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GENERALIZATION OF IDEAL TOPOLOGY IN CHAINS

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This paper deals with the solution of problems, which have set M. Novotný and J. Mayer in their paper, On some topologies on products of ordered sets (Archivum mathematicum, Brno, T. 1 (1965)). The following problems are concerned:

1. Is it possible to construct for every pair of infinite cardinal numbers $\mathfrak{m} < \mathfrak{n}$ an ordered set P such that $\tau_{\mathfrak{m}}(P) \neq \tau_{\mathfrak{n}}(P)$?

2. Is it possible to construct for every cardinal number $\mathfrak{m} > \aleph_1$ such an $\mathfrak{m}$-directed set P that for every pair of cardinal numbers $\mathfrak{p} < \mathfrak{n} < \mathfrak{m}$ the inequality $\tau_{\mathfrak{p}}(P) \neq \tau_{\mathfrak{n}}(P)$ holds?

1. IRREDUCIBLE $\mathfrak{m}$-IDEALS IN COMPLETE LATTICES AND ORDERED CONTINUA

1.1. Definition. Let $\mathfrak{m}$ be an infinite cardinal number, P an ordered set. A subset $I \subset P$ is called an $\mathfrak{m}$-ideal of P iff for every set $M, \emptyset \neq M \subset \subset I$ with card $M < \mathfrak{m}$ the inclusion $M^{*+} \subset I$ holds, where M^{*+} is the set of all lower bounds of the set of all upper bounds of M.

If P is any partially ordered set and if $x \in P$, then let $)x]$ denote the set of $t \in P$ such that $t \leqq x$.

1.2. Definition. Let P be any partially ordered set. A subset A of P is called a semi-ideal of P if, for every $x \in A$, $)x] \subseteq A$ (see [3]).

1.3. Lemma. *Let S be a complete lattice, $\mathfrak{m} = \aleph_\mu$ an infinite cardinal number, $A \subseteq S$ an semi-ideal. Then the following statements are equivalent:*

(α) *A is an $\mathfrak{m}$-ideal.*

(β) *For any set $\emptyset \neq M \subset A$,* card $M < \mathfrak{m}$, sup *$M \in A$ holds.*

1.4. Remark. Let S be a simply ordered complete lattice. Then, there are semi-ideals of two types in S:

$$)x] = \{t \mid t \in S,\ t \leqq x\}, \qquad)x) = \{t \mid t \in S,\ t < x\}.$$

1.5. Definition. Let $\mathfrak{m}$ be an infinite cardinal number, P an ordered set, $I \subset P$ an $\mathfrak{m}$-ideal. This ideal is called irreducible iff for every family $I_\mu (\mu \in N \neq \emptyset)$ of $\mathfrak{m}$-ideals with the property $I = \bigcap_{\mu \in N} I_\mu$ there exists an index $\mu_0 \in N$ such that $I_{\mu_0} = I$.

1.6. Lemma. *Let S be an ordered continuum (in the sense of* [6], *p. 370), $\mathfrak{m} = \aleph_\mu$ an infinite cardinal number, $x \in S$ a point, which is not the last element in S. Then $)x]$ is an $\mathfrak{m}$-ideal which is not irreducible.*

Proof. From 1.3. and 1.4. it follows that $)x]$ is an $\mathfrak{m}$-ideal. Let us choose a decreasing sequence of points $\{x_\alpha\}$, $\alpha \in N \neq \emptyset$, $x \neq x_\alpha \in S$ for all $\alpha \in N$, with the property $\inf \{x_\alpha\} = x$. Let us construct the intersection of all $\mathfrak{m}$-ideals $)x_\alpha]$. Then $\bigcap_{\alpha \in N})x_\alpha] =)x] \neq)x_\alpha]$ for all $\alpha \in N$, because $)x] \subset)x_\alpha]$. Consequently $)x]$ is not irreducible.

1.7. Definition. Let S be an ordered continuum. Say that a point $x \in S$ which is not the first element in S has the left-hand character $c_{\varrho,*}(x)$ iff ω_ϱ is the smallest ordinal number such that there exists in S an increasing sequence of the type ω_ϱ, that is cofinal with $)x)$.

