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ON ISOTONE AND HOMOMORPHIC MAPS OF ORDERED PARTIAL GROUPOIDS

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In this paper the following problem is studied: Let G, G' be ordered sets which are simultaneously partial groupoids. Under what circumstances is the family of all isotone maps of G into G' identical with the family of all homomorphic maps of G into G'? The problem is solved in the case of commutative partial operations in G and G'.

Our problem is studied in more special form in [1], where it is assumed that G, G' are ordered sets and simultaneously so called o-groupoids. Therefore the main result of [1] is a corollary of our Theorem 1.

An ordered set G in which $x, y \in G$, $x \leqq y$ implies $x = y$ is called an *antichain*.

Let G be a set. If to some pairs of elements $a, b \in G$ an element $c \in G$, written $c = ab$, is assigned, then G is called a *partial groupoid*. A partial groupoid G in which $a, b \in G$, ab defined implies that ba is defined and $ba = ab$ holds is called *commutative*. A map of a partial groupoid G into a partial groupoid G' such that $a, b \in G$, ab defined implies that $f(a) f(b)$ is defined and $f(ab) = f(a) f(b)$ holds is called *homomorphic*.

In the following it is assumed that G, G' are ordered sets and commutative partial groupoids. I denotes the family of all isotone maps of G into G', H the family of all homomorphic maps of G into G'.

Lemma 1. *Let ab be defined for a pair of elements $a, b \in G$ and let $I \subseteq H$. Then a'^2 is defined and $a'^2 = a'$ holds for arbitrary $a' \in G'$.*

Proof. Put $f(t) = a'$ for all $t \in G$, where a' is an arbitrary element of G'. It is $f \in I$, consequently $f \in H$. Therefore $f(a) f(b) = a'^2$ is defined and $a'^2 = f(a) f(b) = f(ab) = a'$ holds.

Lemma 2. *Let ab be defined for a pair of elements $a, b \in G$ such that $a < b$ and let $I \subseteq H$. Then for each pair of elements $a', b' \in G'$ such that $a' < b'$ $a'b'$ is defined and*

$$a'b' = \begin{cases} a' & \text{for } ab \leqq a \\ b' & \text{for } ab \nleqq a \end{cases}.$$

Proof. Put

$$f(t) = \begin{cases} b' & \text{for } t \nleqq a \\ a' & \text{for } t \leqq a \end{cases}.$$

It is $f \in I$, consequently $f \in H$. Therefore $f(a)f(b) = a'b'$ is defined and

$$a'b' = f(a)f(b) = f(ab) = \begin{cases} b' & \text{for } ab \nleqq a \\ a' & \text{for } ab \leqq a. \end{cases}$$

Lemma 3. *Let $I \subseteq H$ and let G' fail to be an antichain. Let ab be defined for a pair of elements $a, b \in G$. Then neither $ab \nleqq a$, $ab \nleqq b$, nor $ab \ngeqq a$, $ab \ngeqq b$.*

Proof. Assume that $ab \nleqq a$, $ab \nleqq b$. Let $a', b' \in G'$, $a' < b'$. Put

$$f(t) = \begin{cases} b' & \text{for } t \ngeqq ab \\ a' & \text{for } t \geqq ab \end{cases}$$

Then $f \in I$, so that $f \in H$ and we have $b'^2 = f(a)f(b) = f(ab) = a'$, which is a contradiction with Lemma 1.

The second part of the proof is analogical.

