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ON A TRIVARIATE POISSON DISTRIBUTION

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Summary. A four parameter trivariate Poisson distribution is considered. Recurrences for the probabilities and the partial derivatives of the probabilities with respect to the parameters are derived. Solutions of the maximum likelihood equations are obtained and the determinant of their asymptotic covariance matrix is given. Applications of the maximum likelihood estimation technique to simulated data sets are also examined.

Keywords: Trivariate Poisson distribution, recurrence relationships, estimation, information matrix, maximum likelihood, simulation.

AMS Subject Classification: 62E15, 62F10.

0. INTRODUCTION

The correlated bivariate Poisson distribution was introduced by Campbell [1] as a limiting case of the bivariate binomial distribution. An alternative derivation was given by Holgate [2] using sums of independent Poisson random variables.

Multivariate generalizations of the Poisson distribution as limiting forms of multivariate binomial distributions were considered in [5], [7] and [15]. In addition, Mahamunulu [9] pointed out that a k-variate Poisson distribution with $2^k - 1$ nonnegative parameters can be constructed by considering appropriate sums over k overlapping subsets of a set of $2^k - 1$ independent Poisson random variables. The corresponding seven parameter trivariate Poisson model was discussed in [4], [8] and [9].

Johnson and Kotz [3], Ch. 11 § 4, observed that a random vector $(X_1, X_2, \ldots, X_k)$ has a k-variate Poisson distribution with $k + 1$ parameters, when

$$X_i = X_i' + T\,, \quad i = 1, \ldots, k \tag{1}$$

where X_i', $i = 1, \ldots, k$ and T are independent Poisson random variables with parameters $a_i' = a_i - d$, $i = 1, \ldots, k$ and d, respectively. This multivariate Poisson model was also considerer by Šidák in [12], [13] and [14] who derived various probability inequalities.

In this paper we consider a four parameter trivariate Poisson distribution with the structure (1). Various properties of the distribution are derived including recurrences for the probabilities and the partial derivatives of the probabilities with respect to the parameters. Parameter estimation by the method of maximum likelihood is discussed and the information matrix is derived. Finally, applications of the maximum likelihood estimation technique to trivariate Poisson simulated data are also given.

1. THE FOUR PARAMETER TRIVARIATE POISSON DISTRIBUTION

The random vector (X, Y, Z) has a trivariate Poisson distribution with parameters a, b, c, d, if

$$
\begin{aligned}
X &= X' + T \\
Y &= Y' + T \qquad (2)\\
Z &= Z' + T
\end{aligned}
$$

where X', Y', Z' and T are independent Poisson variables with parameters $a - d$, $b - d$, $c - d$ and d respectively. The probability generating function (p.g.f.) of the trivariate Poisson with the structure (2) is

$$
\begin{aligned}
(3) \qquad G_{X,Y,Z}(u, v, w) &= \mathsf{E}(u^X v^Y w^Z) = \mathsf{E}(u^{X'})\,\mathsf{E}(v^{Y'})\,\mathsf{E}(w^{Z'})\,\mathsf{E}\{(uvw)^T\} \\
&= \exp\{(a - d)(u - 1) + (b - d)(v - 1) + (c - d)(w - 1) + \\
&\quad + d(uvw - 1)\}\,,
\end{aligned}
$$

$a, b, c > 0, 0 < d \leqq \min(a, b, c)$.

The marginal distributions are Poisson with

$$\mathsf{E}(X) = a\,, \quad \mathsf{E}(Y) = b\,, \quad \mathsf{E}(Z) = c$$

and

$$\operatorname{Cov}(X, Y) = \operatorname{Cov}(X, Z) = \operatorname{Cov}(Y, Z) = d\,.$$

1.1 Probabilities and Recurrence Relationships

Expressions for the probabilities $\mathsf{P}_{x,y,z} = \mathsf{P}(X = x, Y = y, Z = z)$ can be obtained using the corresponding p.g.f. For the four parameter trivariate Poisson we have

$$
\begin{aligned}
G_{X,Y,Z}(u, v, w) &= \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \sum_{z=0}^{\infty} \mathsf{P}_{x,y,z} u^x v^y w^z \\
&= \exp\{(a - d)(u - 1) + (b - d)(v - 1) + (c - d)(w - 1) + \\
&\quad + d(uvw - 1)\} = \exp\{-A\} \exp\{(a - d)\,u\} \exp\{(b - d)v\} \\
&\quad . \exp\{(c - d)\,w\} \exp(duvw) \\
&= \exp\{-A) \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \sum_{z=0}^{\infty} \sum_{i=0}^{\infty} \frac{(a - d)^x}{x!} \frac{(b - d)^y}{y!} \frac{(c - d)^z}{z!} \frac{d^i}{i!} u^{x+i} v^{y+i} w^{z+i}
\end{aligned}
$$

