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AREA OF CONTRACTION OF NEWTON'S METHOD APPLIED TO A PENALTY TECHNIQUE FOR OBSTACLE PROBLEMS

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1. THE OBSTACLE PROBLEM AND ITS DISCRETIZATION

The obstacle problem can be considered as a variational problem with inequality constraints. The discretization by piecewise linear finite elements leads to large scale optimization problems with a special structured objective functional and with simple bounds for the variable as constraints. In this paper we include the constraints into an auxiliary objective functional by means of a penalty technique. The unconstrained problems obtained are nonlinear variational equations. These can be solved by Newton's method. As a well known fact the auxiliary problems generated in penalty techniques are ill conditioned in the limit. Thus an important question is to estimate the area of contraction of Newton's method in dependence of the penalty and of the discretization parameter.

In the first part we summarize some basic facts on obstacle problems, its discretization and on an adapted penalty technique.

Let $\Omega \subset \mathbb{R}^2$ be some open polyphedron and we denote $U := H_0^1(\Omega)$, $V := H^1(\Omega)$. With given functions $f \in L_\infty(\Omega)$, $g \in W_\infty^2(\Omega)$ we investigate the following constrained variational problem

$$J(v) := \frac{1}{2}a(v,v) - (f,v) \to \min! \qquad \text{subject to } v \in G \tag{1}$$

with $G := \{v \in U : v \geqslant g\}$. Here $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and $(f,\cdot) : V \to \mathbb{R}$ are defined by

$$a(u,v) := \int_\Omega \nabla u \nabla v \, \mathrm{d}x \qquad \forall u,v \in V \tag{2}$$

and

$$(f,v) := \int_\Omega fv \, \mathrm{d}x \qquad \forall v \in V, \tag{3}$$

respectively. The semiordering "$\geqslant$" is given by the natural one almost everywhere in Ω. Additionally we suppose

$$g\big|_\Gamma \leqslant 0$$

on the boundary Γ of the region Ω. Thus, the feasible set $G \subset U$ is nonempty, convex and closed. Furthermore the objective functional $J(\cdot)$ of (1) is continuous and strongly convex. This results in

Lemma 0.1. *Problem* (1) *possesses a unique solution $u \in G$ and an element $u \in G$ forms a solution of* (1) *if and only if the variational inequality*

$$a(u, v - u) \geqslant (f, v - u) \qquad \text{for any } v \in G \tag{4}$$

is satisfied.

Now, we relate a Lagrange functional $L(\cdot,\cdot)\colon V \times V^* \to \mathbf{R}$ to the original problem (1). Let $\langle \cdot,\cdot\rangle$ denote the dual pairing between $V^* = H^{-1}(\Omega)$ and V. With the cone K given by

$$K = \{q \in H^{-1}(\Omega) \colon \langle q, v\rangle \geqslant 0 \quad \text{for any } v \in H_0^1(\Omega), v \geqslant 0\}. \tag{5}$$

we define the Lagrange functional L related to (1) by

$$L(v, q) := \frac{1}{2} a(v, v) - (f, v) + \langle q, g - v\rangle \qquad \text{for any } v \in U,\ q \in K. \tag{6}$$

Let us assume throughout the paper the regularity that $u \in H^2(\Omega)$. Furthermore we also assume that the following smoothing property holds: $\tilde{f} \in L_2(\Omega)$ implies $\tilde{u} \in H^2(\Omega)$ for the solution of the related elliptic variational equality

$$\tilde{u} \in U\colon \qquad a(\tilde{u}, v) = (\tilde{f}, v) \quad \text{for any } v \in U.$$

This can be guaranteed by the convexity of the region Ω e.g. Now, we can set

$$\langle p, v\rangle = \int_\Omega p(x) v(x)\,\mathrm{d}x \tag{7}$$

with

$$p(x) = \begin{cases} -[\Delta u + f](x), & \text{if } u(x) = g(x) \\ 0, & \text{if } u(x) > g(x). \end{cases} \tag{8}$$

Thus, the pair $(u, p) \in U \times K$ satisfies the system

$$\begin{aligned} a(u, v) - \langle p, v\rangle &= (f, v), && \text{for any } v \in U \\ \langle q - p, u\rangle &\leqslant \langle q - p, g\rangle, && \text{for any } q \in K \end{aligned} \tag{9}$$

which is just a mixed formulation related to (1) (compare [4], [9], [8] e.g.) and p is the optimal Lagrange multiplier of the problem (1). Condition (9) is necessary and sufficient for $(u,p) \in U \times K$ to be a saddle point of the Lagrange functional L.

