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GLOBAL IN TIME SOLUTIONS TO QUASILINEAR TELEGRAPH EQUATIONS INVOLVING OPERATORS WITH TIME DELAY

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Summary. The existence of small global (in time) solutions to an abstract evolution equation containing a damping term is proved. The result is then applied to fully nonlinear telegraph equations and to nonlinear equations involving operators with time delay.

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Providing important mathematical models for a variety of physical phenomena, telegraph equations represent a subject many authors have made contributions towards. The present paper addresses the case when both the coefficients and the right-hand side of the equation are nonlinear operators allowed to contain time delay.

Let us denote $\mathscr{L}u = u_{tt} + du_t - cu_{xx}$ where $c, d > 0$. With $u = u(x, t)$ a function defined for $x \in [0, l]$, $t \in \mathbb{R}^1$ we study the equation

$$\text{(E)} \qquad \mathscr{L}u + \mathscr{P}^1(u, \lambda)\, u_{tt} + \mathscr{P}^2(u, \lambda)\, u_{xt} + \mathscr{P}^3(u, \lambda)\, u_{xx} = \mathscr{F}(u, \lambda)$$

for $x \in [0, l]$, $t \in [0, +\infty)$ together with the conditions

$$\text{(B)} \qquad u(0, t) = u(l, t) = 0\,, \quad t \in \mathbb{R}^1$$

$$\text{(I)} \qquad u(x, t) = u_t(x, t) = 0\,, \quad x \in [0, l]\,, \quad t \in (-\infty, 0]\,.$$

Here the operators $\mathscr{P}^i$, $i = 1, 2, 3$, $\mathscr{F}$ are supposed to depend on a parameter λ belonging to a Banach space Λ.

To begin with, we intend to prove the existence and uniqueness of the solution u on condition that $\lambda \in \Lambda(\eta) = \{\lambda \mid \lambda \in \Lambda,\ \|\lambda\|_\Lambda \leqq \eta\}$, $\eta > 0$ lying close to zero. Taking advantage of this result, a unique function which is bounded and solves (**E**), (**B**) on the whole real axis $t \in \mathbb{R}^1$ can be found. Apparently, this fact is of great interest if the corresponding time-periodic problem is involved.

According to the choice of the operators $\mathscr{P}^i$, $\mathscr{F}$, various problems may be attacked.

If, for instance, $\mathscr{P}^i$, $\mathscr{F}$ are substitution (Nemytski) operators, the existence theorems can be obtained concerning ordinary *quasilinear telegraph equations* (cf. Matsumura [10], Kato [7]).

Following Shibata-Tsutsumi [15], *the fully nonlinear telegraph equation* can be transformed to a systêm consisting of a quasilinear hyperbolic equation and a nonlinear elliptic one. Using the Green operator related to the elliptic part we are able to cope with this problem as well. For example, the existence of a time-periodic solution may be stated avoiding the use of the hard implicit function theorem (cf. [8], [12], [14]). Moreover, as an added benefit of the method employed, the uniqueness of the solution is obtained. Recently, Štědrý [16] has achieved more general results working directly in the space of periodic functions and making use of the Schauder fixed point theorem.

Dealing with integral operators $\mathscr{P}^i$, $\mathscr{F}$, our results apply to *hyperbolic problems involving time delay* (see Aliev [1], Kamont-Turo [6], Poorkarimi-Wiener [13] etc.). Moreover, the operators of the form $\int_0^l u_x^2(x, t)\,dx$ can be treated arising in *equations of a vibrating string* (see Arosio [2], Biler [4], Medeiros [11], Feireisl [5]).

To carry out the program outlined, we proceed as follows. As to the basic notation, function spaces and auxiliary lemmas, we refer to Section 1.

The precise hypotheses concerning the operators in question as well as the main results will appear in Section 2.

Addressing related linear problems Section 3 represents the bulk of the paper. It is worth noting that our requirements concerning regularity of the coefficients appearing in the equation are slightly more general and correspond with [3].

Section 4 is devoted to the proof of the results claimed in Section 2.

Finally, we mention some applications and examples in Section 5.

1. NOTATION, FUNCTION SPACES, AUXILIARY RESULTS

The notation is standard. All function spaces appearing are supposed to be real. Throughout the text, the symbols c_i, $i = 1, 2, \ldots$ stand for strictly positive real constants, h_i, $i = 1, 2, \ldots$ denote positive, continuous, nondecreasing functions on $[0, +\infty)$.