Lemma 4. *Let $I \subseteq H$ and let G' fail to be an antichain. Let ab be defined for a pair of elements $a, b \in G$. Then the elements a, b are comparable.*

Proof. Let $a', b' \in G'$, $a' < b'$. Assume that, on the contrary, the elements a, b are not comparable. Define

$$f_1(t) = \begin{cases} b' & \text{for } t \nleqq a \\ a' & \text{for } t \leqq a \end{cases}$$

and

$$f_2(t) = \begin{cases} b' & \text{for } t \nleqq b \\ a' & \text{for } t \leqq b \end{cases}$$

We have $f_1, f_2 \in I$, consequently $f_1, f_2 \in H$. Let $ab \leqq a$. Then $a'b' = f_1(a)f_1(b) = f_1(ab) = a'$. If it were $ab \nleqq b$, $a'b' = b'a' = f_2(a)f_2(b) = f_2(ab) = b'$ would hold, which is not possible. Therefore $ab \leqq b$. Since $a \parallel b$ and $ab \leqq a$, $ab \leqq b$, it is necessarily $ab < a$, $ab < b$ and we have a contradiction with Lemma 3. Consequently it is not possible that $ab \leqq a$ should hold; therefore $ab \nleqq a$. If we interchange the role of the elements a, b, we obtain $ab \nleqq b$. Analogically it would be shown $ab \ngeqq a$, $ab \ngeqq b$. Consequently it is $a \parallel ab \parallel b$ and we have again a contradiction with Lemma 3. The supposition $a \parallel b$ leads to a contradiction in all cases, therefore the elements a, b are comparable.

Lemma 5. *Let $I \subseteq H$ and let G' fail to be an antichain. Let ab be defined for a pair of elements $a, b \in G$. Then ab equals either a or b.*

Proof. Since ab is defined, the elements a, b are comparable by Lemma 4. It suffices to consider the case $a \leqq b$ with regard to the commutativity. Admit $a \neq ab \neq b$. Let $a', b' \in G'$, $a' < b'$. If it is $a \nleqq ab$, $b \nleqq ab$, we have a contradiction with Lemma 3. Say that $a \leqq ab$. Then $a < ab$. If it is $b \leqq ab$, we have $b < ab$, which is again a contradiction with Lemma 3. Thereby the lemma is proved for $a = b$. Consequently it remains to consider the case $a < b$. Let it be $a < ab$, $b \nleqq ab$. Define

$$f_1(t) = \begin{cases} b' & \text{for } t \geqq b \\ a' & \text{for } t \ngeqq b \end{cases}.$$

We have $f_1 \in I$, so that $f_1 \in H$ and $a'b' = f_1(a)f_1(b) = f_1(ab) = a'$. Further define

$$f_2(t) = \begin{cases} b' & \text{for } t > a \\ a' & \text{for } t \ngtr a \end{cases}.$$

Since $f_2 \in I \subseteq H$, we have $a'b' = f_2(a)f_2(b) = f_2(ab) = b'$. This is a contradiction with the preceding result, therefore $a < ab$ cannot hold. The last possibility $a \nleqq ab$, $b < ab$ is excluded by the transitivity.

Lemma 6. *Let $I \subseteq H$ and let G' fail to be an antichain. Let $a, b, c, d \in G$, $a < b$, $c < d$ $(a > b, c > d)$. Let the products ab, cd be defined and let $ab = a$. Then $cd = c$.*

Proof. It is either $cd = c$ or $cd = d$ by Lemma 5. Admit $cd = d$. Let $a', b' \in G'$, $a' < b'$. According to Lemma 2 we have partly $a'b' = a'$ $(a'b' = b')$, for $ab \leqq a$ $(ba \nleqq b)$, partly $a'b' = b'$ $(a'b' = a')$, for $cd \nleqq c$ $(dc \leqq d)$. Thereby we passed to a contradiction.

Remark 1. Lemmas 1, 2, 3, 5, 6 are valid even in the case that the partial operations in G and G' are not commutative. The proofs of Lemmas 1, 2, 3, 6 are the same, but Lemma 5 must be proved in another way.

