Marian Kwapisz

On modification of Samoilenko's numerical-analytic method of solving boundary value problems for difference equations

Applications of Mathematics, Vol. 38 (1993), No. 2, 133–144

Persistent URL: http://dml.cz/dmlcz/104540

38 (1993) APPLICATIONS OF MATHEMATICS No. 2, 133–144

ON MODIFICATION OF SAMOILENKO'S NUMERICAL-ANALYTIC METHOD OF SOLVING BOUNDARY VALUE PROBLEMS FOR DIFFERENCE EQUATIONS

MARIAN KWAPISZ, Gdańsk

(Received April 6, 1992)

Summary. In the paper a modification of Samoilenko's numerical analytic method is adapted for solving of boundary value problems for difference equation. Similarly to the case of differential equations it is shown that the considered modification of the method requires essentially less restrictive condition—than the original method—for existence and uniqueness of solution of auxiliary equations which play a crucial role in solving the boundary value problems for difference equations.

Keywords: difference equations, boundary value problems, numerical methods

AMS classification: 65Q05, 39A70

1.

In recent years the interest in various problems for difference equations has been growing. It includes the boundary value problems for such equations. Because of the close paralellism between the theories of difference equations and ordinary differential equations many results can be transfered from one theory to the other.

The purpose of the present paper is to consider a modification of Samoilenko's numerical-analytic method adapted for solving boundary value problems for difference equations. For the method applied to ordinary differential equations consult the papers [17–19], [4], [9–10] and the books [20], [21]. For some results concerning boundary value problems for difference equations consult the papers [1–3], [5–8], [14–16]. The discrete version of Samoilenko's method appeared in the papers [7–8], [13] where the problem of a periodic solution for the scalar difference equation [13] and the general linear two point boundary value problems [7], [8] were dealt with.

2.

Let X be a linear space equipped with an m-dimensional vector norm $|\cdot|$, i.e. $|\cdot|\colon X \to \mathbb{R}_+^m$, $\mathbb{R}_+ = [0, +\infty)$. It is clear that X becomes a Banach space if the norm in X is defined by the equation $\|\cdot\| = \||\cdot|\|_{\mathbb{R}^m}$, where $\|\cdot\|_{\mathbb{R}^m}$ denotes a fixed norm in $\mathbb{R}^m$.

Let $N \in \{1, 2, \ldots\}$, $J_N = \{0, 1, 2, \ldots, N\}$, and let us consider the space $\mathscr{F}_N = \mathscr{F}(J_N, X)$ of all functions from J_N to X. Define the difference operator $\Delta\colon \mathscr{F}_N \to \mathscr{F}_{N-1}$ and the summation operator $S\colon \mathscr{F}_{N-1} \to \mathscr{F}_N$ by the equations

$$\Delta x(n) = x(n+1) - x(n), \quad n \in J_{N-1},$$

$$Sx(n) = \big(Sx(\cdot)\big)(n) = \sum_{i=0}^{n-1} x(i), \quad n = 1, 2, \ldots N, \quad Sx(0) = 0.$$

Let a mapping $f\colon \mathscr{F}_N \to \mathscr{F}_{N-1}$ and operators C, $D \in L(X)$ ($L(X)$ is the space of all linear and bounded operators from X to X) be given. we consider the boundary value problem

$$\Delta x(n) = fx(n), \quad n \in J_{N-1}, \tag{1}$$

$$Cx(0) + Dx(N) = d \tag{2}$$

for a given $d \in X$. The problem (1)–(2) is a specific system of equations considered in X^{N+1}. We will treat this system using its specific form. We start our discussion with the simple case $d = 0$ and $C = -D = I$, where I stands for the identity operator. It is the case of the periodic boundary condition. First of all we observe that (1)–(2) is equivalent to the system

$$x(n) = x(0) + (Sfx)(n), \quad n \in J_n, \tag{3}$$

$$Cx(0) + Dx(N) = d. \tag{4}$$

Now if we introduce a parameter $x_0 = x(0)$ then (3)–(4) can be viewed as a system with an unknown pair $\big(x_0, x(\cdot)\big) \in X \times \mathscr{F}_N$. The standard shooting method of solving the problem reads: find a solution $\varphi(\cdot, x_0)$ of equation (3) (we call it an auxiliary equation) and use equation (4) (we call it the determining equation) to determine the parameter x_0^* to get a solution $\varphi(\cdot, x_0^*)$ of the problem. The determining equation can be written as