For a (possible) vector function $v = (v_1, \ldots, v_m)$ of x, t, we denote by $D^k v$ the vector of components

$$\left\{\frac{\partial^{i+j} v_l}{\partial x^i\, \partial t^j}\,\middle|\; l = 1, \ldots, m,\ i, j \geqq 0,\ i + j \leqq k\right\}$$

and by $D_y^k v$ (where $y = x$ or $y = t$)

$$\left\{\frac{\partial^i v_l}{\partial y^i}\,\middle|\; l = 1, \ldots, m,\ 0 \leqq i \leqq k\right\}.$$

Here (and always) i, j, k, l are nonnegative integers.

Let $L_p = L_p(0, l)$, $p \in [1, +\infty)$ be the Lebesgue spaces of integrable functions with the norm $\| \ \|_p$ defined in the standard way. For $v = (v_1, \ldots, v_m)$ we set

$$|v| = \max \{\|v_l\|_2 \mid l = 1, \ldots, m\} .$$

$H^k = H^k(0, l)$ are the Sobolev spaces consisting of functions having derivatives up to the order k in L_2. Further we set $H_0^1 = \{v \mid v \in H^1, v(0) = v(l) = 0\}$.

Next, we will make use of vector functions ranging in a Banach space B. $I \subset \mathbb{R}^1$ being an interval, we consider the spaces $W_p^k(I, B)$ containing functions whose derivatives up to the order k with respect to $t \in I$ belong to $L_p(I, B)$ (for exact definitions see [17]). Let $\mathscr{C}^k(I, B)$, $\mathscr{B}^k(I, B)$ be the spaces consisting of all functions having derivatives up to the order k continuous or continuous and bounded on I, respectively (see [17] for details).

To keep the notation simple, we introduce the spaces

$$X^k = \{v \mid \text{each component of } D^k v \text{ belongs to } \mathscr{B}(\mathbb{R}^1, L_2)\} ,$$

$$X_0^k = \left\{v \mid v \in X^k, \ \frac{\partial^i v}{\partial t^i} \in \mathscr{B}(\mathbb{R}^1, H_0^1), \ i = 0, \ldots, k - 1\right\} ,$$

$$X^k(\varepsilon) = \{v \mid v \in X^k, \ |D^k v(t)| \leqq \varepsilon \text{ for all } t \in \mathbb{R}^1\}$$

and

$$Y^k = \{v \mid \text{each component of } D^k v \text{ belongs to } L_\infty(\mathbb{R}^1, L_2)\} ,$$

$$Y_0^k = \left\{v \mid v \in Y^k, \ \frac{\partial^i v}{\partial t^i} \in L_\infty(\mathbb{R}^1, H_0^1), \ i = 0, \ldots, k - 1\right\} ,$$

$$Y^k(\varepsilon) = \{v \mid v \in Y^k, \ |D^k v(t)| \leqq \varepsilon \text{ for a.e. } t \in \mathbb{R}^1\} .$$

We conclude with a short review of auxiliary results. Seeing that the spaces H^1, H^2 are Banach algebras and due to the embedding relation $H^1 \hookrightarrow \mathscr{C}[0, l]$ (see [17]), we arrive at the following assertion.

Lemma 1. *Let $k = 1$ or $k = 2$, $v, w \in Y^k$.*

Then $vw \in Y^k$ and we have the estimate

$$|D^k vw(t)| \leqq c_1 |D^k v(t)| \, |D^k w(t)| \tag{1.1}$$

for a.e. $t \in \mathbb{R}^1$.

Combining Lemma 1 with the Taylor expansion formula one obtains.

Lemma 2. *Consider a function $\Phi: U \subset \mathbb{R}^m \to \mathbb{R}^1$ where U is an open ball centered at $0 \in \mathbb{R}^m$, $\Phi \in \mathscr{C}^4(\bar{U}, \mathbb{R}^1)$. Let functions $v = (v_1, \ldots, v_m)$, $w = (w_1, \ldots, w_m)$ range in U, $v_i, w_i \in Y^k$, $i = 1, \ldots, m$ where $k = 1$ or $k = 2$.*

Then $\Phi \circ v$, $\Phi \circ w \in Y^k$ and

$$|D^k(\Phi \circ v - \Phi \circ w)(t)| \leqq h_1(z) \, |D^k(v - w)(t)| \tag{1.2}$$

holds for a.e. $t \in \mathbb{R}^1$. *If, moreover,* $\Phi'(0) = 0$, *then*

$$(1.3) \qquad |D^k(\Phi \circ v - \Phi \circ w)(t)| \leqq h_2(z)\, z |D^k(v - w)(t)|$$

for a.e. $t \in \mathbb{R}^1$. *Here the symbol* z *stands for*

$$z = z(t) = \max\{|D^k v(t)|, |D^k w(t)|\}\,.$$

2. SUMMARY OF RESULTS

To begin with, we specify the conditions imposed upon the operators $\mathscr{P}^i$, $i = 1, 2, 3$, $\mathscr{F}$. Setting $\mathscr{G} = \mathscr{P}^i$ or $\mathscr{G} = \mathscr{F}$ we assume

