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ON ISOTONE AND HOMOMORPHIC MAPPINGS

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In the paper there are given necessary and sufficient conditions for the set of all isotone mappings of an ordered set G into an ordered set G' to be equal to the set of all homomorphic mappings of the o-groupoid G into the o-groupoid G'.

A non-empty set G will be called a *partial groupoid* if to certain pairs of elements $a, b \in G$ an element ab is assigned, the so called product of the element a with the element b. In what follows, the word "groupoid" will always denote "partial groupoid".

Let G, G' be groupoids and let f be a mapping of G into G'. We say that f is a *homomorphic mapping* if f has the following property: if $a, b \in G$ and ab is defined then $f(a) f(b)$ in G' is also defined and $f(ab) = f(a) f(b)$.

A groupoid G will be called a *commutative groupoid* if the existence of the product ab implies the existence of ba and $ab = ba$. G will be called an *associative groupoid* if it has the following property: if for the elements a, b, c:

1. the products $(ab) c$ and bc are defined, then the product $a(bc)$ is also defined

2. the products ab and $a(bc)$ are defined, then the product $(ab)c$ is also defined and in both cases $(ab) c = a(bc)$.

A groupoid G will be called an *o-groupoid* if G is commutative, associative and has these properties:

1. for any $a \in G$ the product aa is defined

2. if $a, b \in G$ and ab is defined, then $ab = a$ or $ab = b$.

Lemma 1. *Let G be an o-groupoid. Put for any two elements $a, b \in G$ $a \leqq b$ if and only if $ab = a$. Then the relation $\leqq$ is an ordering relation on G.*

Proof. For any $a \in G$ we have $aa = a$ so that $a \leqq a$ and the relation $\leqq$ is reflexive. If $a, b \in G$, $a \leqq b$ and $b \leqq a$ then $ab = a$ and $ba = b$. But G is commutative so that $a = ab = ba = b$ and $\leqq$ is antisymmetric. Let $a, b, c \in G$, $a \leqq b$, $b \leqq c$. Then $ab = a$, $bc = b$ so that $a(bc)$ is defined. As G is associative the product $(ab) c$ is also defined and we have $ac = (ab) c = a(bc) = ab = a$ so that $a \leqq c$ and $\leqq$ is transitive. Thus, $\leqq$ is really an ordering relation.

If G is an o-groupoid and $\leqq$ is an ordering relation defined on G in the same way like in Lemma 1 we say that $\leqq$ is *derived from the multiplication* in G. This ordering relation will be denoted π.

Lemma 2. *Let G be a non-empty ordered set with the ordering relation $\leqq$. Then it is possible to define a multiplication on G so that G is an o-groupoid with respect to this multiplication, and that the ordering derived from this multiplication is the same as $\leqq$.*

Proof. Put for any two elements $a, b \in G$, $ab = ba = a \Leftrightarrow a \leqq b$. Then G is a groupoid; this groupoid is clearly commutative. For any $a \in G$ there is $a \leqq a$ so that aa is defined. If ab is defined, then a, b are comparable so that $a \leqq b$ or $b \leqq a$. In the first case we have $ab = a$, in the second one $ab = b$. It is left to prove that G is associative. Assume that a, b, c are three elements of G such that $(ab)\,c$ and bc are defined. Then the elements a and b, b and c and ab and c are comparable. We shall distinguish two cases:

1. $b \leqq c$. Then $bc = b$ so that $a(bc) = ab$ is defined; at the same time $ab \leqq b \leqq c$ so that $(ab)\,c = ab$ and we have $a(bc) = (ab)\,c$.

2. $b \geqq c$. Then $bc = c$; if $a \leqq b$ then $ab = a$ so that $a(bc) = ac = (ab)\,c$ is defined and $a(bc) = (ab)\,c$; if $a \geqq b$ then $ab = b$ and $a \geqq c$ so that $a(bc) = ac$ is defined and $a(bc) = ac = c = bc = (ab)\,c$.

In a similar way one can prove that if $a, b, c \in G$ and ab, $a(bc)$ are defined then $(ab)\,c$ is also defined and $(ab)\,c = a(bc)$. Thus, G is an o-groupoid. If π is an ordering derived from the multiplication then π is equal to $\leqq$ for

$$a\pi b \Leftrightarrow ab = a \Leftrightarrow a \leqq b.$$

Hence "o-groupoid" and "ordered set" are equivalent concepts. We shall solve the following problem: Let G, G' be o-groupoids and let ϱ be an ordering on G, ϱ' an ordering on G' (these orderings are not necessarily derived from the multiplication). Denote I the system of all isotone mappings of (G, ϱ) into (G', ϱ') and H the system of all homomorphic mappings of the o-groupoid G into G'. Find the necessary and sufficient conditions for $I = H$.