Lemma 7. *Let $I = H$ and let G, G' fail to be antichains. Let $a', b' \in G'$, $a' \parallel b'$. Then $a' \neq a'b' \neq b'$.*

Proof. Admit $a'b' = a'$. We choose an arbitrary element $a \in G$ such that it is not maximal and define

$$f_1(t) = \begin{cases} b' & \text{for } t \nleqq a \\ a' & \text{for } t \leqq a \end{cases}.$$

Clearly $f_1 \notin I$, consequently $f_1 \notin H$. Therefore there exist elements a_1, $b_1 \in G$ such that a_1b_1 is defined, but either $f_1(a_1)f_1(b_1)$ is not defined or $f_1(a_1)f_1(b_1) \neq f_1(a_1b_1)$ holds. It cannot be $a_1 = b_1$, for then $f_1^2(a_1) = f_1(a_1^2)$ would hold by Lemma 1 and 5. According to Lemma 4 $a_1 \parallel b_1$ cannot hold. We may suppose $a_1 < b_1$. Neither $a_1 \leqq a$, $b_1 \leqq a$ nor $a_1 \nleqq a$, $b_1 \nleqq a$ can hold simultaneously, for it would be $f_1(a_1)f_1(b_1) = f_1(a_1b_1)$ by Lemma 1 and 5. It cannot be $a_1 \nleqq a$, $b_1 \leqq a$. Consequently $a_1 \leqq a$, $b_1 \nleqq a$. If it were $a_1b_1 = a_1$, then $f_1(a_1)f_1(b_1) = a'b' = a' = f_1(a_1b_1)$ would hold, which is a contradiction. Therefore $a_1b_1 = b_1$. Define

$$f_2(t) = \begin{cases} a' & \text{for } t \nleqq a \\ b' & \text{for } t \leqq a \end{cases}.$$

Again $f_2 \notin I$, so that $f_2 \notin H$. Therefore there exist elements a_2, $b_2 \in G$

such that a_2b_2 is defined, but either $f_2(a_2)f_2(b_2)$ is not defined or $f_2(a_2)f_2(b_2) \neq f_2(a_2b_2)$ holds. Similarly as in the preceding it cannot be $a_2 = b_2$ or $a_2 \| b_2$. We may suppose again $a_2 < b_2$ and obtain that it can be only $a_2 \leqq a$, $b_2 \nleqq a$. Since $a_1b_1 = b_1$, $a_2b_2 = b_2$ holds by Lemma 6. But then we have $f_2(a_2)f_2(b_2) = b'a' = a' = f_2(a_2b_2)$, which is a contradiction. The supposition $a'b' = b'$ leads also to a contradiction, for if we interchange the elements a', b', we obtain the preceding case.

Agreement. ϱ denotes the relation on G defined in the following way: For $a, b \in G$ $a\varrho b$ holds if and only if $ab = a$. $\sigma = \varrho \cup \{(a, a) \mid a \in G\}$, $\sqsubseteq$ denotes the transitive hull of the relation σ. $\preceq$ denotes the relation on G' defined in the following way: For $a', b' \in G'$ $a' \preceq b'$ holds if and only if $a'b' = a'$.

Theorem 1. *Let G, G' fail to be antichains. Then the following statements are equivalent:*

(A) $I = H$.

(B) For arbitrary elements $a, b \in G$ such that ab is defined $ab = a$ or $ab = b$ holds; the relaiton $\sqsubseteq$ is identical with the ordering on G and the relation $\preceq$ is identical with the ordering on G' or the relation $\sqsubseteq$ is dual to the ordering on G and the relation $\preceq$ is dual to the ordering on G'.

Proof. I. Let (B) hold true.

1. Let $f \in I$, $a, b \in G$, $ab = a$. Then $a \sqsubseteq b$, consequently $a \leqq b (a \geqq b)$. Thence $f(a) \leqq f(b)$ $[f(a) \geqq f(b)]$ and in both cases $f(a)f(b) = f(ab)$. If $a, b \in G$, $ab = b$, the proof is analogous.

