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ESTIMATION OF VARIANCE COMPONENTS IN MIXED LINEAR MODELS

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Summary. The MINQUE of the linear function $f'\vartheta$ of the unknown variance-components parameter ϑ in mixed linear model under linear restrictions of the type $R\vartheta = c$ is defined and derived. As an illustration of this estimator the example of the one-way classification model with the restrictions $\vartheta_1 = k\vartheta_2$, where $k \geqslant 0$, is given.

Keywords: MINQUE, MINQUE with linear restrictions, mixed linear model, one-way classification model with restrictions

AMS classification: 62F10, 62J05

1. QUADRATIC ESTIMATORS OF LINEAR FUNCTION OF ϑ

The mixed linear model is given as follows:

$$Y = X\beta + \varepsilon \tag{1}$$

where the mean of the vector ε is given by $E(\varepsilon) = 0$ and the covariance matrix is $V(\vartheta) = \sum\limits_{i=1}^{p} \vartheta_i V_i$. The matrices X and V_i, $i = 1, \ldots, p$ are known of dimensions $n \times k$ and symmetric $n \times n$, respectively, and the vector parameters β and $\vartheta = (\vartheta_1, \ldots, \vartheta_p)'$ are unknown, $\beta \in R^k$, $\vartheta \in \Theta \subset R^p$, Θ contains an open set. The estimation problem of $f'\vartheta$ has been considered by many authores. The entire theory is found e.g. in Rao, Kleffe (1988), where the locally optimal unbiased and invariant quadratic estimators of $f'\vartheta$ are given and the MINQUE theory based by Rao in 1971 is described.

The usual way of estimating the variance components is to consider a quadratic or linear-quadratic estimator with the additional property of invariance, unbiasedness, or invariance and unbiasedness. The class of estimators considered is $\mathcal{Q} = \{Y'AY : A = A'\}$. The class of quadratic invariant estimators is given as $\mathcal{Q}_I = \{Y'AY = (Y - X\alpha)'A(Y - X\alpha) \text{ for all } \alpha \in R^k\} = \{Y'AY : A = A', AX = 0\}$. The class of quadratic unbiased invariant estimators of the function $f'\vartheta$ is denoted by $\mathcal{Q}_{IU} = \{Y'AY\colon A = A', AX = 0, \operatorname{tr} AV_i = f_i, i = 1, \dots, p\}$. Denote the prior value of the chosen vector parameter by ϑ^0 and the corresponding matrix by $V(\vartheta^0) = V_0$. For convenience let us suppose that V_0 is p.d.. In 1970 and 1971 Rao investigated the MINQUE as the invariant unbiased estimator $Y'AY \in \mathcal{Q}_{IU}$ of $f'\vartheta$ which minimizes the expression $\operatorname{tr} AV(\vartheta^0)AV(\vartheta^0)$ for prior ϑ^0 chosen from the parametric space. It is clear that if the vector Y is normally distributed then the variance of the quadratic form $Y'AY \in \mathcal{Q}_{IU}$ is $2 \operatorname{tr} AV(\vartheta)AV(\vartheta)$. Consequently, the minimization of this variance locally at ϑ^0 under normality of the vector Y is the same as the minimization of the given norm $\|A\|^2_{V(\vartheta^0)} = \operatorname{tr} AV(\vartheta^0)AV(\vartheta^0)$. The MINQUE of $f'\vartheta$ is then given as $f'\vartheta = \sum \lambda_i q_i$, where λ is the solution of the consistent matrix equation $Q'Q\lambda = f$ for $Q'Q$ with elements

$$\{Q'Q\}_{i,j} = \operatorname{tr}(MV_0M)^+V_i(MV_0M)^+V_j, \tag{2}$$

and the vector $q = (q_1, \dots, q_p)'$ equals

$$q = [Y'(MV_0M)^+V_1(MV_0M)^+Y, \dots, Y'(MV_0M)^+V_p(MV_0M)^+Y]', \tag{3}$$

where the matrix $M = I - XX^+$.

