Archivum Mathematicum

Igor Kluvánek

# Characterization of scalar-type spectral operators

*Archivum Mathematicum*, Vol. 2 (1966), No. 4, 153--156

Persistent URL: http://dml.cz/dmlcz/104618

**Terms of use:**

© Masaryk University, 1966

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

# CHARACTERIZATION OF SCALAR-TYPE SPECTRAL OPERATORS

Igor Kluvánek, Košice

Received: May 21, 1966

Much attention is being devoted in literature to the problem of characterization of spectral operators in the Dunford sense [1, 2] since this problem has not yet been solved in a satisfactory way. A characterization in terms of operational calculus (i. e. in terms of a homomorphism of an algebra of functions into the algebra of operators) is a very useful one and permits various generalizations (see e.g. [4, 6]). The main theorem of the present note gives a characterization of spectral operators in terms of this kind. The corollaries of this theorem present generalizations of some criteria known from literature (especially [5]).

Let $X$ be a Banach space, $L(X)$ the algebra of all bounded linear operators in $X$. If $T \in L(X)$ is an operator $\sigma(T)$ stands for its spectrum. The set of all complex numbers will be denoted by $C_1$. We use the notations $R_1 = \{\lambda \mid \lambda \text{ real}, \lambda \in C_1\}$, $K_1 = \{\lambda \mid |\lambda| = 1, \lambda \in C_1\}$.

Let $K$ be a set and $A$ an algebra of complex-valued functions on $K$. Let $t : A \to L(X)$ be an homomorphism of $A$ into $L(X)$. The homomorphism $t$ is said to be weakly $(K, X)$-compact if, for every $x \in X$, the set

$$\{t(f)\, x \mid \quad f \in A,\ \sup_{\lambda \in K} |f(\lambda)| \leqq 1\}$$

is relatively weakly compact in $X$.

If $K$ is a topological space (e.g. a subset of $C_1$), then $C(K)$ will stand for the algebra of all complex bounded continuous functions on $K$ and $B(K)$ will stand for the system of all Borel subsets of $K$.

We denote by $j$ the identity function on $C_1$ (i.e. $j(\lambda) = \lambda$ for $\lambda \in C_1$) or, sometimes, its restriction to a subset of $C_1$.

A spectral measure is a strongly $\sigma$-additiveand multiplicative $L(X)$-valued function $E$ on a $\sigma$-algebra of subsets of a set $K$ such that $E(\emptyset) = O$ and $E(K) = I$, where $O$ is the zero operator and $I$ is the identity operator.

An operator $T \in L(X)$ is said to be a scalar-type spectral operator if there exists a spectral measure $E : B(\sigma(T)) \to L(X)$ such that

$$T = \int j \,\mathrm{d}E. \tag{1}$$

A scalar-type operator $T$ is called pseudohermitian if $\sigma(T) \subset R_1$ and pseudounitary if $\sigma(T) \subset K_1$.

**Theorem.** *An operator $T \in L(X)$ is a scalar-type spectral operator if and only if there exists a compact space K and a weakly $(K, X)$-compact* *homomorphism* $t : C(K) \to L(X)$ *such that there exists a function* $f_0 \in C(K)$ *for which* $t(f_0) = T$.

Proof. If $T$ is a scalar type spectral operator we put $K = \sigma(T)$ and $t(f) = \int f \, dE$, $f \in C(\sigma(T))$, where $E$ is the spectral measure from (1). It is known that $t$ is an homomorphism of $C(\sigma(T))$ into $L(X)$. According to [3; VI. 7. 3] the mapping $f \to t(f)\, x$ from $C(\sigma(T))$ into $X$ is weakly compact for every $x \in X$. Therefore $t$ is a weakly $(K, X)$-compact homomorphism.

Now suppose $K$ is a compact space and $t : C(K) \to L(X)$ a weakly $(K, X)$-compact homomorphism. By [3; VI. 7. 3], for every $x \in X$, there exists a regular $X$-valued measure $m_x$ such that $t(f)\, x = \int f \, dm_x$, $f \in C(K)$, and $\sup_{\tau \in B(K)} || m_x(\tau) || \leqq || t || \; || x ||$. Because $m_x(\tau)$ is determined uniquely by $\tau$, $x$ and depends linearly and continuously on $x$, for every $\tau \in B(K)$, we may put $F(\tau)\, x = m_x(\tau)$. Evidently $F(\tau) \in L(X)$, $\tau \in B(K)$. The function $F : B(K) \to L(X)$ is an operator-valued measure such that $t(f) = \int f \, dF$ in the sense that $t(f)\, x = \int f(s) \, dF(s)\, x$, for every $x \in X$. It is easy to prove (by passing to limits) that $F$ is multiplicative (see e.g. [7; Lemma 6]). It follows that $F$ is a spectral measure. By hypothesis $T = t(f_0) = \int f_0 \, dF$. According to [1], $T$ is a scalar-type spectral operator and we have (1) if we define $E(\varrho) = F(\{s \mid f_0(s) \in \varrho\})$ for $\varrho \in B(\sigma(T))$.