where $A = a + b + c - 2d$. By identifying the coefficients of $u^x v^y w^z$ in the above expansion we obtain

$$\mathsf{P}_{x,y,z} = e^{-A} \sum_{i=0}^{\min(x,y,z)} \frac{(a-d)^{x-i}}{(x-i)!} \frac{(b-d)^{y-i}}{(y-i)!} \frac{(c-d)^{z-i}}{(z-i)!} \frac{d^i}{i!} .$$

Recurrence relationships for the probabilities which facilitate the use of computer programming are easily derived by differentiating the p.g.f. once with respect to a generating variable and then equating coefficients. Illustrations of the method are given in [6] and [10]. For the trivariate Poisson we find

$$(x+1)\,\mathsf{P}_{x+1,y,z} = (a-d)\,\mathsf{P}_{x,y,z} + d\mathsf{P}_{x,y-1,z-1}\,, \tag{4}$$

$$(y+1)\,\mathsf{P}_{x,y+1,z} = (b-d)\,\mathsf{P}_{x,y,z} + d\mathsf{P}_{x-1,y,z-1}\,, \tag{5}$$

$$(z+1)\,\mathsf{P}_{x,y,z+1} = (c-d)\,\mathsf{P}_{x,y,z} + d\mathsf{P}_{x-1,y-1,1,z}\,, \tag{6}$$

with

$$\mathsf{P}_{0,0,0} = G(0,0,0) = \exp\{-(a+b+c-2d)\}\,.$$

2. MAXIMUM LIKELIHOOD ESTIMATION

2.1. Recurrence Relations for the Partial Derivatives of the Probabilities

To evaluate the maximum likelihood equations and the terms of the information matrix (the inverse of the asymptotic covariance matrix for the maximum likelihood estimators) we need the partial derivatives of the probabilities with respect to the parameters. Differentiating the p.g.f. (3) once with respect to a parameter and then equating coefficients of $u^x v^y w^z$, we find that

$$\frac{\partial \mathsf{P}_{x,y,z}}{\partial a} = \mathsf{P}_{x-1,y,z} - \mathsf{P}_{x,y,z} \tag{7}$$

$$\frac{\partial \mathsf{P}_{x,y,z}}{\partial b} = \mathsf{P}_{x,y-1,z} - \mathsf{P}_{x,y,z} \tag{8}$$

$$\frac{\partial \mathsf{P}_{x,y,z}}{\partial c} = \mathsf{P}_{x,y,z-1} - \mathsf{P}_{x,y,z} \tag{9}$$

$$\frac{\partial \mathsf{P}_{x,y,z}}{\partial d} = \mathsf{P}_{x-1,y-1,z-1} - \mathsf{P}_{x-1,y,z} - \mathsf{P}_{x,y-1,z} - \mathsf{P}_{x,y,z-1} + 2\mathsf{P}_{x,y,z}\,. \tag{10}$$

In addition

$$\frac{\partial \mathsf{P}_{0,0,0}}{\partial a} = \frac{\partial \mathsf{P}_{0,0,0}}{\partial b} = \frac{\partial \mathsf{P}_{0,0,0}}{\partial c} = -\exp\{-(a+b+c-2d)\}$$

and

$$\frac{\partial \mathsf{P}_{0,0,0}}{\partial d} = 2\exp\{-(a+b+c-2d)\}\,.$$

2.2 Maximum Likelihood Equations and Estimators

The form for the maximum likelihood equations is

$$\sum_{x,y,z} \frac{1}{\mathsf{P}_{x,y,z}} \frac{\partial \mathsf{P}_{x,y,z}}{\partial \theta} = 0$$

where θ represents a parameter.