Denote by $\zeta_\varrho : \mathbb{R} \to \mathbb{R}_+$ the following functions

$$\zeta_\varrho(t) := t + \sqrt{t^2 + \varrho}, \; t \in \mathbb{R} \tag{10}$$

which approximates uniformly the well known penalty term $2 \cdot \max\{0, t\}$. Now, we define the auxiliary objective functional $T_{\alpha\varrho} \colon V \to \mathbb{R}$ by

$$T_{\alpha\varrho}(v) := J(v) + \alpha \int\limits_\Omega \zeta_\varrho(g(x) - v(x)) \mathrm{d}x. \tag{11}$$

Here $\alpha, \varrho > 0$ denote fixed parameters. Because of the continuity and of the growth behaviour of ζ_ϱ the functional $T_{\alpha\varrho}$ is well defined on V. Furthermore $T_{\alpha\varrho}$ is continuous and strongly convex on U. Thus, the penalty problems

$$T_{\alpha\varrho}(v) \to \min! \qquad \text{s.t. } v \in U \tag{12}$$

possess unique solutions $u_{\alpha\varrho} \in U$ for any fixed $\alpha, \varrho > 0$. The functional $T_{\alpha\varrho}$ is differentiable and an element $u_{\alpha\varrho} \in U$ solves (12) if and only if

$$\langle T'_{\alpha\varrho}(u_{\alpha\varrho}), v \rangle = 0 \qquad \text{for any } v \in U \tag{13}$$

holds. The supposed smoothing property guarantees the regularity $u_{\alpha\varrho} \in H^2(\Omega)$. Thus, by the continuous embedding $H^2(\Omega) \hookrightarrow L_\infty(\Omega)$ we also have $u_{\alpha\varrho} \in L_\infty(\Omega)$. Due to (11) the Fréchet derivative of $T_{\alpha\varrho}$ is given by

$$\langle T'_{\alpha\varrho}(u), v \rangle = a(u,v) - (f,v) - \alpha \int\limits_\Omega \zeta'_\varrho(g(x) - u(x)) v(x) \mathrm{d}x \qquad \text{for any } u, v \in U. \tag{14}$$

Now, we use a piecewise linear finite element discretization on a quasi-uniform triangulation $\mathcal{T}_h$ of Ω, i.e. a circle with the radius $\underline{\delta} h$ can be inscribed into each of the triangles $T_i \in \mathcal{T}_h$ and each T_i can be inscribed into a circle with the radius $\overline{\delta} h$, where $\overline{\delta} > \underline{\delta} > 0$ denote some given constants. Furthermore we assume that the triangulation is of the weakly acute type, i.e. no angle of the triangles contained in the triangulation exeeds $\pi/2$.

Let x_i, $i = 1(1)N$ denote the inner grid points and let x_i, $i = 1(1)\overline{N}$ denote all grid points of the discretization. We use the usual Lagrange base $\{\varphi_i\}_{i=1}^{\overline{N}}$ of piecewise linear functions which satisfy

$$\varphi_i(x_j) = \delta_{ij} \qquad i,j = 1(1)\overline{N},$$

with Kronecker's δ_{ij}. Instead of the spaces U, V the finite element discretization uses the subspaces

$$U_h := \operatorname{span}\{\varphi_i\}_{i=1}^{N}, \qquad V_h := \operatorname{span}\{\varphi_i\}_{i=1}^{\overline{N}}.$$