$$(\mathbf{PF}_1) \qquad \mathscr{G} = \mathscr{G}(v, \lambda): X^{k+1}(\varepsilon) \times \Lambda(\eta) \to X^k$$

where $k = 1, 2,\ \varepsilon, \eta > 0$;

$$(\mathbf{PF}_2) \qquad \mathscr{G}(0, \lambda) \to \mathscr{G}(0, 0) = 0 \quad \text{in} \quad X^2$$

whenever $\|\lambda\|_\Lambda \to 0$.

The following (Lipschitz) conditions resemble those appearing in Lemma 2 related to substitution operators. Let $\lambda \in \Lambda(\eta)$, $v, w \in X^{k+1}(\varepsilon)$ for $k = 1, 2$. We require

$$(\mathbf{P}_3) \qquad |D^k(\mathscr{P}^i(v, \lambda) - \mathscr{P}^i(w, \lambda))(t)| \leqq$$
$$\leqq c_2 \sup\{|D^{k+1}(v - w)(s)| \mid s \in (-\infty, t]\}$$

and

$$(\mathbf{F}_3) \qquad |D^k(\mathscr{F}(v, \lambda) - \mathscr{F}(w, \lambda))(t)| \leqq$$
$$\leqq \varrho(\eta, \varepsilon) \sup\{|D^{k+1}(v - w)(s)| \mid s \in (-\infty, t]\}$$

for all $t \in \mathbb{R}^1$, $i = 1, 2, 3$. Here ϱ is a function such that $\varrho(\eta, \varepsilon) \to \mathscr{B}$ whenever $\eta, \varepsilon \to 0$.

The main results of the present paper can be summarized as follows.

Theorem 1. *Let* $\mathscr{P}^i$, $i = 1, 2, 3$, $\mathscr{F}$ *satisfy* $(\mathbf{PF}_1)$, $(\mathbf{PF}_2)$, $(\mathbf{P}_3)$, $(\mathbf{F}_3)$. *Let the compatibility condition*

$$(\mathbf{C}) \qquad \mathscr{F}(0, \lambda) \in H_0^1$$

hold for all $\lambda \in \Lambda(\eta)$.

The number $\eta > 0$ *being chosen small enough, there exists a unique solution* $u \in X_0^3(\varepsilon)$ *of the problem* (**E**), (**B**), (**I**) *for every fixed* $\lambda \in \Lambda(\eta)$.

Theorem 2. *Let* $\mathscr{P}^i$, $i = 1, 2, 3$, $\mathscr{F}$ *satisfy* $(\mathbf{PF}_1)$, $(\mathbf{PF}_2)$, $(\mathbf{P}_3)$, $(\mathbf{F}_3)$.

Then there is $\eta > 0$ *such that for every* $\lambda \in \Lambda(\eta)$ *there exists a function* $u \in X_0^3(\varepsilon)$ *satisfying* (**E**), (**B**) *for all* $x \in [0, l]$, $t \in \mathbb{R}^1$. *Moreover, the function* u *is the only (global) solution lying in* $X^3(\varepsilon)$.

Corollary (*time-periodic solutions*). *Assume that for* $\mathscr{G} = \mathscr{P}^i$, $\mathscr{G} = \mathscr{F}$

$$\mathscr{G}(v(t + T), \lambda) = \mathscr{G}(v, \lambda)(t + T), \quad t \in \mathbb{R}^1$$

where T *is a fixed positive number. Then the solution* u, *the existence of which is claimed in Theorem* 2, *is* T-*periodic with respect to the variable* t.

Remark. Seeing that $X^3 \subsetneq \mathscr{B}^2([0, l] \times \mathbb{R}^1)$ and $\mathscr{G}(u, \lambda) \in \mathscr{B}([0. l] \times \mathbb{R}^1)$, all solutions mentioned above are, in fact, classical.

3. THE LINEAR PROBLEM

Now we focus our attention on the linear problem associated to (E), (**B**), (**I**). We look for a function v,

$$\text{(L)} \qquad \mathscr{L}v + a^1 v_{tt} + a^2 v_{xt} + a^3 v_{xx} = f$$

for $x \in [0, l]$, $t \in [0, +\infty)$,

$$(\mathbf{B_L}) \qquad v(0, t) = v(l, t) = 0, \quad t \in [0, +\infty)$$

$$(\mathbf{I_L}) \qquad v(x, 0) = v^0(x), \quad v_t(x, 0) = v^1(x), \quad x \in [0, l].$$

Our goal is to prove the following theorem.