1.8. Lemma. *Let S be an ordered continuum, $\mathfrak{m} = \aleph_\mu$ an infinite cardinal number, $x \in S$ a point which is not the first in S. Then, $)x)$ is an $\mathfrak{m}$-ideal iff for the left-hand character $c_{\varrho,*}(x)$ of the point x, $\varrho \geqq \alpha$ holds where*

$$\alpha = \begin{cases} \mu & \textit{for a regular } \aleph_\mu \\ \mu + 1 & \textit{for an irregular } \aleph_\mu. \end{cases}$$

Proof. I. Let $)x)$ be an $\mathfrak{m}$-ideal. Following 1.4. $)x)$ is an semi-ideal. Then, by 1.3, $\sup M \in)x)$ holds for every set $\emptyset \neq M \subset)x)$, card $M < \mathfrak{m}$. Hence, it follows that an arbitrary ordinal number which $)x)$ is cofinal with, fails to be cofinal with an ordinal number of cardinality $< \mathfrak{m}$. The smallest ordinal number possessing this property is ω_α, where

$$\alpha = \begin{cases} \mu & \text{for a regular } \aleph_\mu \\ \mu + 1 & \text{for an irregular } \aleph_\mu. \end{cases}$$

Thus, if $)x)$ is cofinal with a regular ordinal number $\omega_\varrho \geqq \omega_\alpha$ then the point x has the left-hand character $c_{\varrho,*}(x)$ and $\varrho \geqq \alpha$ holds.

II. Let a point $x \in S$ which is not the first element in S have the left-hand character $c_{\varrho,*}(x)$, $\varrho \geqq \alpha$ where

$$\alpha = \begin{cases} \mu & \text{for a regular } \aleph_\mu \\ \mu + 1 & \text{for an irregular } \aleph_\mu. \end{cases}$$

By 1.4. $)x)$ is a semi-ideal. With respect to the assumption concerning the left-hand character of the point x, $)x)$ is cofinal with ω_ϱ, $\varrho \geqq \alpha$. ω_ϱ is regular and, due to this fact, it is not cofinal with any smaller ordinal number and, consequently it fails to be cofinal with any ordinal number of cardinality $< \mathfrak{m}$. From this it follows that for every set $\emptyset \neq M \subset)x)$, card $M < \mathfrak{m}$, the formula $\sup M \in)x)$ holds. According to 1.3. $)x)$ is an $\mathfrak{m}$-ideal.

1.9. Lemma. *Let S be a chain, $x \in S$ a point which is not the first in S, $\mathfrak{m}$ an infinite cardinal number. If $)x)$ is an $\mathfrak{m}$-ideal, then it is irreducible.*

Proof. Indirect. Let $)x)$ be a reducible $\mathfrak{m}$-ideal. Then a system of m-ideals $A_\alpha \neq)x)$, $\alpha \in N \neq \emptyset$ exists such that $)x) = \bigcap_{\alpha\in N} A_\alpha$. Assuming $A_\alpha \supset)x)$, $A_\alpha \neq)x)$, we have $x \in A_\alpha$ for every $\alpha \in N$. Hence $x \in \bigcap_{\alpha\in N} A_\alpha$ and, consequently $)x) \neq)x] \subseteqq \bigcap_{\alpha\in N} A_\alpha =)x)$ which is a contradiction. Thus $)x)$ is irreducible.

2. CERTAIN PROPERTIES OF ORDERED CONTINUA

2.1. Definition. Let $\vartheta = \aleph_\pi$ be an infinite regular cardinal number. Let us denote C_π the lexicographically ordered set of all sequences $\{x_\xi\}$ composed of 0 and 1 of the type ω_π such that there does not exist $0 < \delta < < \omega_\pi$ with the property $x_{\delta-1} = 0$, $x_\xi = 1$ for $\delta \leqq \xi < \omega_\pi$.

2.2. Remark. C_π is an ordered continuum.

2.3. Lemma. (*A*) *Let $x \in C_\pi$ be an arbitrary element which is not the first. Then, for the left-hand character $c_{\varrho,*}(x)$ we have $0 \leqq \varrho \leqq \pi$.*

(*B*) *To every $0 \leqq \varrho_0 \leqq \pi$ with the property that ω_{ϱ_0} is regular, there exists a point x in C_π that is not the first and for whose left-hand character $c_{\varrho,*}(x)$, $\varrho = \varrho_0$ holds.*

Proof. (A). Let y be the last element in C_π. Then $\varrho = \pi$ holds for its left-hand character, because $)y)$ is cofinal with regular ordinal number ω_π. Let $x \in C_\pi$, $x \neq y$ be a point, which is not the first element in C_π. If $)x)$ is cofinal with some regular ordinal number ω_{ϱ_0}, then the point x has the left-hand character $c_{\varrho_0,*}(x)$. Let $)x)$ be cofinal with irregular ordinal number ω_β. Because this ordinal number ω_β is cofinal with some regular ordinal number ω_α, $\omega_\alpha < \omega_\beta$, ordinal number ω_α is cofinal with $)x)$ and the point x has the left-hand character $c_{\alpha,*}(x)$. In C_π there does not exist an increasing sequence of the type ω_ϱ for $\varrho > \pi$ (see [5]). Hence it follows that $0 \leqq \alpha \leqq \pi$.