2. Linear model in the vector parameter ϑ

Following the same technique as used in Seely (1970) and Verdooren (1979) we get: The vector $V_0^{-1/2}Y$ has the identity covariance matrix at the value ϑ^0. Then the original model transformed by the matrix $V_0^{-1/2}$ is $V_0^{-1/2}Y = V_0^{-1/2}X\beta + V_0^{-1/2}\varepsilon$. The maximal invariant in the model is then the vector $Z = M_0V_0^{-1/2}Y$, where the matrix M_0 of the form $M_0 = I - V_0^{-1/2}X(X'V_0^{-1}X)^-X'V_0^{-1/2}$ is the projection matrix onto the orthogonal complement of the column space of the matrix $V_0^{-1/2}X$.

Consider the vector $\operatorname{vec} M_0V_0^{-1/2}YY'V_0^{-1/2}M_0$. Its expectation is

$$E_\vartheta(\operatorname{vec} M_0V_0^{-1/2}YY'V_0^{-1/2}M_0) = (\operatorname{vec} M_0V_0^{-1/2}V_1V_0^{-1/2}M_0, \dots, \operatorname{vec} M_0V_0^{-1/2}V_pV_0^{-1/2}M_0)\vartheta.$$

For simplification we use the notation

$$E_\vartheta(\operatorname{vec} ZZ') = Q\vartheta. \tag{4}$$

Let us consider the linear function of the vector $\operatorname{vec} ZZ'$ of the form $L' \operatorname{vec} ZZ'$ where L is the n^2 dimensional vector. It can be shown that $L' \operatorname{vec} ZZ' \in \mathcal{Q}_I$ for all $L \in R^{n^2}$ (in further considerations the invariance of the estimator is used in this sense), and moreover if $f'\vartheta$ is the parametric function then $L' \operatorname{vec} ZZ' \in \mathcal{Q}_{IU}$ for $f'\vartheta$ if and only if $Q'L = f$.

The covariance matrix of the vector $\operatorname{vec} ZZ'$ is in general dependent on the 4th moments of the vector Y and is denoted by $\Sigma(\vartheta)$. The ordinary least squares estimator of the linear function $f'\vartheta$ with $f \in \mathcal{R}(Q'Q)$ based on the vector $\operatorname{vec} ZZ'$ is the MINQUE of $f'\vartheta$, since $\widehat{f'\vartheta} = f'(Q'Q)^{-}Q' \operatorname{vec} ZZ' = \sum \lambda_i q_i$, where the vector q is given by (3) and the vector λ is the solution of the system $Q'Q\lambda = f$.

Let us have another look at the MINQUE. Working within this linear model we know from the linear theory that the BLUE (best linear unbiased estimator, here the term linear means a linear function of the vector $\operatorname{vec} ZZ'$) depends only on the variance matrix $\Sigma(\vartheta)$ of the vector $\operatorname{vec} ZZ'$. We shall show that the ordinary least squares estimator of the linear function $f'\vartheta$ with $f \in \mathcal{R}(Q'Q)$ is the BLUE if the vector $\operatorname{vec} ZZ'$ is transformed from a normally distributed vector Y with a variance matrix V_0. It means that the variance of such an estimator is minimal. Since this estimator belongs to the class $\mathcal{Q}_{IU}$ (as we have shown) and has the minimal variance locally at ϑ^0 for a normally distributed vector Y, it fulfils all conditions required by the MINQUE. Below we will give the definition of the MINQUE based on the properties mentioned.

Lemma 1. *Let us consider the model $E_\vartheta(\operatorname{vec} ZZ') = Q\vartheta$, let a vector Y be normally distributed and $\operatorname{var} Y = V_0$. Then the ordinary least squares estimator of a linear function $f'\vartheta$ with $f \in \mathcal{R}(Q'Q)$ is the* BLUE.