Using the recurrence relations (4)–(6) and (7)–(10) the four maximum likelihood equations become

$$\text{(11)} \qquad \frac{\bar{x}}{a-d} - \frac{d}{a-d}\bar{R} - 1 = 0\,,$$

$$\text{(12)} \qquad \frac{\bar{y}}{b-d} - \frac{d}{b-d}\bar{R} - 1 = 0\,,$$

$$\text{(13)} \qquad \frac{\bar{z}}{c-d} - \frac{d}{c-d}\bar{R} - 1 = 0\,,$$

$$\text{(14)} \qquad \frac{\bar{x}}{a-d} + \frac{\bar{y}}{b-d} + \frac{\bar{z}}{c-d} - \left(1 + \frac{d}{a-d} + \frac{d}{b-d} + \frac{d}{c-d}\right)\bar{R} - 2 = 0,$$

where

$$\text{(15)} \qquad R_{x,y,z} = \frac{\mathsf{P}_{x-1,y-1,z-1}}{\mathsf{P}_{x,y,z}}\,,$$

$$\bar{R} = \frac{1}{n}\sum_{x,y,z} R_{x,y,z}\,,$$

and n represents the sample size.

Substituting (11), (12) and (13) into (14) we obtain

$$\text{(16)} \qquad \bar{R} = 1\,.$$

Substitution of (16) into (11), (12) and (13) yields, respectively

$$\text{(17)} \qquad \hat{a} = \bar{x}\,, \quad \hat{b} = \bar{y}\,, \quad \hat{c} = \bar{z}\,.$$

The required estimator for the parameter d is obtained from the solution of equation (16). In order to solve this equation we can either use a standard iterative procedure, such as the routine RZERO provided by the Cern Computer Program Library, or the method of scoring (see, for example, Rao [11], § 5g).

The method of scoring is applied as follows: given an initial value d_0 of the estimate of d, a better approximation is obtained by $d_0 + \delta d$, where

$$\text{(18)} \qquad \delta d = \frac{[\bar{R}(d_0) - 1]}{\mathsf{E}\left[\left.\frac{\partial \bar{R}}{\partial d}\right|_{d = d_0}\right]}\,.$$

For the trivariate Poisson distribution using the relations (4)–(6) and (7)–(10)

$$\mathsf{E}\left[\frac{\partial R}{\partial d}\right] = [(ab + ac + bc - 2ad - 2bd - 2cd + 3d^2) + (ad^2 + bd^2 + cd^2 - abc - 2d^3)(Q - 1)] \times \times 1/[(a - d)(b - d)(c - d)] \equiv B_1 \tag{19}$$

where,

$$Q = \sum_{x,y,z} \frac{\mathsf{P}^2_{x-1,y-1,z-1}}{\mathsf{P}_{x,y,z}} . \tag{20}$$

The probability table is computed using the maximum likelihood estimates of a, b and c and an approximation d_0 of the maximum likelihood estimate of d. Consequently by computing the values of the expressions (15), (20) and (19) the correction δd can also be evaluated from (18).

In our studies we have used as d_0 the moment estimate of d given by

$$d_0 \equiv \tilde{d} = \frac{1}{n} \sum_{x,y,z} (x - \bar{x})(y - \bar{y})(z - \bar{z}) . \tag{21}$$

2.3 Derivation of the Information Matrix

The terms of the information matrix $\sum = (n\sigma_{ij})$ with $i, j = 1, 2, 3, 4$ are

$$n\sigma_{11} = n \sum_{x,y,z} \frac{1}{\mathsf{P}_{x,y,z}} \left(\frac{\partial \mathsf{P}_{x,y,z}}{\partial a}\right)^2$$

$$n\sigma_{12} = n \sum_{x,y,z} \frac{1}{\mathsf{P}_{x,y,z}} \frac{\partial \mathsf{P}_{x,y,z}}{\partial a} \frac{\partial \mathsf{P}_{x,y,z}}{\partial b} ,$$

etc. Making use of the relations (4)–(10) and (20) the terms of the information matrix are given by

$$n\sigma_{11} = n(a - d)^{-2} \{a - 2d + d^2(Q - 1)\} ,$$
$$n\sigma_{12} = n(a - d)^{-1} (b - d)^{-1} B_2 ,$$
$$n\sigma_{13} = n(a - d)^{-1} (c - d)^{-1} B_2 ,$$
$$n\sigma_{14} = n(a - d)^{-1} dB_1 ,$$
$$n\sigma_{22} = n(b - d)^{-2} \{b - 2d + d^2(Q - 1)\} ,$$
$$n\sigma_{23} = n(b - d)^{-1} (c - d)^{-1} B_2 ,$$
$$n\sigma_{24} = n(b - d)^{-1} dB_1 ,$$
$$n\sigma_{33} = n(c - d)^{-2} \{c - 2d + d^2(Q - 1)\} ,$$
$$n\sigma_{34} = n(c - d)^{-1} dB_1 ,$$

