, the above conclusion still holds when* $f(x) \equiv F(x)$, $a < x < b$ *(so that* $y(x)$, $Y(x)$ *are solutions of the same differential equation), provided assumption* (15) *is replaced by the stronger condition*

$$\lim_{x\to a+} \{y'(x)\, Y(x) - y(x)\, Y'(x)\} > 0. \tag{15+}$$

The supplementary conclusion is essentially trivial, since the Wronskian of any pair of solutions of the equation $y'' + f(x)y = 0$ is a constant; in view of (15+), a positive constant for $Y$, $y$. Thus,

$$\left(\frac{y(x)}{Y(x)}\right)' = \frac{y'Y - yY'}{Y^2} > 0,$$

so that $y(x)/Y(x)$ increases for $a < x < b$.

The proof of the main conclusion follows a familiar pattern (and can be modified in an obvious way to incorporate the supplementary conclusion):

$$\begin{aligned}(y'Y - yY')' &= y''Y - yY'' \\ &= \{F(x) - f(x)\}\, y(x)\; Y(x) > 0,\ a < x < b,\end{aligned}$$

so that $y'Y - yY'$ increases for $a < x < b$. In view of (15), this implies that $y'Y - yY'$ is positive in the interval $(a, b)$. Hence $(y/Y)' > 0$, $a < x < b$, showing that $y(x)/Y(x)$ is an increasing function. The conclusion follows now from (16).

For the proof of (13) and (14), the following modification of (I) is used; its proof is essentially the same as that of (I):

(II) *Let $y$, $Y$ be positive solutions of the same differential equations as in* (I), *with, again, $f(x) < F(x)$, $a < x < b$, and suppose that*

$$\lim_{x \to b-} \{y'(x)\; Y(x) - y(x)\; Y'(x)\} \leqq 0, \tag{17}$$

$$\lim_{x \to b-} \frac{y(x)}{Y(x)} \geqq 1. \tag{18}$$

*Then $y(x) > Y(x)$, $a < x < b$, and, if equality holds in* (18), *$y(x)/Y(x) \downarrow 1$ as $a < x \uparrow b$.*

**3. Preliminaries.** Before applying the comparison theorems to prove Theorems 1 and 3 (the argument for Theorem 2 does not depend directly on a comparison theorem but is similar in spirit), some additional information is needed.

From the standard asymptotics of $M_{\varkappa,\nu}(x)$ [3, Vol. 1, p. 264 (1) and p. 278 (3)] and of $W_{\varkappa,\nu}(x)$ [3, Vol. 1, p. 262 (2) and p. 278 (1)], it is clear that

$$m_{\varkappa,\nu}(x) = e^{\frac{1}{2}x} x^{-\varkappa}[1 + O(1/x)],\ x \to \infty, \tag{19}$$

when $\frac{1}{2} + \nu - \varkappa \neq 0, -1, -2, \ldots$ (for $\varkappa \neq 0$), and, always,

$$W_{\varkappa,\nu}(x) = e^{-\frac{1}{2}x} x^{\varkappa}[1 + O(1/x)],\ x \to \infty. \tag{20}$$

In particular, both $m_{\varkappa,\nu}(x)$ and $W_{\varkappa,\nu}(x)$ are positive for all sufficiently large $x$. Thus, $m_{\varkappa,\nu}(x) > 0$ when $x > z(\varkappa, \nu)$ and $W_{\varkappa,\nu}(x) > 0$ when $x > \zeta(\varkappa, \nu)$.

Formula (2.4.24) of [4, p. 25] states that

$$x W'_{\varkappa,\nu}(x) = (\tfrac{1}{2}x - \varkappa)\; W_{\varkappa,\nu}(x) - W_{\varkappa+1,\nu}(x) \tag{21}$$

and so

$$\begin{aligned}W'_{\varkappa,\nu+\varepsilon}(x) W_{\varkappa,\nu}(x) - W'_{\varkappa,\nu}(x) W_{\varkappa,\nu+\varepsilon}(x) = x^{-1} W_{\varkappa+1,\nu}(x)\, W_{\varkappa,\nu+\varepsilon}(x) - \\ - x^{-1} W_{\varkappa+1,\nu+\varepsilon}(x)\; W_{\varkappa,\nu}(x)\end{aligned}$$

which, in view of (20), shows that

(22) $W'_{\varkappa,\nu+\varepsilon}(x)\, W_{\varkappa,\nu}(x) - W'_{\varkappa,\nu}(x)\, W_{\varkappa,\nu+\varepsilon}(x) = O(e^{-x}x^{2\varkappa-1}) = o(1),$

as $x \to \infty$.