Proof. The general form of the variance matrix $\Sigma(\vartheta) = \operatorname{var}_\vartheta(\operatorname{vec} ZZ')$ is known. Let Φ and Ψ denote the matrices of the 3rd and 4th moments of the vector Y, respectively. Then the variance matrix at ϑ^0 is

$$\begin{aligned}\Sigma(\vartheta^0) &= \operatorname{var}_{\vartheta^0}(\operatorname{vec} ZZ') = \operatorname{var}_{\vartheta^0}(\operatorname{vec} M_0 V_0^{-1/2} YY' V_0^{-1/2}) \\ &= \operatorname{var}_{\vartheta^0}\left[(M_0 V_0^{-1/2} \otimes M_0 V_0^{-1/2}) \operatorname{vec} YY'\right] \\ &= (M_0 V_0^{-1/2} \otimes M_0 V_0^{-1/2}) \operatorname{var}_{\vartheta^0}(\operatorname{vec} YY')(M_0 V_0^{-1/2} \otimes M_0 V_0^{-1/2})'.\end{aligned}$$

Let F_{nn} be the unique matrix such that $F_{nn} \operatorname{vec} A = \operatorname{vec} A'$ for all $n \times n$ matrices A, see Rao, Kleffe (1988). Note that for symmetric matrices A we have $(I + F_{nn}) \operatorname{vec} A = 2 \operatorname{vec} A$. Hence the covariance matrix of the vector $\operatorname{vec} YY' = Y \otimes Y$ at ϑ^0 is

$$\begin{aligned}\operatorname{var}_{\vartheta^0}(\operatorname{vec} YY') = \Psi - \operatorname{vec} V_0(\operatorname{vec} V_0)' + (X\beta \otimes \Phi')(I + F_{nn}) + (I + F_{nn})(\beta' X' \otimes \Phi) \\ + (I + F_{nn})(X\beta\beta' X' \otimes V_0)(I + F_{nn}).\end{aligned}$$

In particular, for a normal vector Y we have

$$\operatorname{var}_{\vartheta^0}(\operatorname{vec} YY') = (I + F_{nn})(V_0 \otimes V_0) + (I + F_{nn})(X\beta\beta' X' \otimes V_0)(I + F_{nn}).$$

From the theory of linear models, see Zyskind (1967), it is known that the ordinary least squares estimator of $f'\vartheta$ coincides with the BLUE at ϑ^0 iff $\mathcal{R}(\Sigma(\vartheta^0)Q) \subseteq \mathcal{R}(Q)$. We have

$$\begin{aligned}\Sigma(\vartheta^0)Q &= (M_0 V_0^{-1/2} \otimes M_0 V_0^{-1/2})\, \mathrm{var}_{\vartheta^0}(\mathrm{vec}\, YY')(M_0 V_0^{-1/2} \otimes M_0 V_0^{-1/2})' Q \\ &= (M_0 V_0^{-1/2} \otimes M_0 V_0^{-1/2}) \\ &\quad \times [(I + F_{nn})(V_0 \otimes V_0) + (I + F_{nn})(X\beta\beta' X' \otimes V_0)(I + F_{nn})] \\ &\quad \times (M_0 V_0^{-1/2} \otimes M_0 V_0^{-1/2})' \\ &\quad \times (\mathrm{vec}\, M_0 V_0^{-1/2} V_1 V_0^{-1/2} M_0, \dots, \mathrm{vec}\, M_0 V_0^{-1/2} V_p V_0^{-1/2} M_0).\end{aligned}$$