Theorem 3. *Let us assume*

$$(3.1) \qquad a^i \in Y^2(\alpha), \quad i = 1, 2, 3$$

$$(3.2) \qquad f \in Y^k \quad \text{for} \quad k = 1 \quad \text{or} \quad k = 2$$

$$(3.3) \qquad v^0 \in H^{k+1} \cap H_0^1, \quad v^1 \in H^k \cap H_0^1.$$

In case $k = 2$, *the compatibility condition is added*:

$$(\mathbf{C_L}) \qquad v^2 = v_{tt}(0) \in H_0^1$$

the function v^2 *being determined with the help of* v^1, v^2, (**L**).

If the number $\alpha > 0$ *is sufficiently small, then there exists a unique function* $v \in X_0^{k+1}$ *satisfying* (**L**), ($\mathbf{B_L}$), ($\mathbf{I_L}$). *Moreover, we have the decay estimate*

$$(3.4) \qquad |D^{k+1} v(t)|^2 \leqq \exp(-\delta t)\, c_3 |D^{k+1} v(0)|^2 + \\ + c_4 \sup\{|D^k f(s)|^2 \mid s \in (-\infty, t]\}$$

for a certain $\delta > 0$ *and* $t \in [0, +\infty)$.

Since the methods of the proof follow the line of standard arguments (cf. [3] and the literature listed here), we point out the principal ideas only.

STEP 1 (strong solutions). We start with a slightly more general equation

$$(\mathbf{L}') \qquad \mathscr{L}v + a^1 v_{tt} + a^2 v_{xt} + a^3 v_{xx} + b^1 v_t + b^2 v_x + b^3 v = f$$

together with $(\mathbf{B_L})$, $(\mathbf{I_L})$. As to the coefficients b^i, we assume

$$(3.5) \qquad b^i \in Y^1(\alpha)\,, \quad i = 1, 2, 3\,.$$

Uniqueness: Arguing similarly as in [3], [9], the uniqueness of the solution v may be proved in the class

$$v \in L_{1,\mathrm{loc}}(0, +\infty; H_0^1) \cap W^1_{1,\mathrm{loc}}(0, +\infty; L_2)$$

on condition that $f \in L_{1,\mathrm{loc}}(0, +\infty; L_2)$, $v^0 \in H_0^1$, $v^1 \in L_2$. The equation is satisfied, of course, in a weak sense.

Existence:

Lemma 3. *Let* (3.1)–(3.3) *hold for* $k = 1$. *Then there is a unique solution* v *to* $(\mathbf{L}')$, $(\mathbf{B_L})$, $(\mathbf{I_L})$ *satisfying* (3.4) *for* $k = 1$ *whenever* $\alpha > 0$ *is chosen small enough. As to regularity, the function* v *may be prolonged for all* $t \in \mathbb{R}^1$ *in such a way that* $v \in X_0^2$.

Proof.

(a) Taking a „basis" $\{\sin(n\pi l^{-1}x)\}_{n=1}^{\infty}$, the standard Faedo-Galerkin method yields the existence of a weak solution of our problem. Since the coefficients as well as the right-hand side of the equation are defined on the whole real axis, the unique solution v may be assumed to exist on the interval $t \in (-\gamma, +\infty)$, more precisely

$$v \in \mathscr{C}(-\gamma, +\infty; H_0^1) \cap \mathscr{C}^1(-\gamma, +\infty; L_2)$$

for a certain $\gamma > 0$ (see [3] for more general results).

(b) Keeping (3.2), (3.3) in mind, we are allowed to differentiate the equation $(\mathbf{L}')$ (in fact its Galerkin approximation) with respect to t. Repeating the arguments from (a), the regularity of the derivative v_t is obtained:

$$v_t \in \mathscr{C}(-\gamma, +\infty; H_0^1) \cap \mathscr{C}^1(-\gamma, +\infty; L_2)\,.$$

Note that, due to the choice of the "basis", the approximate problems admit the use of the operator $\partial^2/\partial x^2$. Consequently, the terms containing v_{xx}, v_x can be estimated with the help of the equation $(\mathbf{L}')$.

Finally, using $(\mathbf{L}')$ again, we infer $v_{xx} \in \mathscr{C}(-\gamma, +\infty; L_2)$. Clearly, the function v can be prolonged on $\mathbb{R}^1$ to get $v \in X_0^2$.

(c) The relation (3.4) represents a standard energy decay inherent to this kind of equations. It can be easily deduced by means of multiplying the equation by $v_t + \delta v$ or $v_{tt} + \delta v_t$. Since we work with the Feado-Galerkin approximations, this step is fully justified. Note that we need the number $\alpha > 0$ to be small.

