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FINITE NONDENSE POINT SET ANALYSIS

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Summary. The paper deals with the decomposition and with the boundary and hull construction of the so-called nondense point set. This problem and its applications have been frequently studied in computational geometry, raster graphics and, in particular, in the image processing (see e.g. [3], [6], [7], [8], [9], [10]). We solve a problem of the point set decomposition by means of certain relations in graph theory.

Keywords: nondense point set, decomposition, boundary, hull, stabilized matrix

AMS classification: 68U10 (68R10, 05C12)

1. POINT SET DECOMPOSITION

Let $\mathscr{X}$ be a set of points in plane, i.e. $\mathscr{X} = \{A;\, A_i \in E_2, i = 1, 2, \ldots, n, n > 2\}$. Denote d the distance of two points from the set Z and let $R = \max|A_iA_j|$, where $A_i, A_j \in \mathscr{X}$. A non-oriented graph $G = (V, H, w)$ is created, where V is the set of points A (vertices), H is the set of segments A_iA_j (edges) and $w(e) = A_iA_j = d$, $e \in H$, $e = \{A_iA_j\}$.

Let $\mathscr{D} = \{d, 0 < d \leqslant R;\, G$ is a connected graph$\}$, i.e. D is the set of distances of two points from the set $\mathscr{X}$. Elements of the set $\mathscr{D}$ are less than or equal to $R = \max|A_iA_j|$ such, that G is a connected graph. Denote $d_m(G) = \min \mathscr{D}$. Obviously $d(G)$ is the weight of the longest edge of the graph spanning tree, [4].

The graph G can be defined using an incidence (Boolean) matrix $\mathscr{G}$. It is a symmetric n by n square matrix (n is the number of the graph vertices) in which the element

$$g_{ij} = \begin{cases} 1, & \text{if } |A_iA_j| \leqslant d \text{ vertices are linked up by an edge} \\ 0. & \text{otherwise} \end{cases}$$

Form a matrix

$$\mathscr{S}^k = E + \mathscr{G} + \mathscr{G}^2 + \ldots + \mathscr{G}^k,$$

where element $s_{ij} = s_{ji} = 1$ expressed that there exists a path between the vertices A_i, A_j whose length (in the sense of graph theory) is equal to or less than k, $k \geqslant 1$, E is the unit matrix. Recurrently, it is possible to put

$$\mathscr{S}^k = \mathscr{S}^{k-1} \cdot \mathscr{G} + E,$$

where $\mathscr{S}^0 = E$ means that each vertex links itself and this way length 0.

Theorem 1. *For a finite graph $\mathscr{G}$ there always exists a number k, $0 \leqslant k \leqslant u$, $u = H$ is a cardinal number, such that $S^k = S^{k+1} = \ldots$ The number k is the diameter of the graph $\mathscr{G}$.*

Addition and multiplication of matrix is boolean.

The matrix S from Theorem 1 will be called the stabilized matrix of graph G, [1].

Theorem 2. *If there exists at least one element $s_{ij} = 0$ in $\mathscr{S}^k$, then $\mathscr{S}^k$ generates a decomposition of the graph G into components.*

See [1] for details.

Note. *If $d \leqslant d_m(G)$ then there exist at least two components.*

2. Outer boundary construction: Algorithm I

A polygon is said to be the boundary of a set of points $\mathscr{M}$, if each point of $\mathscr{M}$ either insides with the interior of the boundary or is a vertex of it.

Let $\mathscr{M}_t = \{A_i, A_i \in \mathscr{X}, A_i$ are the vertices of the component H_t of the graph $G\}$, $t = 1, 2, \ldots, n$, where n is the number of components, $n \geqslant 2$. Denote by ℓ the length of the maximal weight of edge of the graph spanning tree of the component H_t, i.e. $\ell = d_m(H_t)$. We construct the outer boundary as a closed polygon $P_0 \ldots P_n$:

1. Choose a point $A_i \in \mathscr{M}_t$ having the smallest x coordinate and denote it by P_0.
2. From the points $A_i \neq P_0$, where $P_0 A_i \leqslant \ell$, we choose the one—denoting it by P_1—which is incident with the arm of the greatest oriented angle (orientation opposite to the clockwise direction) the first arm of which is formed by the x-axis and the other passes through the point P_0. If there are more than one such points we choose one of those with the smallest x.
3. Let P_r be a vertex of the boundary for any natural number $r > 0$. From points $A_i \neq P_r$, for which $P_r A_i \leqslant \ell$ we choose that one denoting it P_{r+1} which incides

with the arm of the greatest oriented angle (in opposite the clockwise direction) $P_{r-1}P_rP_{r+1} = \varphi$, $\varphi \leqslant 2\pi$ (Fig. 1), while

A if there are more such points A_i inciding with this arm we choose that with the smallest distance from vertex P_r

B1 if segment P_rA_i intersects a side of the boundary polygon in its inner point, or

B2 if no point $A_i \neq P_{r-1}$ in the circle domain with the diameter ℓ exists, we choose point P_{r-1} to be a boundary polygon vertex, it means $P_{r+1} = P_{r-1}$.

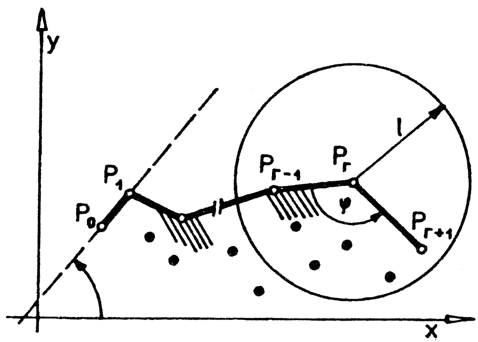

Fig. 1

It follows from the construction that there always exists such a point $P_{n+1} = P_0$ for which the boundary polygon $P_0 \dots P_r$ is closed. An example of the boundary of the set M_t generated by one graph component H_t for given ℓ is in Fig. 2.

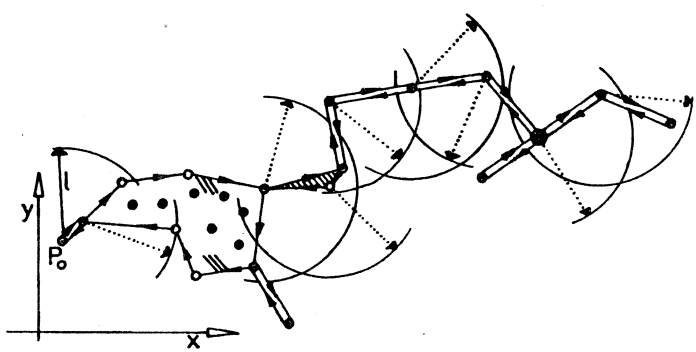

Fig. 2

Note 1. *Obviously, for an arbitrary motion of the set $\mathcal{M}_t$ in the plane, point P_0 is according to this construction the vertex of the angle $P_nP_0P_1 = \varphi$, where $0 \leqslant \varphi \leqslant 2\pi$. The boundary polygon is one-correspondent at any motion.*

Note 2. *Algorithm I shortly describes a method of boundary construction only. The complexity of computing does not exceed $O(n^2)$ in any case, because the complexicity of the construction algorithm of a vertex P_{r+1} using the greatest angle is linear. Computation of the angle φ value is connected either with the elementar arithmetic operationes or with set operations on a point set. Some algorithms of a boundary construction (for convex hull) e.g. Jarvis's march, Graham's algorithm, Quickhull techniques and other are described in* [11].

3. Determination of two components incidence

Let H_1, H_2 be two components of graph G and $\mathcal{M}_1$, $\mathcal{M}_2$ sets of these components vertices. Let $\mathcal{V} = \{v;\ v = A_iA_j;\ A_i \in \mathcal{M}_1 \text{ and } A_j \in \mathcal{M}_2\}$ and let $v_d = \min \mathcal{V}$ (v_d defines the deviation of the $\mathcal{M}_1$ and $\mathcal{M}_2$).