$$Cx_0 + D\big(x_0 + \big(Sf\varphi(\cdot, x_0)\big)(n)\big) = d. \tag{5}$$

The main point of the discrete version of Samoilenko's method is that in the periodic case the auxiliary equation is replaced by

$$x(n) = x_0 + Sfx(n) - n \cdot N^{-1} \cdot Sfx(N), \quad n \in J_N, \tag{6}$$

and is considered together with the determining equation

$$Sf\varphi(\cdot, x_0)(N) = 0. \tag{7}$$

The advantage of the approach is that any solution $\varphi(\cdot, x_0)$ of equation (6) satisfies the periodic boundary condition. It becomes a solution of the problem when x_0 is determined by equation (7). On the other hand, we have to observe that equation (6) is more difficult to solve than equation (3) and rather restrictive condition $3^{-1}KN < 1$ is required to guarantee the existence of a solution to equation (6) (here K denotes the Lipschitz constant for f, see [7], [13]).

We will show that the restrictive condition can be relaxed when the function $n \cdot N^{-1}$ in equation (6) is replaced by some function $\omega\colon J_N \to [0,1], \omega(0) = 0, \omega(N) = 1$. The same is possible for the general boundary value problem (1)–(2) which is equivalent to the system

$$x(n) = x_0 + Sfx(n) + \omega(n)\big(\varphi(x_0) - Sfx(N)\big), \quad n \in J_N, \tag{8}$$

$$\varphi(x_0) - Sfx(N) = 0, \tag{9}$$

if D^{-1} exists and $\varphi(x_0) = D^{-1}[d - (C + D)x_0]$.

Observe that for any $x_0 \in X$ the solutions of the auxiliary equation (8) satisfy boundary condition (2), and equation (9) serves for determing x_0 such that $x(\cdot, x_0)$ satisfies equation (1).

3.

Equation (8) is a fixed point equation in the space $\mathscr{F}_N$ with the parameter x_0. The space $\mathscr{F}_N$ can be normed by making use of the weighted norm in the following way. For $x \in \mathscr{F}_N$ and $\varrho \in \mathscr{F}(J_N, \mathbb{R}^m_+)$ with positive coordinates define (here the maximum of a vector means the vector of the maxima of its coordinates)

$$|x|_n = \max[|x(k)|, 0 \leqslant k \leqslant n], \quad n \in J_N, \tag{10}$$

$$\|x\|_\varrho = \inf_{s>0}[|x|_n \leqslant s\varrho(n), n \in J_N]. \tag{11}$$

It is clear that $(\mathscr{F}_N, \|\cdot\|_\varrho)$ is a Banach space. We will show that the operator F defined by the right hand side of equation (8) is a contraction when the function ϱ is properly chosen and f satisfies the following condition:

there is a matrix function $K \in \mathscr{F}(J_{N-1}, \mathbb{R}_+^{m\times m})$ such that

$$|fx(n) - fy(n)| \leqslant K(n)|x - y|_n, \quad n \in J_{N-1} \tag{12}$$

for any $x,\ y \in \mathscr{F}_N$.

Let us search for the proper weight function ϱ. We have

$$\begin{aligned}|Fx(k) - Fy(k)| &= |Sfx(k) - \omega(k)Sfx(N) - Sfy(k) + \omega(k)Sfy(N)| \\ &= |S(fx - fy)(k) - \omega(k)[S(fx - fy)(N)]| \\ &\leqslant SK|x - y|(k) + \omega(k)SK|x - y|(N).\end{aligned}$$

This in view of the inequality $|x|_i \leqslant \|x\|_\varrho \cdot \varrho(i)$, $i \in J_N$, implies the relations

$$|Fx(k) - Fy(k)| \leqslant \|x - y\|_\varrho[SK\varrho(k) + \omega(k)SK\varrho(N)], \quad k \in J_N, \tag{13}$$

$$|Fx - Fy|_n \leqslant \|x - y\|_\varrho[SK\varrho(n) + \omega(n)SK\varrho(N)], \quad n \in J_N, \tag{14}$$

provided ω is nondecreasing. From inequality (14) we conclude that the function ϱ will be a proper weight function if it is positive nondecreasing and there is $\lambda \in (0, 1)$ such that

$$\lambda\varrho(n) \geqslant SK\varrho(n) + \omega(n)SK\varrho(N), \quad n \in J_N. \tag{15}$$