Q.E.D.

STEP 2 (regularity). To complete the proof of Theorem 3, we have but to show the higher regularity of the solution v corresponding to (3.2), (3.3), $(\mathbf{C_L})$ for $k = 2$. At this stage, we are going to treat the original problem $(\mathbf{L})$, $(\mathbf{B_L})$, $(\mathbf{I_L})$.

The main idea is to differentiate $(\mathbf{L})$ with respect to t and apply Lemma 3 to the function $w = v_t$. Using $(\mathbf{L})$ we can express

$$v_{xx} = (c - a^3)^{-1} (v_{tt} + dv_t + a^1 v_{tt} + a^2 v_{xt} - f) .$$

Taking advantage of the above relation and setting $w = v_t$ we deduce that w is a (unique) weak solution of the problem $(\mathbf{L''})$, $(\mathbf{B_L})$, $(\mathbf{I'_L})$:

$$(\mathbf{L''}) \qquad \mathscr{L}w + a^1 w_{tt} + a^2 w_{xt} + a^3 w_{xx} + b^1 w_t + b^2 w_x + b^3 w = = f_t + a^3_t (c - a^3)^{-1} f ,$$

$$(\mathbf{I'_L}) \qquad w(x, 0) = v^1(x) , \quad w_t(x, 0) = v^2(x) , \quad x \in [0, l]$$

where the coefficients b^i, $i = 1, 2, 3$ are determined by a^i and satisfy (3.5) in view of Lemma 1, Lemma 2.

By virtue of (3.2), (3.3) and $(\mathbf{C_L})$, Lemma 3 yields the regularity of the function w.

To obtain the regularity of v with respect to x, we have to use the equation $(\mathbf{L})$.

The estimate (3.4) for $k = 2$ can be proved in a similar way.

Thus Theorem 3 has been proved.

Consider now a function $\psi_n \in \mathscr{C}_\infty(\mathbb{R}^1)$, $|\psi''_n| \leqq c_5$,

$$\psi_n = \begin{array}{lll} 0 & \text{on} & (-\infty, -n] \\ \in [0, 1] & \text{on} & [-n, -n + 1] \\ 1 & \text{on} & [-n + 1, +\infty) . \end{array}$$

In view of Theorem 3, we are able to solve the initial-boundary value problems:

$$(\mathbf{L}_n) \qquad \mathscr{L}v^n + a^1 v^n_{tt} + a^2 v^n_{tx} + a^3 v^n_{xx} = \psi_n f$$

for $x \in [0, l]$, $t \in [-n, +\infty)$

$$(\mathbf{B}_n) \qquad v^n(0, t) = v^n(l, t) = 0 , \quad t \in \mathbb{R}^1$$

$$(\mathbf{I}_n) \qquad v^n(x, t) = v^n_t(x, t) = 0 , \quad x \in [0, l] , \quad t \in (-\infty, -n] .$$

We get the existence of a unique solution $v^n \in X^{k+1}_0$ satisfying

$$(3.6) \qquad |D^{k+1} v^n(t)|^2 \leqq c_6 \sup \{ |D^k \psi_n f(s)|^2 \mid s \in (-\infty, t] \} \leqq \leqq c_7 \sup \{ |D^k f(s)|^2 \mid s \in (-\infty, t] \}$$

for all $t \in \mathbb{R}^1$ whenever (3.2) holds for $k = 1, 2$.

By means of the weak-star topology on the space $L_\infty(\mathbb{R}^1, L_2)$, we infer there is an accumulation point $v \in Y^{k+1}_0$ of the sequence $\{v^n\}_{n=1}^\infty$ and v satisfies (3.6) for a.e. $t \in \mathbb{R}^1$.

Dealing with a linear problem we check easily that v solves $(\mathbf{L})$, $(\mathbf{B_L})$ for $x \in [0, l]$, $t \in \mathbb{R}^1$. Finally, thanks to the regularity result achieved in Theorem 3, we get, in fact, $v \in X_0^{k+1}$.

Thus we have obtained the following theorem.

Theorem 4. *Let the conditions* (3.1), (3.2) *hold for* $k = 1, 2$, *and* $\alpha > 0$ *sufficiently small.*

Then there exists a unique global solution $v \in X_0^{k+1}$ *to the problem* $(\mathbf{L})$, $(\mathbf{B_L})$ *on* $[0, l] \times \mathbb{R}^1$ *satisfying*

$$(3.7) \qquad |D^{k+1} v(t)|^2 \leqq c_8 \sup \{|D^k f(s)|^2 \mid s \in (-\infty, t]\}$$

for all $t \in \mathbb{R}^1$.

Note that uniqueness in the above theorem follows immediately via the estimate (3.4).