Remark. Following [3; VI. 7. 6], if $X$ is a weakly (sequentially) complete space, an homomorphism $t : C(K) \to L(X)$ is weakly $(K, X)$-compact if and only if it is continuous (in the strong operator topology). For this case the theorem is given in [6].

**Corollary 1.** *$T$ is a scalar-type spectral operator if and only if there exists a weakly $(T, X)$-compact homomorphism $t : C(\sigma(T)) \to L(X)$ such that $t(j) = T$.*

If $K \subset C_1$ is a compact set and $t : C(K) \to L(X)$ is a weakly $(K, X)$-compact homomorphism such that $t(j) = T$, then $T$ is ascalar-type spectral operator.

In the sequel we give some corollaries of the theorem in which the criterium of spectrality is given in terms of operational calculus $f \to f(T)$ defined for holomorphic functions by the means of Cauchy formula [3; VII. 3].

Let $K \subset C_1$ be a compact set and $A$ an algebra of holomorphic functions on $K$ (i.e. for every $f \in A$ there exists an open set $U_f \supset K$ such that $f$ is holomorphic on $U_f$). If the uniform closure of the algebra consisting of restrictions of functions belonging to $A$ is identical with $C(K)$, then the set $K$ is called an $A$-set.

**Corollary 2.** *Let $K \subset C_1$ ve a compact set, $T \in L(X)$, $\sigma(T) \subset K$. Let $A$ be an algebra consisting of functions holomorphic on $K$. Let $K$ be an $A$-set. The operator $T$ is a scalar-type spectral operator if and only if the ope-*

*rational calculus $f \to f(T)$, $f \in A$, is a weakly $(K, X)$-compact homomorphism of $A$ into $L(X)$.*

Proof. $K$ being an $A$-set the homomorphism $f \to f(T)$ can be extended uniquely by continuity on a homomorphism $t : C(K) \to L(X)$. The set $\{t(f)\, x \mid f \in C(K),\ \sup_{\lambda \in K} |f(\lambda)| \leqq 1\}$ is a part of the closure of $\{f(T)\, x \mid f \in A,\ \sup_{\lambda \in K} |f(\lambda)| \leqq 1\}$ and, therefore, by Eberlein-Šmuljan theorem [3; V. 6. 1] it is relatively weakly compact in $X$.

Let $N$ be the set of all integers and $\mathbf{N}$ the system of all its finite subsets. Denote by $P$ the set of all functions of the form

$$f(\lambda) = \sum_{n \in \nu} a_n \lambda^n$$

where $a_n$ are complex numbers and $\nu \in \mathbf{N}$.

**Corollary 3.** *An operator $T \in L(X)$ is pseudounitary if and only if the homomorphism $f \to f(T)$ of $P$ into $L(X)$ is weakly $(K_1, X)$-compact.*

Proof. It is known, that $K_1$ is a $P$-set.

Denote by $Q$ the algebra of all trigonometric polynomials, i.e. of functions of the type

$$f(\lambda) = \sum_{n \in \nu} a_n e^{in\lambda}$$

where $a_n$ are complex numbers and $\nu \in \mathbf{N}$.

**Corollary 4.** *Let $T \in L(X)$, $\| T \| < \pi$. The operator $T$ is pseudohermitian if and only if the homomorphism $f \to f(T)$ of the algebra $Q$ into $L(X)$ is weakly $(R_1, X)$-compact.*

Proof. Denote by $\alpha = \| T \|$ and choose $\beta$ so that $\alpha < \beta < \pi$. The segment $\langle -\beta, \beta \rangle$ is a $Q$-set and $f(T) = g(T)$ if $f(\lambda) = g(\lambda)$ for $|\lambda| \leqq \beta$.

Remarks. 1. If $X$ is a weakly complete space in all corollaries the weak $(K, X)$-compactness of considered homomorphisms may be replaced by the requirement of its continuity in the uniform-norm topology of respective algebra of functions.

2. If $X$ is a reflexive space (in this case it is also weakly complete), the criterium contained in Corollary 4 presents a simplification of criteria from [5] (Theorem 4). In [5] there is exploited the group $e^{itT}$, $t \in R_1$, or the algebra of operators generated by this group. Since $T$ is a bounded operator it suffices to consider the subgroup $e^{in\beta T}$, $n \in N$, where $\beta < \pi / \| T \|$, i.e. to consider the powers of the operator $e^{i\beta T}$.

## REFERENCES


[1] Dunford N.: Spectral operators, Pacific J. Math. *4* (1954), 321—354.

[2] Dunford N.: A survey of the theory of spectral operators, Bull. Amer. Math. Soc. *64* (1958), 217—274.

[3] Dunford N., Schwarz J. T.: Linear Operators. Part I. General Theory. Interscience Publ. New York 1958.
[4] Foiaş C.: Une application des distributions vectorielles à la théorie spectrale, Bull. Sci. Math. *84* (1960), 147—158.
[5] Kantorowitz S.: On the characterization of spectral operators, Trans. Amer. Math. Soc. *111* (1964), 152—181.
[6] Kantorowitz S.: Classification of operators by means of their operational calculus, Trans. Amer. Math. Soc. *115* (1965), 194—224.
[7] Kluvánek I.: Characterization of Fourier—Stieltjes transforms of vector and operator valued measures, Czech. Math. J. (to appear).