The second term in the sum vanishes because of the orthogonality of the matrices X and $V_0^{-1/2} M_0$. Further,

$$\begin{aligned}\Sigma(\vartheta^0)Q &= (M_0 V_0^{-1/2} \otimes M_0 V_0^{-1/2})(I + F_{nn}) \\ &\quad \times (\mathrm{vec}\, V_0^{1/2} M_0 V_0^{-1/2} V_1 V_0^{-1/2} M_0 V_0^{1/2}, \dots, \mathrm{vec}\, V_0^{1/2} M_0 V_0^{-1/2} V_p V_0^{-1/2} M_0 V_0^{1/2}) \\ &= 2(M_0 V_0^{-1/2} \otimes M_0 V_0^{-1/2}) \\ &\quad \times (\mathrm{vec}\, V_0^{1/2} M_0 V_0^{-1/2} V_1 V_0^{-1/2} M_0 V_0^{1/2}, \dots, \mathrm{vec}\, V_0^{1/2} M_0 V_0^{-1/2} V_p V_0^{-1/2} M_0 V_0^{1/2}) \\ &= 2(\mathrm{vec}\, M_0 V_0^{-1/2} V_1 V_0^{-1/2} M_0, \dots, \mathrm{vec}\, M_0 V_0^{-1/2} V_p V_0^{-1/2} M_0) \\ &= 2Q.\end{aligned}$$

Hence $\mathcal{R}(\Sigma(\vartheta^0)Q) = \mathcal{R}(Q)$ and Zyskind's condition is fulfilled which means that the ordinary least squares estimator coincides with the BLUE at ϑ^0. □

Definition 1. Let us consider the model $E_\vartheta(\mathrm{vec}\, ZZ') = Q\vartheta$. The MINQUE of $f'\vartheta$ is the ordinary least squares estimator of the estimable function $f'\vartheta$.

Remark 1. We note that in case of a singular V_0 we can use the transformation $V_0^{+1/2} Y$ where $V_0^{+1/2}$ is the matrix for which $(V_0^{+1/2})' V_0^{+1/2} = V_0^+$ with V_0^+ the Moore-Penrose inverse of the matrix V_0. Further, we can express the projection matrix onto the orthogonal complement of the column space of the matrix $V_0^{+1/2} X$ as

$$M_0 = V_0^{+1/2} V_0 (V_0^{+1/2})' - V_0^{+1/2} X (X' V_0^+ X)^- X' (V_0^{+1/2})'.$$

All results which are valid for a regular matrix V_0 are also valid if V_0 is singular.

In this section we start to investigate the estimation of a linear function $f'\vartheta$ in the model with linear restrictions on the parameter ϑ of the form $R\vartheta = c$, where the matrix R and the vector c are given.

Lemma 2. *The linear function $f'\vartheta$ is unbiasedly invariantly estimable in the linearized model* $(\operatorname{vec} ZZ', Q\vartheta, \Sigma(\vartheta))$ *under restrictions $R\vartheta = c$ iff $f \in \mathcal{R}(Q', R')$.*

For the proof see Rao, Mitra (1971), Chap. 7.

Lemma 3. *Let the vector f satisfy $f \in \mathcal{R}(Q', R')$. Let the vector Y be normally distributed. The locally best linear unbiased invariant estimator of $f'\vartheta$ at ϑ^0 in the model* $(\operatorname{vec} ZZ', Q\vartheta, \Sigma(\vartheta))$ *under $R\vartheta = c$ coincides with the locally best linear unbiased invariant estimator at ϑ^0 in model* $(\operatorname{vec} ZZ', Q\vartheta, I)$ *under $R\vartheta = c$ (which means the ordinary least squares estimator).*