2. Let $f \in H$, $a, b \in G$, $a \leqq b (a \geqq b)$. If $a = b$, $f(a) = f(b)$ holds in both cases. If $a \neq b$, there exist mutually different elements $a_0, a_1, \ldots, a_n \in G$ such that $a_0 = a$, $a_n = b$ and it is $a_0a_1 = a_0$, $a_1a_2 = a_1$, $\ldots$, $a_{n-1}a_n = a_{n-1}$. Thence $f(a_i) = f(a_ia_{i+1}) = f(a_i)f(a_{i+1})$ for $i = 0, 1, \ldots, n-1$. Consequently $f(a_i) \leqq f(a_{i+1})$ $[f(a_i) \geqq f(a_{i+1})]$ for $i = 0, 1, \ldots, n-1$. Therefrom we have $f(a) = f(a_0) \leqq f(a_1) \leqq \ldots \leqq f(a_n) = f(b)$ $[f(a) = f(a_0) \geqq f(a_1) \geqq \ldots \geqq f(a_n) = f(b)]$. Therefore $f \in I$.

Therefore (A) holds true.

II. Let (A) hold true. Since G fails to be an antichain, there exists a pair of elements $\bar{a}, \bar{b} \in G$, $\bar{a} < \bar{b}$ such that $\bar{a}\bar{b}$ is defined, for otherwise each map of G into G' would be homomorphic. According to Lemma 5 and 6, then, for each pair of elements $a, b \in G$, $a < b$ such that ab is defined either $ab = a$ or $ab = b$ holds. Thereby the first part of the statement (B) is proved. Let $ab = a$ $(ab = b)$ hold.

1. Let $a, b \in G$, $a \sqsubseteq b$. If $a = b$, then also $a \leqq b$ $(a \geqq b)$. Consequently let $a \neq b$. Then there exist mutually different elements $a_0, a_1, \ldots, a_n \in G$ such that $a_0 = a$, $a_n = b$ and $a_0a_1 = a_0$, $a_1a_2 = a_1$, $\ldots$, $a_{n-1}a_n = a_{n-1}$. By Lemma 4 it is either $a_i < a_{i+1}$ or $a_i > a_{i+1}$ for $i = 0, 1, \ldots, n-1$. But by Lemma 6 it is necessarily $a_i < a_{i+1}$ $(a_i > a_{i+1})$ for all i, $i = 0, 1, \ldots, n-1$. Consequently $a_0 < a_1$, $a_1 < a_2$, $\ldots$, $a_{n-1} < a_n$

$(a_0 > a_1,\ a_1 > a_2,\ \ldots,\ a_{n-1} > a_n)$. From the transitivity if follows $a = a_0 < a_n = b\ (a = a_0 > a_n = b)$.

2. Let $a, b \in G$, $a \leqq b\ (a \geqq b)$. Admit that it is not $a \sqsubseteq b$. Let $a', b' \in G'$, $a' < b' (a' > b')$. We define

$$f(t) = \begin{cases} b' & \text{for } t \text{ non } \sqsubseteq b \\ a' & \text{for } t \sqsubseteq b. \end{cases}$$

Then $f \in H$. In fact, let c, $d \in G$ such that cd is defined. By Lemma 5 it is either $cd = c$ or $cd = d$. If $c \sqsubseteq b$, $d \sqsubseteq b$, it is by Lemma 1 $f(cd) = a' = a'^2 = f(c) f(d)$. Similarly if c non $\sqsubseteq b$, d non $\sqsubseteq b$, it is $f(cd) = b' = b'^2 = f(c) f(d)$. Further let $c \sqsubseteq b$, d non $\sqsubseteq b$. If $cd = c$, then it is $f(cd) = f(c) = a' = a'b' = f(c) f(d)$. But if $cd = d$, we have by the definition $d \sqsubseteq c$. Since $c \sqsubseteq b$ and the relation $\sqsubseteq$ is transitive, it is $d \sqsubseteq b$, which is a contradiction. Finally, let $d \sqsubseteq b$, c non $\sqsubseteq b$. If $cd = c$, it is $c \sqsubseteq d$, therefore $c \sqsubseteq b$ and we have again a contradiction. If $cd = d$, then it is $f(cd) = f(d) = a' = b'a' = f(c) f(d)$. Thereby it is shown that $f \in H$ and therefore $f \in I$. But $a' = f(b) \ngeqq f(a) = b'$ $[a' = f(b) \nleqq f(a) = b']$. Consequently $a \leqq b\ (a \geqq b)$ implies $a \sqsubseteq b$.