Theorem 3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two point sets (in plane) generated by the decomposition of a component graph into components and let $[A_1, A_2]$, $A_i \in \mathcal{M}_1$, $A_j \in \mathcal{M}_2$ is such a pair of points that $A_iA_j = v_d$. Let further p_1 be the boundary of set $\mathcal{M}_1$. Then if $A_i \notin p_1$ and*

1. *if segment A_iA_j does not incide with a point of boundary p_1, according to the algorithms in part 2, $\mathcal{M}_2$ incides with the interior of set $\mathcal{M}_1$, which boundary is the outline p_1.*
2. *if segment A_iA_j intersects boundary p_1, $\mathcal{M}_2$ does not incident with $\mathcal{M}_1$.*

Note. *The theorem can be rewritten for the priority of $\mathcal{M}_2$ to $\mathcal{M}_1$.*

Proof. Assertion 1 is trivial. Let now $m = [a, A_j]$ be a semiplane (see Fig. 3) and circle domains $k_1 = (A_i, v_d)$, $k_2 = (A_j, v_d)$. Form a region $\mathcal{O} = m \cap k_1 - k_2$ (hatching part). If A_i is not a vertex of polygon p_1, there exists, obviously, exactly one point of the neighbouring vertices P_i, $P_{i+1} \in \mathcal{O}$ such that $P_iP_{i+1} \leqslant v_d$ and segment P_iP_{i+1} incides with the interior point of segment A_iA_j. If there existed another pair of points P_i' and P_{i+1}' having the same property, according to the boundary construction I, points P_i and P_i' would either vertices of side of polygon p_1, or one of them lies in region $\mathcal{O}$. The same is valid for P_{i+1} and P_{i+1}'. □

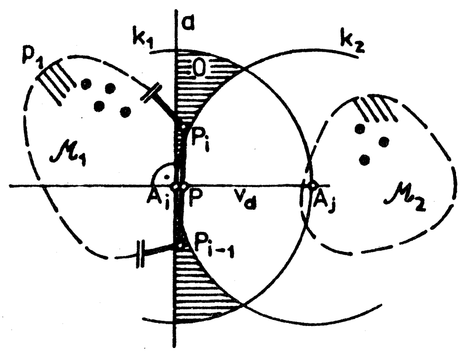

Fig. 3

4. Boundary construction: Algorithms II

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two sets of points in the distance of v_d. Let $A_i \in \mathcal{M}_1$ and $A_j \in \mathcal{M}_2$ be two points such that $A_iA_j \leqslant v_d$. Let us choose in $\mathcal{M}_1$ two points K and L for which $A_iK \leqslant v_d$, $A_iL \leqslant v_d$ and angle $\varphi = A_jA_iK$, $\frac{1}{3}\pi \leqslant \varphi \leqslant \pi$ opposite in clockwise direction is the smallest one, the angle $\psi = A_jA_iL$, $\frac{1}{3}\pi \leqslant \psi \leqslant \pi$ in clockwise direction is the smallest one respectively (Fig. 4). It follows from the Theorem 3 that next point of the boundary lies either in the region $\mathcal{O}$ or in the opposite semiplane to $[a, A_j]$ (see Fig. 3). Therefore the lower bound of the intervals for angles φ and ψ is $\frac{1}{3}\pi$. If some more points satisfying given conditions incide with the arms k or ℓ, we choose those K resp. L, which are closest distance from point A_i. If at least one of angles φ or ψ respectively is equal or greater than $\frac{1}{2}\pi$ (see part 3), it is possible to regard point A_i as a vertex of the boundary polygon; then $K = P_n$ and $L = P_1$ are boundary vertices and other vertices are constructed using the algorithm in part 2.

If both arms make angles $\frac{1}{3}\pi \leqslant \varphi \leqslant \frac{1}{2}\pi$, $\frac{1}{3}\pi \leqslant \psi \leqslant \frac{1}{2}\pi$ and $KL \leqslant v_d$ we choose $K = P_n$ and $L = P_0$, and other vertices according the algorithm in part 2.