The discrete feasible set is defined by

$$G_h := \{v_h \in U_h : v_h(x_i) \geqslant g(x_i), i = 1(1)N\}. \tag{15}$$

This leads to the finite dimensional variational problem

$$J(v_h) \to \min! \qquad \text{s.t. } v_h \in G_h \tag{16}$$

Problem (16) forms a quadratic finite dimensional optimization problem with lower bounds as constraints. We relate to (16) the discrete Lagrange function

$$L_h(v_h, q_h) := J(v_h) + \sum_{i=1}^{N} \operatorname{meas}(D_i)\, q_i(g_i - v_i) \tag{17}$$

where D_i denote a set of the dual division of Ω where $x_i \in D_i$. The dual division of Ω can be generated by the orthogonals to the middle of the edges of the triangulation. We abbreviate $g_i := g(x_i), v_i := v_h(x_i)$. Let $\Psi_i \in L_\infty(\Omega), i = 1(1)N$ denote the functions

$$\Psi_i(x) = \begin{cases} 1, & \text{if } x \in D_i \\ 0, & \text{otherwise} \end{cases} \qquad \text{and let } W_h := \operatorname{span}\{\Psi_i\}_{i=1}^{N}.$$

Thus, W_h forms a subspace of $H^{-1}(\Omega)$ where the related functionals are defined via (7). Thus, the discrete Lagrange function (17) can be considered as the discretization of the continuous one (6) where a mass lumping technique is used to approximate (7) by

$$\langle q_h, v_h \rangle_h := \sum_{i=1}^{N} \operatorname{meas}(D_i) q_i v_i \qquad \text{for any } q_h \in W_h, v_h \in V_h$$

This leads to the natural weighting of the Lagrange multipliers with the areas $\operatorname{meas}(D_i)$ of the sets D_i. Using the piecewise linear interpolant

$$g_h(x) = \sum_{i=1}^{\overline{N}} g_i \varphi_i(x)$$

of g, now, the discrete Lagrange function L_h can be represented by

$$L_h(v_h, q_h) = J(v_h) + \langle q_h, g_h - v_h \rangle_h, \qquad v_h \in U_h, q_k \in K_h \tag{18}$$

where the discrete cone K_h related to K is defined similarly to (5) by

(19) $$K_h := \{q_h \in W_h : \langle q_h, v_h \rangle_h \geqslant 0 \quad \text{for any } v_h \in U_h, v_h \geqslant 0\}$$

With the representation $q_h = \sum_{i=1}^{N} q_i \Psi_i$ we have

$$q_h \in K_h \Longleftrightarrow q_i \geqslant 0, \quad i = 1(1)N.$$

Now, we summarize some important properties of the discretization

Lemma 0.2. *The discrete problem (16) possesses a unique solution $u_h \in G_h$ and the estimation*

$$\|u - u_h\| \leqslant Ch$$

to the solution $u \in G$ of (1) holds with some constant $C > 0$.

The norm used in the error estimation of Lemma 2 is the Sobolev-norm of the underlying space $U = H_0^1(\Omega)$. Baiocchi [3] derived a higher order estimation in the L_∞-norm, namely

Lemma 0.3. *Let $\varepsilon \in (0, 1]$. Then some $C > 0$ exists with*

$$\|u - u_h\|_{0,\infty,\Omega} \leqslant Ch^{2-\varepsilon}.$$

Here u and u_h denote the solution of (1) and of (16) respectively.

By the same arguments as used in [7] we obtain

Lemma 0.4. *There exists a unique $p_h \in K_h$ such that $(u_h, p_h) \in U_h \times K_h$ forms a saddle point of the Lagrange function $L_h(\cdot, \cdot)$, i.e.*

(20) $$L_h(u_h, q_h) \leqslant L_h(u_h, p_h) \leqslant L_h(v_h, p_h) \quad \text{for any } v_h \in U_h, q_h \in K_h.$$

Furthermore there exists some constant $C > 0$ being independent of the discretization parameter $h > 0$ such that

(21) $$\|p_h\|_{0,\infty,\Omega} \leqslant C.$$

R e m a r k . The following condition

(22) $$\begin{aligned} a(u_h, v_h) - \langle p_h, v_h \rangle_h &= (f, v_h), && \text{for any } v_h \in U_h \\ \langle q_h - p_h, u_h \rangle_h &\leqslant \langle q_h - p_h, g_h \rangle_h, && \text{for any } q_h \in K_h \end{aligned}$$

forms a necessary and sufficient criterion for $(u_h, p_h) \in U_h \times K_h$ to be a saddle point of the discrete Lagrangean $L_h(\cdot, \cdot)$. System (22) is a discrete version of the mixed formulation (9).