4. THE PROOF OF THE MAIN RESULTS

In this section we prove the theorems formulated in Section 2 via the iteration method. To begin with, let us estimate the coefficients appearing in the equation $(\mathbf{E})$. Setting $w = 0$, $k = 2$ in $(\mathbf{P}_3)$, we get

$$|D^2 \mathscr{P}^i(v, \lambda)| \leqq c_2 \sup \{|D^3 v(s)| \mid s \in (-\infty, t]\} + |D^2 \mathscr{P}^i(0, \lambda)(t)|$$

for all $t \in \mathbb{R}^1$, $i = 1, 2, 3$.

By virtue of $(\mathbf{PF}_2)$, $\varepsilon, \eta > 0$ may be chosen so small that

$$(4.1) \qquad \mathscr{P}^i(v, \lambda) \in Y^2(\alpha)\,, \quad i = 1, 2, 3\,,$$

where α appears in Theorem 3, whenever $v \in X^3(\varepsilon)$, $\lambda \in \Lambda(\eta)$.

The iteration scheme is constructed as follows. Setting $u^1 = 0$, we determine u^{n+1} as $u^{n+1}(x, t) = 0$ for $t \leqq 0$, and u^{n+1} is the unique solution of the linear problem

$$(\mathbf{L}^{n+1}) \qquad \begin{aligned} &\mathscr{L} u^{n+1} + \mathscr{P}^1(u^n, \lambda)\, u_{tt}^{n+1} + \mathscr{P}^2(u^n, \lambda)\, u_{xt}^{n+1} + \\ &+ \mathscr{P}^3(u^n, \lambda)\, u_{xx}^{n+1} = \mathscr{F}(u^n, \lambda) \end{aligned}$$

for $x \in [0, l]$, $t \in [0, +\infty)$ together with the conditions $(\mathbf{B})$, $(\mathbf{I})$.

In view of (4.1), the procedure just sketched will work if we are able to show

$$(4.2) \qquad u^n \in X_0^3(\varepsilon) \quad \text{for all} \quad n = 1, \ldots.$$

In this case, the above problem can be uniquely solved by means of Theorem 3.

To prove (4.2), assume that we have already stated $u^n \in X_0^3(\varepsilon)$. Evoking the estimate (3.4) together with $(\mathbf{F}_3)$, we have

$$\begin{aligned} &|D^3 u^{n+1}(t)|^2 \leqq c_9 \varrho^2(\eta, \varepsilon) \sup \{|D^3 u^n(s)|^2 \mid s \in \mathbb{R}^1\} + \\ &+ c_{10} \sup \{|D^2 \mathscr{F}(0, \lambda)(s)|^2 \mid s \in \mathbb{R}^1\}\,. \end{aligned}$$

Consequently, η, $\varepsilon > 0$ being small enough, we get (4.2) for u^{n+1}.

Now let us consider the function $v^n = u^{n+1} - u^n \in X_0^3(2\varepsilon)$. Clearly $v^n = 0$ for $t \in (-\infty, 0]$, v^n solves the linear problem $(\mathbf{L}), (\mathbf{B_L}), (\mathbf{I_L})$ with $v^0 = v^1 = 0$, $a^i = \mathscr{P}^i(u^n, \lambda)$ $i = 1, 2, 3$, and

$$f = \mathscr{F}(u^n, \lambda) - \mathscr{F}(u^{n-1}, \lambda) + \\ + [\mathscr{P}^1(u^{n-1}, \lambda) - \mathscr{P}^1(u^n, \lambda)]\, u_{tt}^n + \\ + [\mathscr{P}^2(u^{n-1}, \lambda) - \mathscr{P}^2(u^n, \lambda)]\, u_{xt}^n + \\ + [\mathscr{P}^3(u^{n-1}, \lambda) - \mathscr{P}^3(u^n, \lambda)]\, u_{xx}^n\,. \tag{4.3}$$

Our aim is to apply the estimate (3.4) for $k = 1$ together with Lemma 1, Lemma 2.

According to $(\mathbf{P}_3), (\mathbf{F}_3)$, one deduces

$$|D^1 f(t)| \leqq c_{11}(\varrho(\eta, \varepsilon) + \varepsilon) \sup \{|D^2(u^n - u^{n-1})(s)| \mid s \in \mathbb{R}^1\}\,.$$

For small values of η, ε, we infer from (3.4)

$$\sup \{|D^2(u^{n+1} - u^n)(s)| \mid s \in \mathbb{R}^1\} \leqq \\ \leqq \omega \sup \{|D^2(u^n - u^{n-1})(s)| \mid s \in \mathbb{R}^1\} \tag{4.4}$$

where $\omega < 1$. As a consequence of the contraction mapping principle, we obtain the existence of a function $u \in X_0^2$ – the unique limit of the sequence $\{u^n\}_{n=1}^{\infty}$.