In this case we have

$$\|Fx - Fy\|_\varrho \leqslant \lambda \cdot \|x - y\|_\varrho, \quad x, y \in \mathscr{F}_N, \tag{16}$$

which means that F is a contraction with the contraction coefficient λ. To get a positive solution of inequality (15) we consider the equation

$$\varrho(n) = SK_1\varrho(n) + \omega(n)SK_1\varrho(N) + e, \quad n \in J_N, \tag{17}$$

where $K_1 = \lambda^{-1}K$ and e is the constant vector with each coordinate equal to one. It is clear that each positive solution of equation (17) satisfies (15). Put $v(n) = SK_1\varrho(n)$ and multiply (17) by $K_1(n)$ to get the equation

$$\Delta v(n) = K_1(n)v(n) + \omega(n)K_1(n)v(N) + K_1(n)e, \quad n \in J_{N-1} \tag{18}$$

with the condition $v(0) = 0$. Put

$$M(n, i) = \prod_{s=i+1}^{n-1} (I + K_1(s)), \quad M(n, n-1) = I \tag{19}$$

for $n = 1, 2, \ldots, N$, $i = 0, 1, \ldots, n-2$. Using the elementary results of the theory of difference equations (see [11], [12]) we find

$$v(n) = \left[S\big(M(n,\cdot)\omega K_1\big)(n)\right] v(N) + S\big(M(n,\cdot)K_1 e\big)(n), \quad n \in J_N. \tag{20}$$

Now if we take in this equation $n = N$ then it can be solved with respect to $v(N)$ with a nonnegative $v(N)$ provided the spectral radius of the matrix $S\big(M(N,\cdot)\omega K_1\big)(N)$ is less than one. We write this condition explicitly as follows:

$$r\left(\sum_{i=0}^{N-1} \prod_{s=i+1}^{N-1} (I + K_1(s))\omega(i)K_1(i)\right) < 1. \tag{21}$$

We see that this condition is sufficient to get a nonnegative solution of equation (18) and a positive solution to equation (17) which has the form

$$\varrho(n) = v(n) + \omega(n)v(N) + e, \quad n \in J_N. \tag{22}$$

Observe that because of the continuous dependence of the spectral radius of the matrix on its entries, condition (21) holds of some $\lambda \in (0,1)$ if (21) is satisfied with K_1 replaced by K. Notice also that for given K and $\lambda \in (0,1)$ there always exists a function ω for which (21) holds.

Summarizing above considerations we can formulate

Theorem 1. *If the function ω is nondecreasing and the conditions* (12) *and* (21) *hold (with K_1 replaced by K), then for any $x_0 \in X$ there is a unique solution $x(\cdot, x_0)$ of equation* (8) *and it can be obtained by the successive approximations method.*

4.

Now let us discuss an improvement of the result established in Section 3. First we observe that the operator F defined by the right hand side of equation (8) can be rewritten as

$$Fx(n) = p(n, x_0) + Wfx(n) \tag{23}$$

with

$$Wfx(n) = \big(1 - \omega(n)\big)Sfx(n) - \omega(n)\bar{S}fx(n) \tag{24}$$

and

$$\bar{S}fx(n) = Sfx(N) - Sfx(n), \qquad p(n, x_0) = x_0 + \omega(n)\varphi(x_0).$$

It is clear that $Wfx(0) = Wfx(N) = 0$. Now we see that the auxiliary equation (8) can be rewritten as

$$z(n) = Wf(z+p)(n), \tag{25}$$

with $z(n) = x(n) - p(n, x_0)$ as a new unknown. This is a fixed point equation in the subspace $\mathscr{F}_{N,0} \subset \mathscr{F}_N$ consisting of all $z \in \mathscr{F}_N$ such that $z(0) = z(N) = 0$.