The formula for $M_{\varkappa,\nu}(x)$ corresponding to (21) [4, p. 24 (2.4.12)] is complicated by the presence of the denominator $\frac{1}{2} + \nu - \varkappa$. The situation is simpler when $M_{\varkappa,\nu}(x)$ is replaced by $m_{\varkappa,\nu}(x)$, where the analogue to (21) is

(23) $$xm'_{\varkappa,\nu}(x) = m_{\varkappa-1,\nu}(x) - (\tfrac{1}{2}x - \varkappa)\, m_{\varkappa,\nu}(x).$$

Thus,

$$\begin{aligned} &x[m'_{\varkappa,\nu+\varepsilon}(x)\, m_{\varkappa,\nu}(x) - m'_{\varkappa,\nu}(x)\, m_{\varkappa,\nu+\varepsilon}(x)] \\ &= m_{\varkappa-1,\nu+\varepsilon}(x)\, m_{\varkappa,\nu}(x) - m_{\varkappa-1,\nu}(x)\, m_{\varkappa,\nu+\varepsilon}(x). \end{aligned}$$

Using the power series in (6), we obtain

(24) $$m'_{\varkappa,\nu+\varepsilon}(x)\, m_{\varkappa,\nu}(x) - m'_{\varkappa,\nu}(x)\, m_{\varkappa,\nu+\varepsilon}(x) \cong \gamma_{\varkappa,\nu} x^{2\nu+\varepsilon},\ x \to 0,$$

where

$$\begin{aligned} \gamma_{\varkappa,\nu} &= \frac{\Gamma(\frac{1}{2}+\nu-\varkappa)\,\Gamma(\frac{1}{2}+\nu+\varepsilon-\varkappa)\,\varepsilon}{\Gamma(1+2\nu)\,\Gamma(1+2\nu+2\varepsilon)} \\ &= 2^{-4\nu-2\varepsilon}\,\pi\,\frac{\Gamma(\frac{1}{2}+\nu+\varepsilon-\varkappa)\,\Gamma(\frac{1}{2}+\nu-\varkappa)\,\varepsilon}{\Gamma(\frac{1}{2}+\nu+\varepsilon)\,\Gamma(1+\nu+\varepsilon)\,\Gamma(\frac{1}{2}+\nu)\,\Gamma(1+\nu)}. \end{aligned}$$

Clearly, $\gamma_{0,\nu} > 0$ when $2\nu + \varepsilon \geqq 0$, $\varepsilon > 0$, $\nu > -1$.

**4. Proof of Theorem** 1. Here comparison theorem (I) is used, with $a = z(\varkappa, \nu + \varepsilon)$, $b = \infty$,

$$f(x) = -\tfrac{1}{4} + \frac{\varkappa}{x} + \frac{\frac{1}{4} - (\nu+\varepsilon)^2}{x^2}, \quad F(x) = -\tfrac{1}{4} + \frac{\varkappa}{x} + \frac{\frac{1}{4} - \nu^2}{x^2},$$

$$y(x) = m_{\varkappa,\nu+\varepsilon}(x), \quad Y(x) = m_{\varkappa,\nu}(x).$$

First, we consider the case $z(\varkappa, \nu + \varepsilon) = 0$.

When $2\nu + \varepsilon > 0$, it is clear that $f(x) < F(x)$ so that it remains to verify (15) and (16), the latter with equality, since $m_{\varkappa,\nu}(x)$ and $m_{\varkappa,\nu+\varepsilon}(x)$ are both positive for $x > z(\varkappa, \nu + \varepsilon) \geqq z(\varkappa, \nu)$. But (15) is obvious from (24), and (16), with equality, from (19). Thus (7) follows from comparison theorem (I) when $2\nu + \varepsilon > 0$.

For $\varkappa = 0$, it is clear that $\gamma_{\varkappa,\nu}$ and all other quantities involved in the various calculations are well-defined for $\nu > -1$, and that $z(\varkappa, \nu) = z(\varkappa, \nu + \varepsilon) = 0$. This verifies (8).

When $2\nu + \varepsilon = 0$, the functions $m_{\varkappa,\nu+\varepsilon}(x)$ and $m_{\varkappa,\nu}(x)$ are solutions of the same differential equation, since $f(x) = F(x)$. Therefore the

Wronskian $m'_{\varkappa,\nu+\varepsilon}(x)\, m_{\varkappa,\nu}(x) - m'_{\varkappa,\nu}(x)\, m_{\varkappa,\nu+\varepsilon}(x)$ is a constant. The sign of the derivative of $m_{\varkappa,\nu+\varepsilon}(x)/m_{\varkappa,\nu}(x)$ is the same as the sign of the Wronskian.