Consequence. *If points $P_0, \ldots, P_n$, obtained using this algorithms are the same, designation excluding, the boundary is concerned. If it is not true, the boundary obtained using this construction is inner boundary (Fig. 5).*

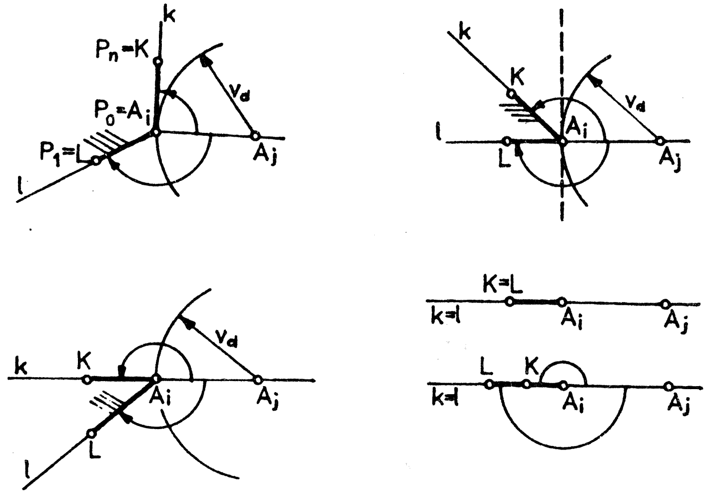

Fig. 4

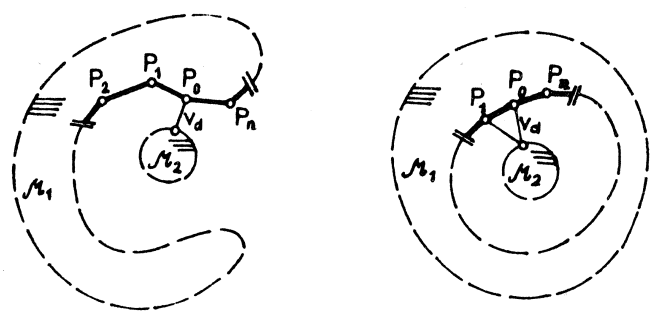

Fig. 5

5. Boolean matrices application to graph decomposition and application

Neighbourhood matrix defining the graph is a Boolean matrix which consists of elements 0 and 1. The occupation of the operation memory of this matrix is not much economizing. For this reason we transformed the Boolean matrix through the binary number system, to reduced matrix, consisting of elements in decadic number system. [5]

Matrix $\mathcal{S}^k = \mathcal{S}^{k-1} \cdot \mathcal{G} + E$ we formed multiplying competent reduced matrices. Problem of the reduced matrices multiplication has been solved and competent algorithm as well as the program realizing graph component aided decomposition of point sets into groups is detail described in [2].

We multiply matrices as long as the stabilized matrix is obtained, i.e. while $\mathcal{S}^k = \mathcal{S}^{k+1} = \mathcal{S}^{k+2} = \dots$

At last, we find out which rows of the stabilized matrix S^k are equal. These equal rows represent single-components of the given graph.

In Fig. 6 is the boundary of convex hull (6a), hull for $d_m(G)$ (6b), the case of boundary for $d_1 < d_m(G)$ (6c) and the hull for $d_2 < d_1$.

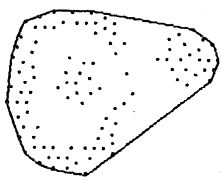

Fig. 6a

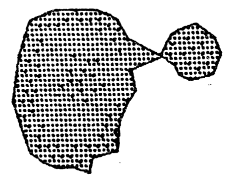

Fig. 6b

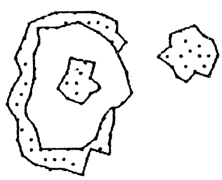

Fig. 6c

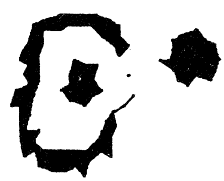

Fig. 6d

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Súhrn

ANALÝZA KONEČNÝCH RIEDKYCH BODOVÝCH MNOŽÍN

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V príspevku sa pojednáva o rozklade tzv. riedkej bodovej množiny spolu s konštrukciou obrysu a obalu. Tento problém a jeho aplikácie sú frekventovanou problematikou v počítačovej geometrii a špeciálne v rastrovej grafike, najmä v oblasti spracovania obrazov. (Pozri napr. [3], [6], [7], [8], [9], [10].)

Problém rozkladu bodovej množiny je riešený pomocou istých vzťahov teórie grafov.

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