A direct discretization of the penalty problem (12) according to

$$T_{\alpha\varrho}(v_h) \to \min! \qquad \text{s.t. } v_h \in U_h$$

leads to a reduction of the order of convergence as a consequence of the ill-posedness of the problem for large α and for $\varrho \to 0$. A proper modification is to apply the penalty technique to the discrete problem (16). This is equivalent to

$$(23) \qquad T_{h\alpha\varrho}(v_h) \to \min! \qquad \text{s.t. } v_h \in U_h$$

where the function $T_{h\alpha\varrho}$ is defined by

$$(24) \qquad T_{h\alpha\varrho}(v_h) := J_h(v_h) + \alpha \sum_{i=1}^{N} \operatorname{meas}(D_i)\, \zeta_\varrho(g_i - v_i)$$

with the penalty parameters $\alpha > 0, \varrho > 0$. Here $J_h(\cdot)$ denotes the functional given by

$$J_h(v_h) := \frac{1}{2}\, a(v_h, v_h) - (f, v_h)_h \qquad \text{for any } v_h \in U_h$$

with

$$(f, v_h)_h := \sum_{i=1}^{N} \operatorname{meas}(D_i)\, f(x_i)\, v_h(x_i).$$

We remark that the discrete auxiliary function $T_{h\alpha\varrho}$ can be obtained by mass lumping from the continuous one $T_{\alpha\varrho}$ also. This guarantees that the Jacobians of $T_{h\alpha\varrho}$ have the same structure as the related stiffness matrix $A_h = (a(\varphi_j, \varphi_i))_{i,j=1}^{N}$ which arises in the same way in variational equation. Thus, efficient solvers for discrete elliptic equations can be applied if Newton's method is used. Furthermore, the auxiliary function $T_{h\alpha\varrho}$ is related to the reduced integration as for improving penalty methods.

In [7] optimal parameter selection rules $\alpha = \alpha(h)$ and $\varrho = \varrho(h)$ have been proposed to adjust the error caused by the penalty terms to the same magnitude as the discretization error of the finite element approximation.

As an immediate consequence of the differentiability and strong convexity of the auxiliary objective function $T_{h\alpha\varrho}(\cdot)$ we have

Lemma 0.5. *For any conform discretization $U_h \subset U$ and for any parameters $\alpha > 0$, $\varrho > 0$ the discrete penalty problem (23) possesses a unique solution $u_{h\alpha\varrho} \in U_h$. The condition*

$$(25) \qquad \langle T'_{h\alpha\varrho}(u_{h\alpha\varrho}), v_h\rangle = 0 \qquad \text{for any } v_h \in U_h$$

is necessary and sufficient for $u_{h\alpha\varrho} \in U_h$ to solve (23).

2. Newton's method applied to the continuous penalty problem

In this part we investigate the convergence behaviour of Newton's method applied to the variational equality (13). To obtain sharper bounds we do not estimate the inverse of F' and the Lipschitz-constant of F'' separately as often used in the convergence analysis of Newton's method for an operator equation $Fy = 0$. We take care of the asymptotic ill-conditioness by a combined estimation similar to [6]. In the case of penalyzed obstacle problems we are interested in the following weak comparison theorem for weakly nonlinear elliptic boundary problems forms a useful tool for the investigation.

Lemma 0.6. *Let $r\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a mapping which is monotone w.r.t. the second variable, i.e.*

$$r(x,s) \leqslant r(x,t) \qquad \text{for any } x \in \Omega,\ s \leqslant s.$$

Furthermore, we suppose that the related Nemyckij operator R defined by

$$[Rv](x) := r(x, v(x)),\ x \in \Omega$$

maps from $L_2(\Omega)$ into $L_2(\Omega)$. Then for any $u, w \in V$ satisfying

$$a(u,v) + \int\limits_\Omega r(x,u(x))\, v(x)\, \mathrm{d}x \leqslant a(w,v) + \int\limits_\Omega r(x,w(x))\, v(x)\, \mathrm{d}x \qquad \text{for any } v \in U,\ v \geqslant 0$$

and $u|_\Gamma \leqslant w|_\Gamma$ the estimation $u(x) \leqslant w(x)$ for almost every $x \in \Omega$ holds.