(B). Let $0 \leqq \varrho_0 \leqq \pi$, ω_{ϱ_0} be regular. From the definition of C_π its follows that in C_π exists an increasing sequence of the type ω_{ϱ_0}. This sequence converges to some point $x \in C_\pi$, and consequently, this point x has the left-hand character $c_{\varrho_0,*}(x)$.

2.4. Definition. A chain A is said quasihomogenous if an arbitrary non-trivial interval $I \subset S$ contains a sub-interval $I' \subset I$ such that I' and S are isomorphic.

2.5. Lemma. *C_π is quasihomogenous.*

Proof. Let $I \subset C_\pi$ be an arbitrary interval with end-points $x = = \{x_\lambda\}_{\lambda<\omega_\pi} < y = \{y_\lambda\}_{\lambda<\omega_\pi}$. Then there exists the smallest $\delta < \omega_\pi$ such that $x_\delta = 0 < 1 = y_\delta$. Let $\varepsilon, \delta < \varepsilon < \omega_\pi$ be an ordinal number with the property $x_\varepsilon = 0$. Let us put $v = \{v_\lambda\}_{\lambda<\omega_\pi}$, $v_\lambda = x_\lambda$ for $\lambda < \varepsilon$, $v_\varepsilon = 1$,

$v_\lambda = 0$ for $\lambda > \varepsilon$. Then $x < v < y$. Furthermore, be ε' the smallest ordinal number having the properties $\varepsilon < \varepsilon' < \omega_\pi$, $x_{\varepsilon'} = 0$. Let us put $u = \{u_\lambda\}_{\lambda<\omega_\pi}$, $u_\lambda = x_\lambda$ for $\lambda < \varepsilon'$, $u_{\varepsilon'} = 1$, $u_\lambda = 0$ for $\lambda > \varepsilon'$. Then $x < u < v < y$ and there holds that $\langle u, v\rangle$ is isomorphic with C_π. Consequently C_π is quasihomogenous.

2.6. Corollary. Let $0 \leqq \varrho_0 \leqq \pi$, ω_{ϱ_0} be regular. Then, the set of all points $x \in C_\pi$ for whose left-hand character $c_{\varrho,*}(x)$ $\varrho = \varrho_0$ holds, is dense in C_π.

2.7. Remark. All Theorems and Definitions can be dualized.

2.8. Definition. Let it be $D_0 = C_0$. Let $p_\pi \in C_\pi$ be the first point in C_π for every $\pi > 0$. Then, we put $D_\pi = C_\pi - \{p_\pi\}$.

2.9. Definition. Let $\aleph_{\varkappa'}$ be an infinite regular cardinal number. We put $E_{\varkappa'} = \sum\limits^*_{0\leqq\lambda\leqq\varkappa'} D_\lambda$, where the symbol Σ^* stands for an ordinal sum over such indices λ for which it holds that ω_λ is regular.

2.10. Remark. $E_{\varkappa'}$ is an ordered continuum.

2.11. Lemma. *Let $\mathfrak{n} = \aleph_\nu$ be an infinite cardinal number, $x \in \sum\limits^*_{0\leqq\lambda<\nu} D_\lambda$ a point which is not the first. Then $\varrho < \nu$ holds for its left-hand character $c_{\varrho,*}(x)$.*

Proof. The statement follows from 2.3.

2.12. Definition. Let M be a set, $\mathfrak{M}$ a system of its subsets and $x, y \in M$, $x \neq y$ be arbitrary points. Say that points x, y are separable by elements of $\mathfrak{M}$, if at least one $M^\alpha \in \mathfrak{M}$ exists such that $x \in M^\alpha$, $y \in M^\alpha$ or vice versa.