We shall need the following lemma.

Lemma 3. *Let G, G' be o-groupoids, let f be a mapping of G into G'. Let π, π' be orderings on G, resp. G' derived from the multiplication. Then f is a homomorphic mapping of G into G' if and only if f is an isotone mapping of (G, π) into (G', π').*

Proof. Let f be a homomorphic mapping and let $a, b \in G$, $a\pi b$. According to the definition of π we have $ab = a$. From this it follows $f(a)\,f(b) = f(ab) = f(a)$ so that $f(a)\,\pi' f(b)$ and f is isotone. Let f be an isotone mapping of (G, π) into (G', π') and let $a, b \in G$, ab be defined. Then $ab = a$ or $ab = b$; assume $ab = a$. Then $a\pi b$ and hence $f(a)\,\pi' f(b)$ so that $f(a)\,f(b)$ is defined and $f(a)\,f(b) = f(a)$; this implies $f(ab) = f(a) =$

$= f(a)f(b)$. Similarly we accomplish the proof in the case $ab = b$. Hence f is a homomorphic mapping of G into G'.

Corollary. *Let G, G' be o-groupoids, let ϱ, ϱ' be orderings on G, resp. G' and let π, π' be orderings derived from the multiplication on G, resp. G'. Then the following statements are equivalent:*

(A) $I = H$

(B) *The system of all isotone mappings of (G, ϱ) into (G', ϱ') is identical with the system of all isotone mappings of (G, π) into (G', π').*

For that reason our problem can be formulated in such a way: *Find* the necessary and sufficient conditions for the system I_ϱ of all isotone mappings of (G, ϱ) into $(G' \varrho')$ to be equal to the system I_π of all isotone mapings of (G, π) into (G', π').
The following lemma is clear.

Lemma 4. *Let (G, ϱ), (G', ϱ') be ordered sets. Let $a \in G$, $a', b' \in G'$, $a'\varrho'b'$, $a' \neq b'$. Put*

$$f(t) = \begin{cases} b' \text{ for } t \in G,\ a\varrho t \\ a' \text{ for } t \in G,\ a\bar{\varrho} t \end{cases}$$

Then f is an isotone mapping of (G, ϱ) into (G', ϱ').

Lemma 5. *Let G, G' be non-empty sets, let ϱ, π be orderings on G, ϱ', π' orderings on G' such that (G, ϱ), (G, π), (G', ϱ'), (G', π') are not antichains.*[1] *Let I_ϱ denote the set of all isotone mappings of (G, ϱ) into (G', ϱ'), I_π the set of all isotone mappings of (G, π) into (G', π'). If $\pi \subseteq \varrho$ and $\varrho' \subseteq \pi'$ or $\pi \subseteq \breve{\varrho}$*[2]*) and $\varrho' \subseteq \breve{\pi}'$ then $I_\varrho \subseteq I_\pi$. If, moreover, $\pi \subset \varrho$ or $\varrho' \subset \pi'$ ($\pi \subset \breve{\varrho}$ or $\varrho' \subset \breve{\pi}'$), then $I_\varrho \subset I_\pi$.*

Proof. Assume that $\pi \subseteq \varrho$ and $\varrho' \subseteq \pi'$ (the case $\pi \subseteq \breve{\varrho}$ and $\varrho' \subseteq \breve{\pi}'$ would be accomplished in a similar way). Let $f \in I_\varrho$, $a, b \in G$, $a\pi b$. Then $a\varrho b$ so that $f(a)\,\varrho' f(b)$ and hence $f(a)\,\pi' f(b)$. Thus $f \in I_\pi$ and $I_\varrho \subseteq I_\pi$. Assume now that $\pi \subset \varrho$, $\varrho' \subseteq \pi'$. Then there exist elements $c, d \in G$ such that $c\varrho d$, $c\bar{\pi} d$. Choose any elements $c', d' \in G'$ such that $c'\varrho' d'$, $c' \neq d'$. Then $c'\pi' d'$ and if we put

$$f(t) = \begin{cases} d' \text{ for } t \in G,\ c\pi t \\ c' \text{ for } t \in G,\ c\bar{\pi} t \end{cases}$$

then $f \in I_\pi$ according to Lemma 4 but $f(c) = d'$, $f(d) = c'$ so that $f(c)\,\bar{\varrho}' f(d)$ and $f \bar{\in} I_\varrho$. Assume that $\pi \subseteq \varrho$, $\varrho' \subset \pi'$. Then there exist $p', q' \in G'$ such that $p'\pi' q'$, $p'\bar{\varrho}' q'$. Choose any $p, q \in G$, $p\pi q$, $p \neq q$ and put

$$g(t) = \begin{cases} q' \text{ for } t \in G,\ q\pi t \\ p' \text{ for } t \in G,\ q\bar{\pi} t \end{cases}$$

[1]) An ordered set is an antichain if any two its distinct elements are incomparable.