First, we investigate the continuous problem (12) which is equivalent to (13). Let us define $S\colon U \times U \to \mathbb{R}$ and $D\colon U \times U \times U \to \mathbb{R}$ by

$$\left.\begin{aligned} S(u,v) &:= -\alpha \int\limits_\Omega \zeta_\varrho'(g-u)\, v\, \mathrm{d}x \\ D(u,w,v) &:= \alpha \int\limits_\Omega \zeta_\varrho''(g-u)\, w\, v\, \mathrm{d}x \end{aligned}\right\} \text{ for any } u, w, v \in U. \tag{26}$$

Let $y \in U \cap L_\infty(\Omega)$ denote some approximation of the wanted solution $u_{\alpha\varrho}$ of (13). One step of Newton's method to improve y for a new approximation $z \in U$ can be described by

$$a(z,v) - (f,v) + S(y,v) + D(y, z-y, v) = 0 \qquad \text{for any } v \in U. \tag{27}$$

With the properties of $S(\cdot,\cdot)$ and $D(\cdot,\cdot,\cdot)$ Lax-Milgram's lemma (compare [5], [8] e.g.) guarantees that (27) has a unique solution $z \in U$ for any given $y \in U$. With (13) we obtain

(28)

$$a(z-u_{\alpha\varrho},v)+D(y,z-u_{\alpha\varrho},v) = S(u_{\alpha\varrho},v)-S(y,v)+D(y,y-u_{\alpha\varrho},v) \qquad \text{for any } v \in U.$$

Next, we estimate the right hand side of (28) using the fact $y, u_{\alpha\varrho} \in L_\infty(\Omega)$. Let $q \in L_\infty(\Omega)$ be defined by

$$q := \zeta'_\varrho(g-y) - \zeta'_\varrho(g-u_{\alpha\varrho}) + \zeta''_\varrho(g-y)\,(y-u_{\alpha\varrho}) \qquad \text{a.e. in } \Omega. \tag{29}$$

For almost every $x \in \Omega$ we can apply Taylor's formula and we obtain

$$q(x) = -\frac{1}{2}\,\zeta'''_\varrho(g(x)-y(x)+\xi(y(x)-u_{\alpha\varrho}(x))\,(y(x)-u_{\alpha\varrho}(x))^2 \tag{30}$$

with some $\xi = \xi(x) \in (0,1)$. The definition (10) of the function ζ_ϱ results in

$$\left|\zeta'''_\varrho(t+\xi\,d)\right| = 3\,\frac{|t+\xi d|\varrho}{((t+\xi d)^2+\varrho)^{5/2}} \leqslant \frac{3\varrho}{((t+\xi d)^2+\varrho)^2} \qquad \text{for any } t,\xi,d \in \mathbb{R}. \tag{31}$$

Now, for fixed $x \in \Omega$ we set

$$t := (g-y)(x), \qquad d := (y-u_{\alpha\varrho})(x) \tag{32}$$

and we distinguish two different cases.

i) $|t| \leqslant 2|d|$ Then (31) trivially leads to the estimation

$$\left|\zeta'''_\varrho(t+\xi d)\right| \leqslant \frac{3}{\varrho} \qquad \text{for any } \xi \in (0,1). \tag{33}$$

ii) $|t| > 2|d|$ In this case we obtain

$$|t+\xi d| \leqslant |t|+|d| < \frac{3}{2}\,|t| \quad \text{and} \quad (t+\xi d)^2 \geqslant (|t|-\xi|d|)^2 \geqslant \left|\frac{t}{2}\right|^2 \qquad \text{for any } \xi \in (0,1).$$

Thus, (31) results in

$$\left|\zeta'''_\varrho(t+\xi d)\right| \leqslant \frac{9\varrho}{(|\frac{t}{2}|^2+\varrho)^2} \qquad \text{for any } \xi \in (0,1). \tag{34}$$

On the base on these estimations we can prove the following

Theorem 0.1. *Let the penalty parameters $\alpha > 0$, $\varrho > 0$ be fixed. If the initial guess $u^0 \in U \cap L_\infty(\Omega)$ satisfies the condition*

$$(35) \qquad \|u^0 - u_{\alpha\varrho}\|_\infty < \frac{1}{36}\,\varrho^{1/2}$$

then Newton's method applied to (13) generates a sequence $\{u^l\} \subset (U \cap L_\infty(\Omega)$ which converges Q-quadratically in the L_∞-norm to the solution $u_{\alpha\varrho}$ of (13).