3. Let $a', b' \in G'$, $a' \leqq b'\ (a' \geqq b')$. By Lemma 2 it is in both cases $a'b' = a'$, therefore $a' \preceq b'$.

4. Let $a', b' \in G'$, $a' \preceq b'$. Then it is by the definition of the relation $\preceq$ $a'b' = a'$. By Lemma 7 it is not $a' \parallel b'$. Consequently either $a' \leqq b'$ or $a' > b'$ (either $a' \geqq b'$ or $a' < b'$) holds. But if $a' > b'$ $(a' < b')$ held, then we should have $a'b' = b'$, which is a contradiction. Therefore $a' \leqq b'\ (a' \geqq b')$.

Therefore the second part of the statement (B) holds true.

Remark 2. In Theorem 1 the statement (B) implies the statement (A) even in the case that at least one of the ordered sets G, G' is an antichain.

Remark 3. The main result of [1] follows from Theorem 1, for the relation $\sqsubseteq$, resp. $\preceq$ is identical with the ordering π, resp. π' derived from the multiplication on G, resp. G'.

Theorem 2. *Let G be an antichain and let G' fail to be an ordered set containing a single element. Then the following statements are equivalent:*

(A) $I = H$.

(B) Either no product is defined in G or the product ab of elements a, $b \in G$ is defined only if $a = b$ and $a^2 = a$ holds and simultaneously for each $a' \in G'$ a'^2 is defined and $a'^2 = a'$ holds.

Proof. I. Let (B) hold true. Since each map of G into G' is isotone, it suffices to show that each map of G into G' is homomorphic. If no product is defined in G, then it is true. Consequently let a product ab of the elements a, $b \in G$ be defined. Then according to the supposition $a = b$ and $a^2 = a$ holds. Let f be an arbitrary map of G into G'. Since for each $a' \in G'$ a'^2 is defined and $a'^2 = a'$ holds, $f^2(a)$ is therefore defined and $f^2(a) = f(a) = f(a^2)$ holds. Consequently f is a homomorphic map and (A) holds true.

II. Let (A) hold true. If no product is defined in G, then (B) holds true. Consequently let a product ab of the elements $a, b \in G$ be defined. Then by Lemma 1 for each $a' \in G'$ a'^2 is defined and $a'^2 = a'$ holds.

1. Let G' fail to be an antichain. Since G is an antichain, it is $a = b$ by Lemma 4 and $a^2 = a$ by Lemma 5.

2. Let G' be an antichain. Admit that $a \neq b$. Let $a', b' \in G'$, $a' \neq b'$. Define

$$f_1(t) = \begin{cases} a' & \text{for } t = a \\ b' & \text{for } t \in G - \{a\} \end{cases},$$

$$f_2(t) = \begin{cases} b' & \text{for } t = a \\ a' & \text{for } t \in G - \{a\} \end{cases},$$

The maps f_1, f_2 are isotone and therefore homomorphic. But if $ab = b$, we have $b' = f_1(ab) = f_1(a)\, f_1(b) = a'b'$ and simultaneously $a' = f_2(ab) = f_2(a)\, f_2(b) = b'a'$, which is a contradiction. If $ab = a$, we have $a' = f_1(ab) = f_1(a)\, f_1(b) = a'b'$ and simultaneously $b' = f_2(ab) = f_2(a) f_2(b) = b'a'$, which is again a contradiction. Therefore $a = b$. Further admit that $a^2 \neq a$. Then it is $b' = f_1(a^2) = f_1^2(a) = a'^2$, which is contradictory to Lemma 1. Consequently $a^2 = a$.