[2]) $\breve{\varrho}$ denotes a relation dual to ϱ (i.e. $a\breve{\varrho}b \Leftrightarrow b\varrho a$).

We have $g \in I_\pi$ according to Lemma 4 but $g(p) = p'$, $g(q) = q'$ and $g(p)\,\bar{\varrho}'g(q)$ so that $g \bar{\in} I_\varrho$. Therefore in both cases we have $I_\varrho \subset I_\pi$.

If (G, ϱ) and (G', ϱ') are ordered sets then we denote by the symbol I_ϱ^2 the set of all isotone mappings f of (G, ϱ) into (G', ϱ') such that card $f(G) = 2$.

Now we shall prove the main theorem.

Theorem 1. *Let G, G' be sets, let ϱ, π be orderings on G, ϱ', π' orderings on G' such that the sets (G, ϱ), (G, π), (G', ϱ'), (G', π') are not antichains. Then the following statements are equivalent:*

(A) $\pi \subseteq \varrho$ *and* $\varrho' \subseteq \pi'$ *or* $\pi \subseteq \breve{\varrho}$ *and* $\varrho' \subseteq \breve{\pi}'$

(B) $I_\varrho \subseteq I_\pi$

(C) $I_\varrho^2 \subseteq I_\pi^2$

Proof. (A) $\Rightarrow$ (B) according to Lemma 5. (B) $\Rightarrow$ (C) is clear. We shall prove (C) $\Rightarrow$ (A). Assume $I_\varrho^2 \subseteq I_\pi^2$ and let $\varrho' \nsubseteq \pi'$, $\varrho' \nsubseteq \breve{\pi}'$. Then there exist either two elements $a', b' \in G'$ such that $a'\varrho'b'$, $a' \,\|_{\pi'} b'$[3]) or four distinct elements $a'_1, b'_1, a'_2, b'_2 \in G'$ such that $a'_1\varrho'b'_1$, $a'_2\varrho'b'_2$, $a'_1\pi'b'_1$, $b'_2\pi'a'_2$. Suppose the first possibility. Choose any two distinct elements $a, b \in G$. If $a\varrho b$, put

$$f(t) = \begin{cases} b' & \text{for } t \in G,\ b\varrho t \\ a' & \text{for } t \in G,\ b\bar{\varrho}t \end{cases}$$

If $a\bar{\varrho}b$, put

$$f(t) = \begin{cases} b' & \text{for } t \in G,\ a\varrho t \\ a' & \text{for } t \in G,\ a\bar{\varrho}t. \end{cases}$$

In both cases we have $f \in I_\varrho^2$ according to Lemma 4 and hence $f \in I_\pi^2$. But in both cases $f(a) \,\|_{\pi'} f(b)$ so that $a \,\|_\pi b$. This implies that (G, π) is an antichain and this is a contradiction. Suppose now the second possibility. Choose any two distinct elements $a, b \in G$. If $a\varrho b$, put

$$f_1(t) = \begin{cases} b_1' & \text{for } t \in G,\ b\varrho t \\ a_1' & \text{for } t \in G,\ b\varrho t \end{cases}, \qquad f_2(t) = \begin{cases} b_2' & \text{for } t \in G,\ b\varrho t \\ a_2' & \text{for } t \in G,\ b\varrho t \end{cases}.$$

If $a\bar{\varrho}b$, put

$$f_1(t) = \begin{cases} b_1' & \text{for } t \in G,\ a\varrho t \\ a_1' & \text{for } t \in G,\ a\varrho t \end{cases}, \qquad f_2(t) = \begin{cases} b_2' & \text{for } t \in G,\ a\varrho t \\ a_2' & \text{for } t \in G,\ a\bar{\varrho}t \end{cases}.$$

In both cases there is $f_1, f_2 \in I_\varrho^2$ and hence $f_1, f_2 \in I_\pi^2$. But this implies $a \,\|_\pi b$ for $a\pi b$, $a\varrho b$ implies $a_2' = f_2(a)\,\pi' f_2(b) = b_2'$, resp. $a\bar{\pi}b$, $a\bar{\varrho}b$ implies $b_1' = f_1(a)\,\pi' f_1(b) = a_1'$ and $b\pi a$, $a\varrho b$ implies $b_1' = f_1(b)\,\pi' f_1(a) = a_1'$, resp.