In solving equation (25) the crucial role is played by a comparison operator $\Omega\colon \mathscr{F}(J_N, \mathbb{R}^m_+) \to \mathscr{F}(J_N, \mathbb{R}^m_+)$ associated with the operator G, $Gz(n) = Wf(z+p)(n)$ by the relation

$$|G(z+w)(n) - Gz(n)| \leqslant \Omega|w|(n), n \in J_N. \tag{26}$$

Under the assumptions about f and ω introduced in the previous sections we conclude that

$$\begin{aligned} |G(z+w)(n) - Gz(n)| &\leqslant \big(1-\omega(n)\big)\big(SK(i)|w|_i\big)(n) + \omega(n)\big(\bar{S}K(i)|w|_i\big)(n) \\ &\leqslant \big(1-\omega(n)\big)\big(SK(i)|w|_i\big)(n) + \omega(n)\big(SK(i)|w|_i\big)(N) \\ &\leqslant \big(SK(i)|w|_i\big)(n) + \omega(n)\big(SK(i)|w|_i\big)(N). \end{aligned} \tag{27}$$

We see that there are three different comparison operators for G; we denote them by Ω_2, Ω_1 and Ω_0 in the order as they appear in the above sequence (27) of inequalities.

In the previous section we have in fact worked with the operator Ω_0. It is clear that the conditions of Theorem 1 are nothing else than sufficient conditions for the operator Ω_0,

$$\Omega_0 u(n) = SKu(n) + \omega(n) SKu(N), \tag{28}$$

to have the spectral radius less than one. The last condition, according to the comparison theory, is sufficient for the unique solvability of equation (25). On the other hand, if we employ the operator Ω_1,

$$\Omega_1 u(n) = \big(1-\omega(n)\big) SKu(n) + \omega(n) SKu(N), \tag{29}$$

instead of Ω_0 then by the same reasoning as that used in the proof of Theorem 1 we obtain that the condition

$$r\Big(\sum_{i=0}^{N-1} \prod_{s=i+1}^{N-1} \Big(I + \big(1-\omega(n)\big)K(s)\Big)\omega(i)K(i)\Big) < 1 \tag{30}$$

is sufficient in order to get the assertion of Theorem 1. Notice that any positive solution of the equation $\varrho(n) = \Omega_1\varrho(n) + e$ is nondecreasing, which can be seen by a direct calculation of $\Delta\varrho(n)$. It is clear that condition (30) is less restrictive than the condition of Theorem 1.

5.

To make use of the fact that the operator G is an operator in the space $\mathscr{F}_{N,0}$ we have to assume a stronger condition for f of the form

$$|fx(n) - fy(n)| \leqslant K(n)|x(n) - y(n)|, \quad n \in J_{N-1}, \tag{12'}$$

which means that $fx(n) = f\big(n, x(n)\big)$, so that there is no hereditary dependence of f on x. In this case instead of (27) we have

$$\begin{aligned} &|g(z+w)(n) - Gz(n)| \\ &\qquad \leqslant \big(1 - \omega(n)\big)\big(SK(i)|w(i)|\big)(n) + \omega(n)\big(\bar{S}K(i)|w(i)|\big)(n). \end{aligned} \tag{31}$$

Now we can imply the operator Ω_2,

$$\begin{aligned} \Omega_2 u(n) &= \big(1 - \omega(n)\big)SKu(n) + \omega(n)\bar{S}Ku(n) \\ &= \big(1 - 2\omega(n)\big)SKu(n) + \omega(n)SKu(N). \end{aligned} \tag{32}$$

Observe that $\Omega_2 u(0) = \Omega_2 u(N) = 0$ for any $u \in \mathscr{F}(J_N, \mathbb{R}^m_+)$.

Suppose that there are $\lambda \in (0,1)$ and $\varrho \in \mathscr{F}(J_N, \mathbb{R}^m_+)$, $\varrho(n) > 0$, $n = 1, 2, \ldots, N-1$, such that the inequality

$$\lambda\varrho(n) \geqslant \Omega_2\varrho(n), \quad n \in J_N \tag{33}$$

holds. Using this function ϱ as weight function we define in $\mathscr{F}_{N,0}$ a norm by the relation

$$\|x\|_{\varrho,0} = \inf_{s>0}[|x(n)| \leqslant s\varrho(n), n \in J_N]. \tag{34}$$