Proof. As remarked earlier Newton's method is well defined if applied to (13). Thus it suffices to show the contraction of Newton's iteration to the solution $u_{\alpha\varrho}$. The quadratic rate of convergence follows by standard arguments using the Taylor expansion as shown in (28)–(30).

With $y := u^l \in (U \cap L_\infty(\Omega)$ the general step of Newton's method is given by $u^{l+1} := z$ where z is determined by (27). The assumed smoothing property guarantees $u^{l+1} \in H^2(\Omega)$ which leads to $u^{l+1} \in L_\infty(\Omega)$. We construct upper and lower bounds via the comparison estimation stated in lemma 6 which can be applied because of the structure of problem (27) and because of the properties of the function $\zeta_\varrho(\cdot)$. Using these bounds we will show that

$$(36) \qquad \|y - u_{\alpha\varrho}\|_\infty \leqslant \mu\varrho^{1/2} \qquad \text{implies} \qquad \|z - u_{\alpha\varrho}\|_\infty \leqslant \delta\|u - u_{\alpha\varrho}\|_\infty$$

with $\delta = 36\mu$ holds. If we select $\mu \in (0, \frac{1}{36})$ then $\delta \in (0,1)$ and by mathematical induction the convergence of Newton's method follows from (36).

We select

$$(37) \qquad \overline{w}(x) := u_{\alpha\varrho}(x) + \delta\|y - u_{\alpha\varrho}\|_\infty.$$

This leads to $\overline{w}|_\Gamma \geqslant u_{\alpha\varrho}|_\Gamma$ and to

(38)

$$a(\overline{w} - u_{\varrho\alpha}, v) + D(y, \overline{w} - u_{\alpha\varrho}, v) = \alpha\delta\,\|y - u_{\alpha\varrho}\|_\infty \int_\Omega \zeta''_\varrho(g-y)\,v\,\mathrm{d}x \qquad \text{for any } v \in U.$$

Again we use the abbreviations given in (32). The definition of the function ζ_ϱ leads to

$$(39) \qquad \zeta''_\varrho(t) = \varrho\,(t^2 + \varrho)^{-3/2}.$$

As in the investigations earlier we distinguish two cases.

i) Let be $x \in \Omega$ such that $|t(x)| \leqslant 2|d(x)|$. With (33) we obtain

$$\zeta''_\varrho(t) \geqslant \frac{\varrho^2}{3}(t^2+\varrho)^{-3/2}\,|\zeta'''_\varrho(t+\xi d)| \geqslant \frac{1}{3}\,\varrho^{1/2}\,|\zeta'''_\varrho(t+\xi d)|.$$

ii) Let $|t(x)| > 2|d(x)|$ hold. With (34), (39) we have

$$\zeta_\varrho''(t) \geqslant \frac{1}{72}\,\varrho^{1/2}\,\left|\zeta_\varrho'''(t+\xi d)\right|. \tag{40}$$

Thus, (40) holds in both cases, i.e. it is valid for almost ever $x \in \Omega$. Estimation (40) results in

$$\begin{aligned} \delta\|t\|_\infty \zeta_\varrho''(t) &\geqslant \frac{1}{72}\,\delta\,\varrho^{1/2}\,\left|\zeta_\varrho'''(t+\xi d)\right| \\ &\geqslant \frac{\delta}{36\mu}\left|\frac{1}{2}\,\zeta_\varrho'''(t+\xi d)\,t^2\right| \\ &\geqslant -\frac{1}{2}\,\zeta_\varrho'''(t+\xi d)\,t^2. \end{aligned}$$

With (28)–(30) and with (38) this leads to

$$a(\overline{w}-u_{\alpha\varrho},v) + D(y,\overline{w}-u_{\alpha\varrho},v) \geqslant a(z-u_{\alpha\varrho},v) + D(y,z-u_{\alpha\varrho},v) \qquad \text{for any } v \in U,\ v \geqslant 0.$$

Furthermore $(\overline{w}-u_{\alpha\varrho})|_\Gamma \geqslant 0 = (z-u_{\alpha\varrho})|_\Gamma$ holds. Using Lemma 6 this results in

$$\overline{w} \geqslant z \quad \text{a.e. in } \Omega. \tag{41}$$

By the same arguments we can show

$$\underline{w} \geqslant z \quad \text{a.e. in } \Omega \tag{42}$$

for

$$\underline{w}(x) := u_{\alpha\varrho}(x) - \delta\|y-u_{\alpha\varrho}\|_\infty. \tag{43}$$