Proof. The condition $f \in \mathcal{R}(Q', R')$ implies the existence of vectors L, M for which $f = Q'L + R'M$. The vector L satisfies $L \in R^{n^2}$ and therefore there exist vectors $u, k \in \ker Q'$ such that $L = Qu + k$. The unbiased invariant estimator of $f'\vartheta$ is of the form $\widehat{f'\vartheta} = L' \operatorname{vec} ZZ' + M'c$. Considering the model $(\operatorname{vec} ZZ', Q\vartheta, \Sigma(\vartheta))$ the variance of $\widehat{f'\vartheta}$ is $L'\Sigma(\vartheta)L$. From $\Sigma(\vartheta^0)Q = 2Q$ it follows that $\operatorname{var}_{\vartheta^0} \widehat{f'\vartheta} = L'\Sigma(\vartheta^0)L = 2u'Q'Qu + k'\Sigma(\vartheta^0)k \geqslant 2u'Q'Qu$, which means that the vector L is to be taken from $\mathcal{R}(Q)$. In that case $2L'L$ coincides with $2\operatorname{var}_{\vartheta^0} \widehat{f'\vartheta}$ in the model $(\operatorname{vec} ZZ', Q\vartheta, I)$. To find an optimal estimator means to find the vector (L', M') for which $L'\Sigma(\vartheta^0)L = 2L'L$ is minimal under the restriction $Q'L + R'M = f$, which completes the proof of the lemma. □

The next definition of the MINQUE with linear restrictions is a natural extension of the one without restrictions, given in the previous section.

Definition 2. Let us consider the model $E_\vartheta(\operatorname{vec} ZZ') = Q\vartheta$ with linear restrictions $R\vartheta = c$. The ordinary least squares estimator of an estimable function $f'\vartheta$ in this model is defined as the MINQUE with linear restrictions and in general is of the form $Y'AY + d$, d being a constant.

Remark 2. It is possible to define the MINQUE of an estimable function $f'\vartheta$ in the model (1) under given restrictions as the ϑ_0-locally best unbiased invariant estimator under the normality assumption. Considering the reparametrization $\vartheta = R^- c + U\eta$ with $RU = 0$ it can be shown that these two approaches coincide.

Remark 3. In Definition 2 we can see some inconsistency with the conditions which are required by MINQUE. In particular, the estimator of the form $Y'AY + d$ is not an element of the class $\mathcal{Q}_{IU}$. However, we note that this problem is connected with the constant vector c. If $c = 0$, the linear restrictions $R\vartheta = c$ form a linear subspace in the parametric space, and the MINQUE with restrictions belongs then to the class $\mathcal{Q}_{IU}$.

Theorem 1. *The linear function $f'\vartheta$ is* MINQUE*-estimable under the model (1) with linear restrictions $R\vartheta = c$ if and only if $f \in \mathcal{R}(Q'Q + R'R)$. The* MINQUE *of a* MINQUE*-estimable function $f'\vartheta$ is then $f'\hat{\vartheta}$, where $\hat{\vartheta}$ is the solution of the equation*

$$
\begin{aligned}
Q'Q\vartheta + R'\nu &= q, \\
R\vartheta &= c.
\end{aligned} \tag{5}
$$

The matrix $Q'Q$ and the vector q are given by (2) and (3), ν is the nuisance vector of Lagrangian multipliers.

Proof. Lemma 2 implies $f \in \mathcal{R}(Q', R') = \mathcal{R}(Q'Q + R'R)$ as stated in the theorem. It follows from the linear theory (see Rao, Mitra (1971)) that the minimization of $L'L$ under $R\vartheta = c$ from Lemma 3 leads to $\widehat{f'\vartheta}$ and it is the same as $f'\hat{\vartheta}$ which minimizes the quadratic form $(\operatorname{vec} ZZ' - Q\vartheta)'(\operatorname{vec} ZZ' - Q\vartheta)$ under the constrains $R\vartheta = c$ by the method of Langrangian multipliers that leads to the system of matrix equations from the theorem which are called normal equations. □

Theorem 2. *One special choice of the* MINQUE *with restrictions of a* MINQUE*-estimable function $f'\vartheta$ is*

$$\widehat{f'\vartheta} = \sum_{i=1}^{p} \kappa_i q_i + \sum_{i=1}^{r} \gamma_i c_i,$$

where κ is the solution of the equation $(M_{R'} W M_{R'})^+ f = \kappa$, where $M_{R'} = I - R'(RR')^- R$ and $W = Q'Q + R'R$. The vector γ is given as $\gamma = (RW^- R')^- RW^- f$, $c = (c_1, \dots, c_r)'$. The matrix $Q'Q$ and the vector q are given by (2) and (3).