Now by a reasoning similar to that of Section 3 we conclude that the operator G is a contraction in $\mathscr{F}_{N,0}$. Finally, we have

Theorem 2. *If condition (12′) is satisfied and there is a function ϱ having the properties mentioned in this section then the assertion of Theorem 1 holds.*

Observe that for any $h \in \mathscr{F}(J_N, \mathbb{R}^m_+)$, $h(0) = h(N) = 0$, and $\lambda \in (0,1)$ the equation

$$\lambda\varrho(n) = \Omega_2\varrho(n) + h(n), \quad n \in J_N, \tag{35}$$

can be explicitly solved in the same way as equation (17) was treated. Finally, one can find that the condition: $\omega(n)$, $n = 1, 2, \ldots, N-1$, small enough is sufficient for the existence of ϱ satisfying (35) and positive for $n = 1, 2, \ldots, N-1$.

However, we can reason in another way as follows. If the spectral radius of the operator Ω_2 is less than one then for some $\lambda \in (0,1)$ there is a solution of (35) positive for $n = 1, 2, \ldots, N-1$. Observe that the operator Ω_2 is associated with the matrix

$$\begin{pmatrix} \omega(1)K(1), & \omega(1)K(2), & \omega(1)K(3), & \ldots, & \omega(1)K(N-1) \\ (1-\omega(2))K(1), & \omega(2)K(2), & \omega(2)K(3), & \ldots, & \omega(2)K(N-1) \\ (1-\omega(3))K(1), & (1-\omega(3))K(2), & \omega(3)K(3), & \ldots, & \omega(3)K(N-1) \\ \ldots & & & & \\ (1-\omega(N-1))K(1), & (1-\omega(N-1))K(2), & (1-\omega(N-1))K(3), & \ldots, & \omega(N-1)K(N-1) \end{pmatrix}$$

We denote it by A_ω. It is clear that $r(\Omega_2) = r(A_\omega)$. From the form of the matrix A_ω it follows that $r(A_\omega) < 1$ if $\omega(i)$ is small for $i = 1, 2, \ldots, N-1$. Indeed, if $\omega(i) = 0$ for $i = 1, 2, \ldots, N-1$ then $r(A_\omega) = 0$ and the assertion is a consequence of the continuous dependence of the spectral radius on the entries of the matrix.

To get a more explicit result one can use the natural splitting of the matrix A_ω into the sum of a lower triangular matrix L and an upper triangular U (L consists of all entries that stand below the diagonal of A_ω). it is clear that $r(L) = 0$. Then for a given $\varepsilon < 1$ one can take a norm $\|\cdot\|_\varepsilon$ such that $\|L\|_\varepsilon \leqslant \varepsilon$. There is a positive constant Q_ε such that

$$\|U\|_\varepsilon \leqslant Q_\varepsilon \max[\omega(i), i = 1, 2, \ldots, N-1].$$

Now we find the condition for ω from the inequality

$$\begin{aligned} \|A_\omega\|_\varepsilon &\leqslant \|L + U\|_\varepsilon \leqslant \|L\|_\varepsilon + \|U\|_\varepsilon \\ &\leqslant \varepsilon + Q_\varepsilon \cdot \max[\omega(i), i = 1, 2, \ldots, N-1] < 1. \end{aligned}$$

6.

There is a special case when a solution ϱ of inequality (33) can be found explicitly. It is the case when the function K is constant, $K(n) = K$ and $\omega(n) = N^{-1}n$. Observe first that for the function $\xi(n) = n(N-n)$ the inequality

$$(36) \qquad (N-2n)S\xi(n) + nS\xi(N) = (N-n)S\xi(n) + n\bar{S}\xi(n) \leqslant 3^{-1}N^2\xi(n)$$

holds (see [7]). Let $b \in \mathbb{R}^m_+$ be a positive eigenvalue of the matrix K satisfying the relation $Kb = r(K)b$ (the existence of such a vector is guaranteed by the well known Perron theorem). Now multiplying (36) by the vector $N^{-1}Kb$ one finds

$$(37) \qquad N^{-1}(N-2n)SK\xi b(n) + N^{-1}nSK\xi b(N) \leqslant 3^{-1}Nr(K)\xi(n)b, \quad n \in J_N.$$

This means that the function ϱ, $\varrho(n) = \xi(n)b$, is a solution of (33) in the case considered here with $\lambda = 3^{-1}Nr(K) < 1$.