Applying the definition (6) to the known value of the Wronskian of $M_{\varkappa,\nu}(x)$, $M_{\varkappa,-\nu}(x)$ with $\nu = -\frac{1}{2}\varepsilon$ (formula (2.4.26) of [4, p. 26]), we find that

$$m'_{\varkappa,\varepsilon/2}(x)\, m_{\varkappa,-\varepsilon/2}(x) - m_{\varkappa,\varepsilon/2}(x)\, m'_{\varkappa,-\varepsilon/2}(x)$$
$$= \frac{\varepsilon\Gamma(\frac{1}{2} - \frac{1}{2}\varepsilon - \varkappa)\, \Gamma(\frac{1}{2} + \frac{1}{2}\varepsilon - \varkappa)}{\Gamma(1-\varepsilon)\, \Gamma(1+\varepsilon)}.$$

The right hand member can be transformed by using the familiar relations $\Gamma(1+\varepsilon) = \varepsilon\Gamma(\varepsilon)$, $\Gamma(\varepsilon)\,\Gamma(1-\varepsilon) = \pi \csc(\pi\varepsilon)$ [3, Vol. 1, p. 3], becoming $\pi^{-1}\,\Gamma(\frac{1}{2} - \frac{1}{2}\,\varepsilon - \varkappa)\,\Gamma(\frac{1}{2} + \frac{1}{2}\,\varepsilon - \varkappa)\sin(\pi\varepsilon)$.

When this constant is positive, it follows that the derivative of $m_{\varkappa,\varepsilon/2}(x)/m_{\varkappa,-\varepsilon/2}(x)$ is also positive. Thus, this function increases to the limit it approaches as $x \to \infty$, which, from (19), is 1.

In case $\varkappa = 0$, we have again that $z(0, \nu) = z(0, \nu + \varepsilon) = 0$, $\nu > -1$. The Wronskian simplifies to the quantity $2 \sin(\frac{1}{2}\varepsilon\pi)$.

This completes the proof of Theorem 1 when $z(\varkappa, \nu+\varepsilon) = z(\varkappa, \nu) = 0$.

When $z(\varkappa, \nu + \varepsilon) = \alpha > 0$, it suffices to establish (15) in considering the case $2\nu + \varepsilon > 0$, since (16) has already been verified. For (15) we have

$$\lim_{x\to\alpha} \{y'(x)\; Y(x) - y(x)\; Y'(x)\}$$
$$= y'(\alpha)\; Y(\alpha) - y(\alpha)\; Y'(\alpha)$$
$$= y'(\alpha)\; Y(\alpha),$$

since $\alpha > 0$ and $y(\alpha) = m_{\varkappa,\nu+\varepsilon}(\alpha) = 0$.

The assumption that $z(\varkappa, \nu) \leqq z(\varkappa, \nu + \varepsilon) = \alpha$ implies that $m_{\varkappa,\nu}(x) > 0$ for $x > \alpha$, so that $Y(\alpha) \geqq 0$. If $Y(\alpha) = 0$, the Wronskian would be zero at $\alpha = z(\varkappa, \nu + \varepsilon)$ and (15) would be verified.

Suppose now that $Y(\alpha) > 0$. Clearly, $y'(\alpha) \neq 0$, since $y(\alpha) = 0$. If $y'(\alpha) < 0$, then $y(x)$ would be negative for some $x > z(\varkappa, \nu + \varepsilon)$. But this would imply the existence of a zero $x_0$ of $y(x)$, $x_0 > z(\varkappa, \nu + \varepsilon)$, since $y(\infty) = +\infty$, contradicting the definition of $z(\varkappa, \nu + \varepsilon)$.

Therefore, $y'(\alpha) > 0$, and (15) is satisfied.

This proves the theorem for the case $z(\varkappa, \nu + \varepsilon) > 0$, $2\nu + \varepsilon > 0$.

All that remains is the case $z(\varkappa, \nu + \varepsilon) > 0$, $2\nu + \varepsilon = 0$.

Here $y(x)$ and $Y(x)$ are linearly independent solutions of the same differential equation (1), since $\Gamma(\frac{1}{2} - \frac{1}{2}\,\varepsilon - \varkappa)\,\Gamma(\frac{1}{2} + \frac{1}{2}\,\varepsilon - \varkappa)\sin(\pi\varepsilon) \neq 0$ according to our hypotheses. Hence $Y(\alpha) \neq 0$, and so $Y(\alpha) > 0$, since again, $Y(\infty) = +\infty$.