Proof. One solution of the equation (5) is given by

$$\begin{pmatrix} Q'Q & R' \\ R & 0 \end{pmatrix}^{-} \begin{pmatrix} q \\ c \end{pmatrix}.$$

We observe that the equations (5) are solvable. If we denote

$$\begin{pmatrix} Q'Q & R' \\ R & 0 \end{pmatrix}^{-} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

the solution $\hat{\vartheta}$ of the equations (5) is given by $C_1 q + C_2 c$. One possible choice of the matrices C_1, C_2, C_3 and C_4 is (see Rao, Mitra 1971)

$$
\begin{aligned}
C_1 &= \left[I - (Q'Q + R'R)^+ R' \left[R(Q'Q + R'R)^+ R' \right]^+ R \right] (Q'Q + R'R)^+, \\
C_2 &= (Q'Q + R'R)^+ R' \left[R(Q'Q + R'R)^+ R' \right]^+, \\
C_3 &= \left[R(Q'Q + R'R)^+ R' \right]^+ R(Q'Q + R'R)^+, \\
C_4 &= I - \left[R(Q'Q + R'R)^+ R' \right]^+.
\end{aligned}
$$

The matrix C_1 can be expressed as

$$\begin{aligned} C_1 &= (Q'Q+R'R)^+ - (Q'Q+R'R)^+R'\left[R(Q'Q+R'R)^+R'\right]^+ R(Q'Q+R'R)^+ \\ &= (M_{R'}WM_{R'})^+ \end{aligned}$$

for $W = Q'Q + R'R$ and $M_{R'} = I - R'(RR')^- R$. Substituting for C_1 and for C_2 in the expression for $\hat{\vartheta}$ and observing that $\widehat{f'\vartheta} = f'\hat{\vartheta}$ we complete the proof of the theorem. □

As we mentioned in Remark 2 we can calculate the MINQUE of the function $f'\vartheta$ under given restrictions $R\vartheta = c$ by the classical method of reparametrization of the model. However, for every set of restrictions we have to recalculate the vector q as well as the criterion matrix $Q'Q$. The definition of the MINQUE given here enables us to calculate estimates with different sets of linear restrictions and to use the same vector q and the same criterion matrix $Q'Q$.

Testing of linear hypotheses of the type H_0: $R\vartheta = c$ against K: $R\vartheta \neq c$ is another area where estimators which meet the condition can be employed. However, this problem is still the subject of study. It seems to be a good idea to base a testing criterion on the vector of residuals calculated with respect to the conditions given by the hypothesis. There is a lot of work needed to get reasonable results in that field.

Example. The unbalanced one-way ANOVA (analysis of variance) model can be regarded as a special case of the general mixed linear model (1). Rao, Kleffe (1988) gave the expressions for the MINQUE in this model. We give the MINQUEs with a special type of restrictions of the form $\vartheta_1 = k\vartheta_2$, which can be written as

$$R\vartheta = c, \tag{6}$$

where $R = (1, -k)$ and $c = 0$.

The unbalanced one-way model is defined as

$$y_{ij} = \beta + \xi_i + \xi_{ij}, \quad i = 1, \dots, m, \ j = 1, \dots, n_i, \tag{7}$$

where $E(\xi_i^2) = \vartheta_1$, $E(\xi_{ij}^2) = \vartheta_2$, $E(\xi_k \xi_{ij}) = 0$.