Consequently, from 1.8. and 2.11. follows

2.13. Corrollary. Let $\mathfrak{n} = \aleph_\nu < \aleph_{\varkappa'}$ be an infinite cardinal number. Let $\mathfrak{A}_\nu$ be a system composed of all irreducible $\mathfrak{n}$-ideals and irreducible dual $\mathfrak{n}$-ideals in $\sum\limits^*_{0\leqq\lambda\leqq\varkappa'} D_\lambda$. Let it be $x, y \in \sum\limits^*_{0\leqq\lambda<\nu} D_\lambda$, $x \neq y$. Then these points are not separable by the elements of $\mathfrak{A}_\nu$.

2.14. Lemma. *Let $\mathfrak{n} < \aleph_{\varkappa'}$, $\mathfrak{n} = \aleph_\nu$ be an infinite cardinal number. Then the set of all points $x \in \sum\limits^*_{\alpha\leqq\lambda\leqq\varkappa'} D_\lambda$, where $\alpha = \begin{cases} \nu & \text{for a regular } \omega_\nu \\ \nu+1 & \text{for an irregular } \omega_\nu, \end{cases}$ for whose left-hand character $c_{\varrho,*}(x)$ $\varrho = \alpha$ holds, is dense in $\sum\limits^*_{\alpha\leqq\lambda\leqq\varkappa'} D_\lambda$.*

Proof. The statement follows from 2.6.

2.15. Corollary. Let it be $x, y \in \sum\limits^*_{\alpha\leqq\lambda\leqq\varkappa'} D_\lambda$, $x \neq y$ and let α possess the required property by Lemma 2.14. Then the points x, y are separable by the elements of $\mathfrak{A}_\nu$.

2.16. Definition. Let $\mathfrak{m}$ be an infinite cardinal number, P an ordered set. Let $(P, \tau_{\mathfrak{m}}(P))$ be a topological space where the topology is defined by taking the family of all irreducible $\mathfrak{m}$-ideals and irreducible dual $\mathfrak{m}$-ideals of P as a subbases for open sets. Then $\tau_{\mathfrak{m}}(P)$ is called the $\mathfrak{m}$-ideal topology on P.

2.17. Definition. Let $\mathfrak{m} < \mathfrak{n}$ be infinite cardinal number. Let us say that these numbers have the property (α), if the following holds: $\mathfrak{m} = \aleph_\mu$ is an infinite irregular cardinal number, $\mathfrak{n} = \aleph_\nu = \aleph_{\mu+1}$.

2.18. Main Theorem. *Let $\aleph_\varkappa > \aleph_1$ be an arbitrary infinite cardinal number, let $\aleph_{\varkappa'} \geqq \aleph_\varkappa$ be an infinite regular cardinal number. Then for every pair of infinite cardinal numbers $\mathfrak{m} < \mathfrak{n} < \aleph_\varkappa$, where $\mathfrak{m}$, $\mathfrak{n}$ have not the property (α), $\tau_{\mathfrak{m}} \neq \tau_{\mathfrak{n}}$ on $E_{\varkappa'}$ holds.*

Proof. The statement follows from 2.13., 2.14. and 2.15.

2.19. Remark. In case $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{m} < \mathfrak{n} < \aleph_\varkappa$ possess the property (α), then $\tau_{\mathfrak{m}} = \tau_\varkappa$ on $E_{\varkappa'}$.

Problem. Is it possible to construct for every cardinal number $\aleph_\varkappa > \aleph_1$ an ordered set P so that for every pair of infinite cardinal numbers $\mathfrak{m} < \mathfrak{n} < \aleph_\varkappa$ where $\mathfrak{m}$, $\mathfrak{n}$ possess the property (α), $\tau_{\mathfrak{m}}(P) \neq \tau_{\mathfrak{n}}(P)$ holds?

BIBLIOGRAPHY

[1] J. Mayer and M. Novotný: *On some topologies on products of ordered sets*, Archivum mathematicum, Brno, T 1, (1965), 251—257.

[2] Orrin Frink: *Ideals in partially ordered sets*, Amer. Math. Monthly, 61 (1954), 223—234.

[3] G. N. Raney: *Completely distributive complete lattices*, Proc. Am. Math. Soc. 3 (1952), 677—680.

[4] F. Hausdorff: *Mengenlehre*, Leipzig 1914.

[5] W. Sierpinski: *Sur une propriété des ensembles ordonnés*, Fundamenta Mathematica, V. 36, (1949), 56—67.

[6] J. Novák: *O někotorych charakteristikach uporjadočennovo kontinuuma*, Čechosl. mat. ž., 2 (77), (1952), 369—386.