In this case, therefore, the Wronskian $y'(\alpha)\, Y(\alpha) - y(\alpha)\, Y'(\alpha) > 0$, so that, again, the derivative of $m_{\varkappa,\, \varepsilon/2}(x)/m_{\varkappa,\, -\varepsilon/2}(x)$ is positive for $x > z(\varkappa, \frac{1}{2}\varepsilon) = \alpha$. This ratio must, then, increase to its limit as $x \to \infty$; the limit is 1.

All parts of Theorem 1 are proved.

**5. Proof of Theorem 2.** As before, the differential equation (1) shows that the Wronskian $m'_{\varkappa,\, \nu+\varepsilon}(x)\, m_{\varkappa,\, \nu}(x) - m'_{\varkappa,\, \nu}(x)\, m_{\varkappa,\, \nu+\varepsilon}(x)$ increases where both $m_{\varkappa,\, \nu}(x)$ and $m_{\varkappa,\, \nu+\varepsilon}(x)$ are positive. Under the present hypothesis that $z(\varkappa, \nu + \varepsilon) < z(\varkappa, \nu) = \beta$, this is the case when $x > \beta$.

When $x = \beta$, this Wronskian is strictly *negative*, since $m_{\varkappa,\, \nu}(\beta) = 0$, $m'_{\varkappa,\, \nu}(\beta) > 0$, $m_{\varkappa,\, \nu+\varepsilon}(\beta) > 0$. (The two inequalities are, as in a similar case before, consequences of $m_{\varkappa,\, \nu}(\infty) = +\infty$.)

Hence, the derivative of the quotient $m_{\varkappa,\, \nu+\varepsilon}(x)/m_{\varkappa,\, \nu}(x)$ is negative for $\beta \leqq x < \alpha_1$, where $\alpha_1 \leqq +\infty$. (We take $\alpha_1$ to be the largest number satisfying these conditions.)

If $\alpha_1 = +\infty$, then (9) follows; if $\alpha_1 < +\infty$, then (10), (11) and (12) hold, since the Wronskian is a strictly increasing function of $x$ whose sign is the same as that of the derivative of $m_{\varkappa,\, \nu+\varepsilon}(x)/m_{\varkappa,\, \nu}(x)$, and this ratio has limit 1.

**6. Proof of Theorem 3.** In applying here comparison theorem (II), we have the same $f(x)$ and $F(x)$ as in § 4, with now

$$a = \max\{\zeta(\varkappa, \nu), \zeta(\varkappa, \nu + \varepsilon)\}, \qquad b = \infty,$$
$$y(x) = W_{\varkappa,\, \nu+\varepsilon}(x), \qquad Y(x) = W_{\varkappa,\, \nu}(x).$$

Thus, $f(x) < F(x)$ when $\nu \geqq 0$, $\varepsilon > 0$. From (22) it is seen that (17) is satisfied, with equality, and from (20) that (18) holds, with equality. Furthermore, $y(x)$ and $Y(x)$ are both positive for $x > a$, as explained in § 3, following (20).

This completes the proof of Theorem 3, except for demonstrating that *the largest positive zero of $W_{\varkappa,\, \nu}(x)$ is a non-increasing function of $\nu$*, i.e.,

$$\zeta(\varkappa, \nu + \varepsilon) \leqq \zeta(\varkappa, \nu), \ \varepsilon > 0. \tag{25}$$

Suppose the contrary, i.e., $\zeta(\varkappa, \nu + \varepsilon) > \zeta(\varkappa, \nu)$. Then $a = \zeta(\varkappa, \nu + \varepsilon)$ and the proof, as thus far given, would show that

$$\frac{W_{\varkappa,\, \nu+\varepsilon}(x)}{W_{\varkappa,\, \nu}(x)} \downarrow 1 \qquad \zeta(\varkappa, \nu + \varepsilon) < x \uparrow \infty;$$

in particular,

$$\frac{W_{\varkappa,\, \nu+\varepsilon}(x)}{W_{\varkappa,\nu}(x)} > 1, \qquad \zeta(\varkappa, \nu + \varepsilon) < x < \infty.$$

Then,

$$\lim_{x \to a} \frac{W_{\varkappa, \nu+\varepsilon}(x)}{W_{\varkappa,\nu}(x)} \geqq 1.$$

But

$$\lim_{x \to a} \frac{W_{\varkappa, \nu+\varepsilon}(x)}{W_{\varkappa, \nu}(x)} = 0, \qquad a = \zeta(\varkappa, \nu + \varepsilon) > \zeta(\varkappa, \nu),$$

a contradiction.

This establishes (25) and, with it, Theorem 3.

**Remarks 1.** Equality can occur in (25), since both quantities can be zero.

2. The monotonicity relation (25) contrasts with a standard theorem [1, p. 211 (Theorem 8.4.4)] on the monotonicity of zeros, whose hypotheses, of course, are not satisfied here.

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