Therefore (B) holds true in both cases.

Theorem 3. *Let G' be an antichain. Let G' fail to be an ordered set containing a single element and G fail to be an antichain. Then the following statements are equivalent:*

(A) $I = H$.

(B) For each $a' \in G'$ a'^2 is defined and $a'^2 = a'$ holds; for each homomorphic map φ of an arbitrary component $K \subseteq G$ into G' $\varphi[K]$ is a set containing a single element; if $a \in K$, $b \in G$, ab defined, then $b \in K$, $ab \in K$.

Proof. I. Let (B) hold true. Let $f \in I$, $a, b \in G$, ab defined. Let $K \subseteq G$ be the component for which $a \in K$ holds. Then $b \in K$, $ab \in K$ and consequently $f(ab) = f(a) = f(b)$. We have therefore $f(ab) = f(a) f(b)$ and $f \in H$. Let $f \in H$. Then for an arbitrary component $K \subseteq G$ the restriction $f \mid K$ of f to K is a homomorphic map of K into G', so that $f[K]$ is a set containing a single element and $f \in I$.

II. Let (A) hold true. Since there exists a map of G into G' which is not homomorphic, there exist elements $a, b \in G$ such that ab is defined. Consequently for each $a' \in G'$ a'^2 is defined and it is $a'^2 = a'$ by Lemma 1. Let $a', b' \in G'$, $a' \neq b'$. Let $K \subseteq G$ be such a component that $a \in K$. Admit $b \in G - K$. We define

$$f_1(t) = \begin{cases} a' & \text{for } t \in K \\ b' & \text{for } t \in G - K \end{cases},$$

$$f_2(t) = \begin{cases} b' & \text{for } t \in K \\ a' & \text{for } t \in G - K \end{cases}$$

It is $f_1, f_2 \in I$, consequently $f_1, f_2 \in H$. If $ab \in K$, we have $a' = f_1(ab) = f_1(a) f_1(b) = a'b'$ and simultaneously $b' = f_2(ab) = f_2(a) f_2(b) = b'a'$, which is a contradiction. If $ab \in G - K$, we have $b' = f_1(ab) = f_1(a) f_1(b) = a'b'$ and simultaneously $a' = f_2(ab) = f_2(a) f_2(b) = b'a'$, which is again a contradiction. Therefore it is $b \in K$. If it now were $ab \in G - K$, we should have $b' = f_1(ab) = f_1(a) f_1(b) = a'^2$, which is impossible. Therefore also $ab \in K$. Let φ be a homomorphic map of a component K into G' such that $\varphi[K]$ fails to be a set containing a single element. Then there exist elements $a_1, a_2 \in K$ such that a_1, a_2 are comparable and $\varphi(a_1) \neq \varphi(a_2)$. Put

$$f_3(t) = \begin{cases} \varphi(t) & \text{for } t \in K \\ \varphi(a_1) & \text{for } t \in G - K. \end{cases}$$

If $a \in K$, $b \in G$, ab defined, then according to the preceding $b \in K$, $ab \in K$. Since φ is a homomorphic map of K into G', it is $f_3(ab) = \varphi(ab) = \varphi(a) \varphi(b) = f_3(a) f_3(b)$. If $a \in G - K$, $b \in G - K$, ab defined, then $ab \in G - K$. Consequently it is $f_3(ab) = \varphi(a_1) = \varphi^2(a_1) = f_3(a) f_3(b)$. It is $f_3 \in H$, so that $f_3 \in I$, but $f_3(a_1) = \varphi(a_1) \neq \varphi(a_2) = f_3(a_2)$, which is a contradiction. Therefore for each homomorphic map φ of K into G' $\varphi[K]$ is a set containing a single element.

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