We can see that the matrix form of the model (7) is a special case of the model (1) where $Y = (y_{11}, \dots, y_{mn_m})'$, the matrix $X = 1_{mn} = (1, \dots, 1)'$ and the vector $\varepsilon = (\varepsilon_{11}, \dots, \varepsilon_{mn_m})'$ with $\varepsilon_{ij} = \xi_i + \xi_{ij}$. Here $E(\varepsilon\varepsilon') = \vartheta_1 V_1 + \vartheta_2 I$ with $V_1 = \mathrm{Diag}(11'_{n_1}, \dots, 11'_{n_m})$. The expressions needed for computing the MINQUE without restrictions with prior values α_1 for ϑ_1 and 1 for ϑ_2 are

$$\begin{aligned} q_1 &= \sum_{i=1}^m L_i^2(\bar{y}_{i.} - \hat{\beta})^2, \quad \hat{\beta} = S_1^{-1}\sum_{i=1}^m L_i \bar{y}_{i.}, \\ q_2 &= \sum_{i=1}^m \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2 + \sum_{i=1}^m n_i^{-1} L_i^2 (\bar{y}_{i.} - \hat{\beta})^2, \end{aligned}$$

where $L_i = n_i/(1+\alpha_1 n_i)$, $\bar{y}_{i\cdot} = n_i^{-1} \sum_{j=1}^{n_i} y_{ij}$ and the elements of the matrix $Q'Q$ are

$$\{Q'Q\}_{11} = S_2 - 2S_3/S_1 + (S_2/S_1)^2,$$
$$\{Q'Q\}_{12} = S_1 - S_2/S_1 - \alpha_1\{Q'Q\}_{11},$$
$$\{Q'Q\}_{22} = n - 1 - \alpha_1(S_1 - S_2/S_1 + \{Q'Q\}_{12})$$

with $n = \sum_{i=1}^{m} n_i$ and $S_k = \sum_{i=1}^{m} L_i^k$.

The MINQUE of the estimable linear function $f'\vartheta$ is $\sum_{i=1}^{2} \lambda_i q_i$, where λ is the solution of the equation $Q'Q\lambda = f$. The MINQUE with restrictions (6) on the vector ϑ of the estimable linear function $f'\vartheta$ is $\sum_{i=1}^{2} \kappa_i q_i$, where κ fulfils the equation $\kappa = (M_{R'}WM_{R'})^{+}f$, where the elements of the matrix $M_{R'}WM_{R'}$ are expressed in terms of the elements of the matrix $Q'Q$:

$$\{M_{R'}WM_{R'}\}_{11} = 1/(1+k^2)^2\left[k^4\{Q'Q\}_{11} + 2k^3\{Q'Q\}_{12} + k^2\{Q'Q\}_{22}\right],$$
$$\{M_{R'}WM_{R'}\}_{12} = 1/(1+k^2)^2\left[k^3\{Q'Q\}_{11} + 2k^2\{Q'Q\}_{12} + k\{Q'Q\}_{22}\right],$$
$$\{M_{R'}WM_{R'}\}_{22} = 1/(1+k^2)^2\left[k^2\{Q'Q\}_{11} + 2k\{Q'Q\}_{12} + \{Q'Q\}_{22}\right].$$

We note only that the second term from the expression for the MINQUE with restrictions from Theorem 2 vanishes because $c = 0$, as given by the condition (6).

The figure shows the scatterplot of 25 realizations of the MINQUE and MINQUE under restrictions of the parameters ϑ_1 vs. ϑ_2. Here $m = 3$, $n_1 = 4$, $n_2 = 6$, $n_3 = 14$ and for simulation we have used $\beta = 5.0$ and $\vartheta = (0.81, 3.24)'$. As a prior value of ϑ for computing MINQUEs and MINQUEs under the condition we have used $\vartheta^0 = (1,4)'$. The parametric space is restricted by the condition $\vartheta_1 = 0.25\vartheta_2$.