Combining (37) and (41)–(43) we obtain

$$\|z-u_{\alpha\varrho}\|_\infty \leqslant \delta\|y-u_{\alpha\varrho}\|_\infty.$$

Thus, (36) is valid and the sequence $\{u^l\}$ generated by Newton's method converges to $u_{\alpha\varrho}$ provided the initial guess u^0 was selected such that $\|u^0-u_{\alpha\varrho}\|_\infty < \frac{1}{36}\,\varrho^{1/2}$ holds. $\square$

3. Iterative solution of the discrete penalty problem

Similar to the investigations of the continuous problems we can estimate the area of contraction of Newton's method applied to the discrete penalty problem (23). The fundamental Lemma 6 can be modified to

Lemma 0.7. *Let $r_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1(1)N$ be continuous and monotone nondecreasing. Then for any $u_h, w_h \in V_h$ satisfying*

$$a(u_h, v_h) + \sum_{i=1}^{N} r_i(u_h(x_i))\, v_h(x_i) \leqslant a(w_h, v_h) + \sum_{i=1}^{N} r_i(w_h(x_i))\, v_h(x_i) \qquad \text{for any } v_h \in U_h,\ v_h \geqslant 0$$

and $u_h(x_i) \leqslant w_h(x_i)$, $i = N + 1(1)\overline{N}$ the estimation $u_h \leqslant w_h$ holds.

This lemma can be proven because the stiffness matrix $A_h = (a(\varphi_i, \varphi_j))_{i,j=1}^{N}$ is an M-matrix which results from the supposed angle condition of the triangulation. Finally the monotonicity of the functions r_i guarantees the stated comparison result (compare [11] e.g.).

We define the discrete versions $S_h : U_h \times U_h \to \mathbb{R}$ and $D_h : U_h \times U_h \times U_h \to \mathbb{R}$ of S and D, respectively, by the application of the mass lumping idea to (26). This leads to

(44)

$$\left.\begin{aligned} S_h(u_h, v_h) &:= -\alpha \sum_{i=1}^{N} \operatorname{meas}(D_i)\, \zeta_\varrho'(g_i - u_i)\, v_i \\ D_h(u_h, w_h, v_h) &:= \alpha \sum_{i=1}^{N} \operatorname{meas}(D_i)\, \zeta_\varrho''(g_i - u_i)\, w_i\, v_i \end{aligned}\right\} \quad \text{for any } u_h,\ w_h,\ v_h \in U_h.$$

Here u_i, w_i, v_i denote the components of the related functions u_h, w_h, $v_h \in U_h$ in the representation by the piecewise linear base $\{\varphi_i\}_{i=1}^{N}$ of U_h.

One step of Newton's method to improve $y_h \in U_h$ for a new approximation $z_h \in U_h$ of the wanted solution $u_{h\alpha\varrho}$ of the discrete penalty problem (23) can be described similarly to (27) by

$$a(z_h, v_h) - (f, v_h)_h + S_h(y_h, v_h) + D_h(y_h, z_h - y_h, v_h) = 0 \qquad \text{for any } v_h \in U_h. \tag{45}$$

Because of the pointwise estimations used in the proof of theorem 1 and because of the mass lumping applied in definition (44) we can carry over the results of Theorem 1 to the discrete case. With lemma 7 we have

Theorem 0.2. *Let the penalty parameters $\alpha > 0$, $\varrho > 0$ be fixed. If the initial guess $u_h^0 \in U_h$ satisfies the condition*

$$\|u_h^0 - u_{h\alpha\varrho}\|_\infty < \frac{1}{36}\varrho^{1/2} \tag{46}$$

then Newton's method applied to (23) generates a sequence $\{u_h^l\} \subset U_h$ which converges Q-quadratically to the solution $u_{h\alpha\varrho}$ of (23).

Finally, we remark that comparison theorems for weakly nonlinear elliptic problems also form a good tool for deriving sharper bounds for the order of convergence of penalty methods applied to continuous and discrete variational inequalities as shown in [1], [8].

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