According to $(\mathbf{P}_3), (\mathbf{F}_3)$, we get

$$\mathscr{P}^i(u^n, \lambda) \to \mathscr{P}^i(u, \lambda)\,, \quad \mathscr{F}(u^n, \lambda) \to \mathscr{F}(u, \lambda) \quad \text{in} \quad Y^1\,. \tag{4.5}$$

Moreover, in view of (4.2), $u \in Y_0^3(\varepsilon)$ by virtue of the weak-star convergence of the derivatives in $L_\infty(\mathbb{R}^1, L_2)$.

Using (4.5) we deduce that u is a solution of $(\mathbf{E}), (\mathbf{B}), (\mathbf{I})$. A regularity argument concerning linear problems (see Theorem 3) gives finally $u \in X_0^3(\varepsilon)$.

In order to complete the proof of Theorem 1, the uniqueness of the solution u is to be proved. Consider two possible solutions u_1, u_2 satisfying $(\mathbf{E})$ on $[0, l] \times [0, T]$ for some $T > 0$, $|D^3 u_i(t)| \leqq \varepsilon$, $i = 1, 2$. Repeating the above arguments, we get similarly as in (4.4)

$$\sup \{|D^2(u_1 - u_2)(s)| \mid s \in (-\infty, T]\} \leqq \\ \leqq \omega \sup \{|D^2(u_1 - u_2)(s)| \mid s \in (-\infty, T]\}\,.$$

Clearly, $u_1(t) = u_2(t)$ for all $t \in (-\infty, T]$.

As to the proof of existence in Theorem 2, we proceed similarly as above using Theorem 4 instead of Theorem 3 and (3.7) in place of (3.4).

The uniqueness of the solution claimed in Theorem 2 is proved analogously.

Our eventual goal is to present some applications of the results stated in Section 2.

Example 1 (fully nonlinear telegraph equations). We examine the problem

$$\mathscr{L}v + F(v_{tt}, v_{xt}, v_{xx}, v_t, v_x, v) = f \tag{5.1}$$

$$v(0, t) = v(l, t) = 0\,, \quad t \in \mathbb{R}^1 \tag{5.2}$$

$$v(x, t + T) = v(x, t)\,, \quad x \in [0, l]\,, \quad t \in \mathbb{R}^1 \tag{5.3}$$

(cf. [8], [12], [14]). Here the function f is supposed to satisfy (5.3) as well.

Using the idea of Shibata-Tsutsumi [15] (cf. also [16]), the problem can be transformed to the system

$$\begin{aligned} &\mathscr{L}u + F_1'(D^1u, D_x^2v)\, u_{tt} + F_2'(D^1u, D_x^2v)\, u_{xt} + \\ &+ F_3'(D^1u, D_x^2v)\, u_{xx} = -F_4'(D^1u, D_x^2v)\, u_t - F_5'(D^1u, D_x^2v)\, u_x - \\ &- F_6'(D^1u, D_x^2v)\, u + f_t \end{aligned} \tag{5.4}$$

$$-cv_{xx} = -u_t - du - F(D^1u, D_x^2v) + f \tag{5.5}$$

$$u(0, t) = u(l, t) = v(0, t) = v(l, t) = 0 \quad \text{for all} \quad t \in \mathbb{R}^1 \tag{5.6}$$

$$u(x, t + T) = u(x, t)\,, \quad v(x, t + T) = v(x, t) \quad \text{for all} \quad x \in [0, l]\,, \quad t \in \mathbb{R}^1 \tag{5.7}$$

via differentiating with respect to t and setting $v_t = u$. Here, of course, the symbols F_i' stand for the derivatives of the function F with respect to the i-th variable, $i = 1, \ldots, 6$.

To apply the results of Section 2, we are forced to require $F \in \mathscr{C}^4(\overline{U})$ for some open ball U centered at $0 \in \mathbb{R}^6$, and

$$F(0) = F_i'(0) = 0\,, \quad i = 1, \ldots, 6\,. \tag{5.8}$$

In agreement with the notation of Theorem 2, we set $\Lambda = X^3$, $\lambda = f$.

To apply Theorem 2, we need the following auxiliary lemma.