We can conclude the following

Corollary (see [7]). *If condition* (12′) *is satisfied with* $K(n) = K$ *and* $\frac{1}{3}Nr(K) < 1$ *then the assertion of Theorem* 1 *is true.*

7.

Finally, let us consider a more general problem of the form

$$(38) \qquad \Delta x(n) = fx(n),$$

$$(39) \qquad \sum_{i=0}^{N} C_i x(i) = d,$$

with $C_i \in L(X)$.

It is easy to see that this problem is equivalent to the system

$$(40) \qquad x(n) = x_0 + Sfx(n) + \omega(n)[\varphi(x_0) - \sum_{i=0}^{N} \bar{C}_i SFx(i)],$$

$$(41) \qquad \varphi(x_0) - \sum_{i=0}^{N} \bar{C}_i Sfx(i) = 0$$

with

$$Q = \sum_{i=0}^{N} \omega(i)C_i, \qquad \varphi(x_0) = Q^{-1}\Big[d - \sum_{i=0}^{N} C_i x_0\Big], \qquad \bar{C}_i = Q^{-1}C_i.$$

It is easy to check that for any $x_0 \in X$, a solution of equation (40) satisfies condition (39). To prove the existence and uniqueness of the solution of equation (40) we assume that for some $\lambda \in (0,1)$ there is a positive solution ϱ of the inequality

$$\lambda\varrho(n) \geqslant SK\varrho(n) + \omega(n)\sum_{i=0}^{N} |\overline{C}_i| SK\varrho(i), \tag{42}$$

where $|\overline{C}_i|$ are matrices satisfying the condition $|\overline{C}_i x| \leqslant |\overline{C}_i||x|$ for all $x \in X$. Then the reasoning remains quite similar to that of Section 3. To get a solution of inequality (42) we consider the equation

$$\varrho(n) = SK_1\varrho(n) + \omega(n)\sum_{i=0}^{N} |\overline{C}_i| SK_1\varrho(i) + e. \tag{43}$$

Multiplying this equation by $K_1(n)$ and substituting $v(n) = SK_1\varrho(n)$ we find

$$\Delta v(n) = K_1(n)v(n) + \omega(n)K_1(n)\sum_{i=0}^{N} |\overline{C}_i| v(i) + K_1(n)e, \tag{44}$$

$$v(n) = \sum_{i=0}^{n-1}\prod_{s=i+1}^{n-1} (I + K_1(s))\omega(i)K_1(i) \cdot \sum_{j=0}^{N} |\overline{C}_j| v(j) + H(n) \tag{45}$$

for some positive H. From (45) it follows that $v(n)$ is positive if the condition

$$r\Big(\sum_{n=0}^{N} |\overline{C}_n| \Big[\sum_{i=0}^{n-1}\prod_{s=i+1}^{n-1} (I + K_1(s))\omega(i)K_1(i)\Big]\Big) < 1 \tag{46}$$

is satisfied. Here as before K_1 can be replaced by K. From this considerations we can conclude the following

Theorem 3. *Assume that conditions* (12) *and* (46) *are satisfied and the matrix* Q^{-1} *exists. Then for any* $x_0 \in X$ *there is a unique solution of equation* (40), *and it can be obtained by the successive approximations method.*

Observe that in a similar way one can discuss the quasilinear boundary value problem provided condition (39) is replaced by

$$\sum_{i=0}^{N} C_i x(i) = d(x) \tag{47}$$

with a given function $d\colon \mathscr{F}_N \to X$. Now the problem (38), (47) is equivalent to the system

$$x(n) = x_0 + Sfx(n) + \omega(n)Q^{-1}\Big[d\big((x_0 + Sfx(\cdot))\big) - \sum_{i=0}^{n} C_i\big(x_0 + Sfx(i)\big)\Big], \tag{48}$$

$$d\Big(\big(x_0 + Sfx(\cdot)\big)\Big) - \sum_{i=0}^{N} C_i\big(x_0 + Sfx(i)\big) = 0. \tag{49}$$

However, we will not discuss details concerning this system.