Note. We note that the model considered in the example and the restrictions given could be investigated as the reparametrized model, which would lead in that case to the model with just one unknown variance parameter. The results in both cases coincide; the example is included just for illustration.

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References

[1] *C. R. Rao*: Estimation of variance and covariance components — MINQUE theory, Journal of Multivariate Analysis *1* (1971), 267–275.

[2] *C. R. Rao and J. Kleffe*: Estimation of Variance Components and Applications volume 3 of Statistics and probability, North-Holland, Amsterdam, New York, Oxford, Tokyo, 1988, first edition.

[3] *C. R. Rao and S. K. Mitra*: Generalized Inverse of Matrices and Its Applications, John Wiley & Sons, New York, London, Sydney, Toronto, 1971, first edition.

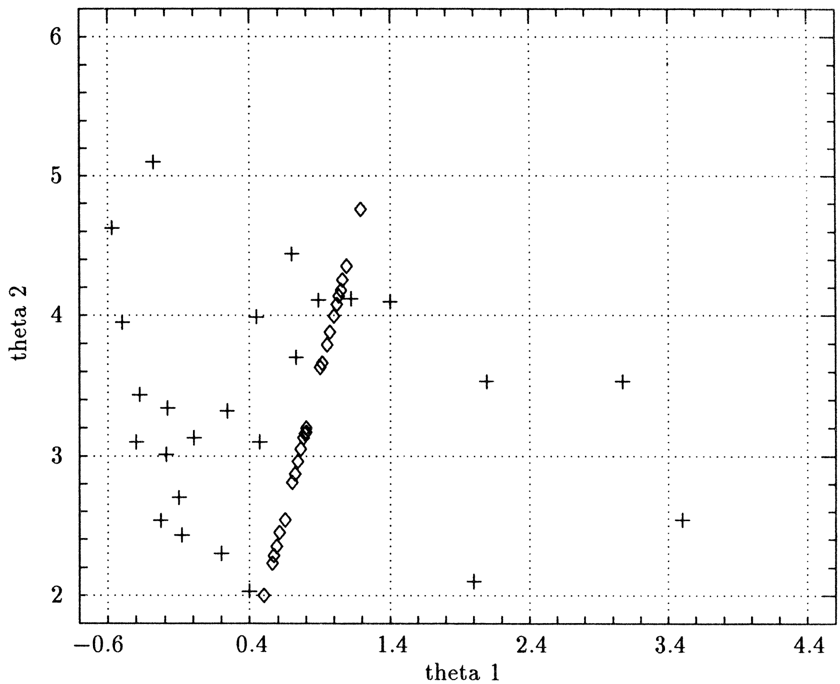

Fig. 1. Scatterplot of MINQUE: + and MINQUE with restrictions: ◇

[4] *J. Seely*: Linear spaces and unbiased estimation, Ann. Math. Stat. *41* (1970), 1725–1734.

[5] *L. R. Verdooren*: Practical aspects of variance component estimation, Invited lecture for the 4th International Summer School on Problems of Model Choice and Parameter Estimation in Regression Analysis Mülhausen, GDR, May 1979.

[6] *G. Zyskind*: On canonical forms, negative covariance matrices and best and simple least square estimator in linear models, Ann. Math. Stat. *38* (1967), 1092–1110.

Súhrn

ODHAD VARIANČNÝCH KOMPONENTOV V ZMIEŠANOM LINEÁRNOM MODELI S LINEÁRNYMI PODMIENKAMI

JÚLIA VOLAUFOVÁ A VIKTOR WITKOVSKÝ

V článku je odvodený MINQUE odhad lineárnej funkcie neznámych variančných komponentov ϑ v zmiešanom lineárnom modeli s lineárnymi podmienkami typu $R\vartheta = c$. Ako ilustračný príklad je uvedený model analýzy rozptylu s jednoduchým triedením a s podmienkami typu $\vartheta_1 = k\vartheta_2$, kde $k \geqslant 0$.

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