Lemma 4. *Let $f \in \Lambda(\eta)$, $u \in X^2(\varepsilon)$ where $\eta, \varepsilon > 0$ are sufficiently small.*

Then there exists a unique (small) solution $v = \mathscr{T}(f, u)$ of the "elliptic" equation (5.5) satisfying the boundary conditions (5.6). Moreover, $D_x^2 v \in X_0^1$, and

$$\begin{aligned} &|D_x^2 D^k(v^1 - v^2)(t)| \leqq c_{12} |D^{k+1}(u^1 - u^2)(t)| \\ &|D_x^2 D^k\, v^0(t)| \leqq c_{13} |D^k f|\,, \quad k = 0, 1 \end{aligned} \tag{5.9}$$

where $v^i = \mathscr{T}(f, u^i)$, $i = 1, 2$, $v^0 = \mathscr{T}(f, 0)$.

If $u \in X^3(\varepsilon)$, then $D_x^2 v \in X_0^2$ and (5.9) holds for $k = 0, 1, 2$.

Proof. First of all, consider a linear problem

$$v_{xx} = h \tag{a}$$

$$v(0, t) = v(l, t) = 0\,, \quad t \in \mathbb{R}^1\,. \tag{b}$$

As to the above problem, we have the following result. If $h \in X^k$, $k = 1, 2$, then there is a unique solution v such that

(c) $$D_x^2 v \in X^k ,$$

and

(d) $$|D_x^2 D^k v(t)| \leqq c_{14} |D^k h(t)| , \quad t \in \mathbb{R}^1 .$$

(Cf. [15] for a more general case.)

Fix for the moment $u \in X^k(\varepsilon)$, $f \in \Lambda(\eta)$. Consider the operator S determined uniquely as a solution of

$$-c(Sv)_{xx} = -u_t - du - F(D^1 u, D_x^2 v) + f$$
$$Sv(0, t) = Sv(l, t) = 0 , \quad t \in \mathbb{R}^1 .$$

Combining (d) with Lemma 2, we get

(e) $$|D_x^2 D^k (Sv^1 - Sv^2)(t)| \leqq$$
$$\leqq c_{15}(\varepsilon + |D_x^2 D^k v^1(t)| + |D_x^2 D^k v^2(t)|)(|D_x^2 D^k (v^1 - v^2)(t)|)$$

(f) $$|D_x^2 D^k Sv| \leqq c_{16}(\varepsilon + \eta + (\varepsilon + |D_x^2 D^k Sv|)^2) .$$

One checks easily by means of the contraction mapping principle that for $\varepsilon, \eta > 0$ small enough, S has a (unique) fixed point. Using the results of Lemma 2 we can show the estimate (5.9).

Q.E.D.

In accordance with the notation of Theorem 2, we can set

$$\mathscr{P}^i(u, \lambda) = F_i'(D^1 u, D_x^2 \mathscr{T}(f, u)) , \quad i = 1, 2, 3$$
$$\mathscr{F}(u, \lambda) = -F_4'(D^1 u, D_x^2 \mathscr{T}(f, u)) u_t - \ldots - F_6'(D^1 u, D_x^2 \mathscr{T}(f, u)) u + f_t .$$

It is a matter of routine, by combining Lemma 1 with Lemma 2, to verify all assumptions required in Theorem 2. Thus the existence of a unique solution to the problem (5.1)–(5.3) can be proved via the results of Section 2.

Example 2 (integral operators). Consider integral operators of the form

$$\mathscr{P}^i(u, \lambda) = \int_0^{+\infty} p^i(s, D^1 u(x, t - s)) \, ds , \quad i = 1, 2, 3 ,$$
$$\mathscr{F}(u, \lambda) = \int_0^{+\infty} F(s, D^1 u(x, t - s)) \, ds + \lambda$$

where $\lambda \in X^3$.

As to the functions $p^i = p^i(s, y)$, $i = 1, 2, 3$, $F = F(s, y)$, we assume that they are defined on $[0, +\infty) \times U$, U being an open neighborhood of the point $0 \in \mathbb{R}^3$.

Further, p^i, F are supposed to be smooth and

$$(5.10) \qquad |D_y^k g(s, y)| \leqq \varphi_k(s) \quad k = 0, 1, \ldots$$

for $g = p^i, F$ where φ_k are functions integrable on the set $[0, +\infty)$.

Moreover, we require

$$(5.11) \qquad D_y^1 F(s, 0) = 0 \quad \text{for all} \quad s \in [0, +\infty).$$

Taking Lemma 1, Lemma 2 into account, we are able to verify all requirements appearing in the theorems of Section 2.

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Souhrn

GLOBÁLNÍ V ČASE ŘEŠENÍ KVAZILINEÁRNÍCH ROVNIC SE ZPOŽDĚNÍM

EDUARD FEIREISL

V práci je dokázána existence malých globálních (v čase) řešení abstraktní evoluční rovnice s tlumícím členem. Výsledek je aplikován na silně nelineární telegrafní rovnici a na rovnice obsahující operátory se zpožděním.

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