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[3]) $a' \,\|_{\pi'} b'$ denotes that the elements a', b' are incomparable in the ordering π'

$b\pi a$, $a\bar{\varrho}b$ implies $a_2' = f_2(b)\,\pi' f_2(a) = b_2'$. Thus, (G, π) is an antichain and this is a contradiction. Hence the assumption $I_\varrho^2 \subseteqq I_\pi^2$ implies $\varrho' \subseteqq \pi'$ or $\varrho' \subseteqq \breve{\pi}'$. Assume now $\varrho' \subseteqq \pi'$ and let $\pi \nsubseteq \varrho$. Then there exist elements $a, b \in G$ such that $a\pi b$, $a\bar{\varrho}b$. Choose any distinct elements $a', b' \in G'$ such that $a'\varrho' b'$ and put

$$f(t) = \begin{cases} b' & \text{for } t \in G,\ a\varrho t \\ a' & \text{for } t \in G,\ a\bar{\varrho}t \end{cases}.$$

Then $f \in I_\varrho^2$, $f \bar{\in} I_\pi^2$ and this is a contradiction. Assume that $\varrho' \subseteqq \breve{\pi}'$ and that $\pi \nsubseteq \breve{\varrho}$. Then there exist elements $a, b \in G$ such that $a\pi b$, $b\bar{\varrho}a$. Choose any distinct elements $a', b' \in G'$ such that $a'\varrho' b'$ and put

$$f(t) = \begin{cases} b' & \text{for } t \in G,\ b\varrho t \\ a' & \text{for } t \in G,\ b\bar{\varrho}t \end{cases}.$$

Then $f \in I_\varrho^2$, $f \bar{\in} I_\pi^2$ and this is a contradiction. Thus, the assumption $I_\varrho^2 \subseteqq I_\pi^2$ implies $\pi \subseteqq \varrho$ and $\varrho' \subseteqq \pi'$ or $\pi \subseteqq \breve{\varrho}$ and $\varrho' \subseteqq \breve{\pi}'$.

Corollary. *Let G, G', ϱ, ϱ', π, π' satisfy the conditions of Theorem* 1. *Then the following statements are equivalent:*

(A) $\varrho = \pi$ *and* $\varrho' = \pi'$ *or* $\varrho = \breve{\pi}$ *and* $\varrho' = \breve{\pi}'$

(B) $I_\varrho = I_\pi$

(C) $I_\varrho^2 = I_\pi^2$

This corollary together with Lemma 3 gives the solution of our problem:

Theorem 2. *Let G, G' be o-groupoids, let ϱ, ϱ' be orderings on G, resp. G' and let π, π' be orderings derived from the multiplication on G, resp. G'. Do not let the sets (G, ϱ), (G, π), (G', ϱ'), (G', π') be antichains. Denote by I the system of all isotone mappings of (G, ϱ) into (G', ϱ'), H the system of all homomorphic mappings of G into G' and I^2, resp. H^2 the system of all isotone, resp. homomorphic mappings f such that card $f(G) = 2$.*

Then the following statements are equivalent:

(A) $\varrho = \pi$ *and* $\varrho' = \pi'$ *or* $\varrho = \breve{\pi}$ *and* $\varrho' = \breve{\pi}'$

(B) $I = H$

(C) $I^2 = H^2$

Note 1. *Let (G, ϱ) be an antichain. Then $H \subseteqq I$. If (G, π) is also an antichain then $I = H$.*

Proof. Let $f \in H$. As (G, ϱ) is an antichain, each mapping of (G, ϱ) into (G', ϱ') is isotone. Thus, $f \in I$ and $H \subseteqq I$. If (G, π) is an antichain then each mapping of G into G' is homomorphic so that also $I \subseteqq H$ and we have $I = H$.

Note 2. *Let* (G', ϱ'), (G', π') *be antichains. Then* $I = H$ *if and only if* (G, ϱ) *and* (G, π) *have the same components.*[4])

Proof. A mapping f of (G, ϱ) into (G', ϱ'), where (G', ϱ') is an antichain, is isotone if and only if f maps each component of (G, ϱ) onto a one-point subset of (G', ϱ'). The same holds for a mapping g of (G, π) into (G', π') where (G', π') is an antichain. From this follows our statement.

REFERENCES

[1] Birkhoff G: *Lattice Theory*, rev. ed. New York 1948

[2] Novotný M. und Skula L.: *Über gewisse Topologien auf geordneten Mengen*, Fund. Math. LVI (1965), 313—324

[3] Skula L.: *Systeme von stetigen Abbildungen*, to appear in Czech. Math. Journ.

[4]) A subset H of an ordered set G is connected if for any two elements $a, b \in H$ there exist elements $t_0, t_1, \ldots, t_n \in H$ such that $t_0 = a$, $t_n = b$ and t_{i-1}, t_i are comparable for $i = 1, \ldots, n$. A component is a maximal connected subset of G.