Let us make another observation: if one has problem (37)–(38) with equation (37) replaced by an equation of the form

$$x(n+1) = A(n)x(n) + fx(n) \tag{50}$$

then one can reduce the problem to the original one by substituting $x(n) = Y(n)y(n)$ with a new unknown y. Here $A\colon J_{N-1} \to L(X)$ and Y is the fundamental solution corresponding to the linear part of (50). In this case one has to assume that the operators $A^{-1}(n)$ exist for $n \in J_{N-1}$.

References

[1] *R. P. Agarval*: On multipoint boundary value problems for discrete equations, J. Math. Anal. Appl. *96* (1983), 520–534.

[2] *R. P. Agarval*: Initial-value methods for discrete boundary value problems, J. Math. Anal. Appl. *100* (1984), 513–529.

[3] *R. P. Agarval, R. C. Gupta*: A new shooting method for multi-point boundary value problems, J. Math. Anal. Appl. *112* (1985), 210–220.

[4] *A. Augustynowicz, M. Kwapisz*: On a numerical-analytic method of solving of boundary value problem for functional-differential equation of neutral type, Mathem. Nachrichten *145* (1990), 255–269.

[5] *P. W. Eloe*: A boundary value problem for a system of difference equations, Nonlinear Anal. 7 (1983), 813–820.

[6] *J. Henderson*: Existence theorems for boundary value problems for nth order nonlinear difference equations, SIAM J. Math. Anal. *20* (1989), 468–478.

[7] *M. Kwapisz*: Some existence and uniqueness results for boundary value problems for difference equations, Applicable Analysis *37* (1990), 169–182.

[8] *M. Kwapisz*: On boundary value problems for difference equations, J. Math. Anal. Appl. *157* (1991), 254–270.

[9] *M. Kwapisz*: Some remarks on integral equations arising in applications of numerical–analytic method of solving of boundary value problems, Ukrain. Mathem. Journal *(44)1* (1992), 128–132.

[10] *M. Kwapisz*: On modifications of the integral equation of Samoilenko's numerical-analytic method of solving boundary value problems, Math. Nachr. *157* (1992), 125–135.

[11] *M. Kwapisz*: Elements of the Theory of Recurrent Equations, The Gdańsk University Publications, Gdańsk, 1983. (In Polish.)

[12] *V. Lakshmikantham, D. Trigiante*: Theory of Difference Equations, Numerical Methods and Applications, Academic Press, New York, 1988.
[13] *R. Musielak, J. Popenda*: On periodic solutions of a first order difference equation, Anale Stiintifice Ale Universitatii "Al. I. Cuza" Din Iasi, 34, s. I a, Matematica *2* (1988), 125–133.
[14] *A Peterson*: Boundary value problems for an nth order linear difference equation, SIAM J. Math. Anal. *15* (1984), 124–132.
[15] *A. Peterson*: Existence and uniqueness theorems for nonlinear difference equations, J. Math. Anal. Appl. *125* (1987), 185–191.
[16] *Y. Rodriguez*: On nonlinear discrete boundary value problems, J. Math. Anal. Appl. *114* (1986), 398–408.
[17] *A. M. Samoilenko*: Numerical-analytic method of investigation of systems of ordinary differential equations, I Ukrain. Math. Journal *17* (1965), 82–93. (In Russian.)
[18] *A. M. Samoilenko*: Numerical-analytic method of investigation of systems of ordinary differential equations, II Ukrain. Math. Journal *18* (1966), 50–59. (In Russian.)
[19] *A. M. Samoilenko, N. I. Ronto*: A modification of numerical-analytic method of successive approximations for boundary value problems for ordinary differential equations, Ukrain. Math. Journal *42* (1990), 1107–1116. (In Russian.)
[20] *A. M. Samoilenko, N. I. Ronto*: Numerical-analytic Method of Investigation of Periodic Solutions, Vyssha Shkola Publications, Kiev, 1976. (In Russian.)
[21] *A. M. Samoilenko, N. I. Ronto*: Numerical-analytic Method of Investigation of Solutions of Boundary value problems, Naukova Dumka Publications, Kiev, 1986. (In Russian.)

Author's address: Marian Kwapisz, Instytut Matematyki, Universytet Gdańsk, Wita Stwosza 57, 80 952 Gdańsk, Poland.