$$n\sigma_{44} = n(a - d)^{-2}(b - d)^{-2}(c - d)^{-2}$$
$$.\{(ad^2 + bd^2 + cd^2 - abc - 2d^3)$$
$$.(ab + ac + bc - 2ad - 2bd - 2cd + 3d^2)$$
$$+ (abc - ad^2 - bd^2 - cd^2 + 2d^3)^2(Q - 1)\},$$

where B_1 is given by equation (19) and

$$B_2 = d^2(Q - 1) - d.$$

2.4 Derivation of the Covariance Matrix of the Maximum Likelihood Estimators

Let $V = (n^{-1}v_{ij})$ for $i, j = 1, 2, 3, 4$ denote the covariance matrix of the maximum likelihood estimators. However, the covariance matrix of the maximum likelihood estimators $(\bar{x}, \bar{y}, \bar{z})$ of (a, b, c) can be easily obtained. Hence the form of V is

$$V = n^{-1}\begin{bmatrix} a & d & d & v_{14} \\ d & b & d & v_{24} \\ d & d & c & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix}$$

To obtain the remaining terms of V we use the relation

$$\sum V = V\sum = I,$$

or

$$\sum_{k=1}^{4} \sigma_{ik}v_{kj} = i_{ij},$$

where $I = (i_{ij})$ is the identity matrix. The required terms of V are

$$v_{14} = v_{24} = v_{34} = d,$$

and

$$v_{44} = C_1/C_2,$$

where

$$C_1 = d^2(ab + ac + bc - 2ad - 2bd - 2cd + 3d^2)(Q - 1)$$
$$+ (a - 2d)(b - 2d)(c - 2d) - d^2(a + b + c) + 4d^3,$$

and

$$C_2 = (abc - ad^2 - bd^2 - cd^2 + 2d^3)(Q - 1)$$
$$- (ab + ac + bc - 2ad - 2bd - 2cd + 3d^2).$$

Finally, the determinant of V is

$$n^{-4}(a - d)^2(b - d)^2(c - d)^2/C_2.$$

3. EXAMPLES

To illustrate the application of the method of maximum likelihood we have decided to use simulated data. To generate trivariate Poisson samples we used the "common elements" method suggested by Loukas and Kemp [8]. This method is based on the structure of the trivariate Poisson distribution and requires the generation of four independent univariate Poisson random variates distributed as: $X' \sim$ Poisson $(a - d)$, $Y' \sim$ Poisson $(b - d)$, $Z' \sim$ Poisson $(c - d)$ and $T \sim$ Poisson (d).

The trivariate Poisson variate is then constructed as

$$(X, Y, Z) = (X' + T, Y' + T, Z' + T).$$

Two sets of simulated trivariate Poisson data, one of size 200 and the other of size 1000, together with the true parameter values and their maximum likelihood estimates are given in Tables 1 and 2 respectively. The selected parameter values were relatively small in order to avoid lengthy tabulations. The required estimates were derived by the method of scoring and by using the subroutine RZERO provided by the Cern Computer Program Library.

Table 1.

Trivariate Poisson ($a = 0{\cdot}5$, $b = 0{\cdot}5$, $c = 0{\cdot}5$, $d = 0{\cdot}1$) data

	$X = 0$					$X = 1$					$X = 2$				
Y/Z	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
0	49	25	5	0	0	23	14	1	2	0	4	1	0	0	0
1	24	14	1	0	0	12	8	0	0	0	1	1	1	1	0
2	2	0	0	0	0	1	1	1	0	0	2	0	2	0	0
3	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0
4	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0

The maximum likelihood estimates are: $\hat{a} = 0{\cdot}465$, $\hat{b} = 0{\cdot}470$, $\hat{c} = 0{\cdot}500$, $\hat{d} = 0{\cdot}0575$. The moment estimate of d calculated from equation (21) is $\tilde{d} = 0{\cdot}0807$. This estimate was used as the starting value d_0 in the method of scoring.

Table 2.

Trivariate Poisson ($a = 0{\cdot}1$, $b = 0{\cdot}1$, $c = 0{\cdot}1$, $d = 0{\cdot}05$) data

	$X = 0$			$X = 1$			$X = 2$		
Y/Z	0	1	2	0	1	2	0	1	2
0	832	41	1	37	2	0	2	0	0
1	39	2	0	1	36	1	0	2	0
2	2	0	0	0	2	0	0	0	0

The maximum likelihood estimates are: $\hat{a} = 0{\cdot}087$, $\hat{b} = 0{\cdot}089$, $\hat{c} = 0{\cdot}089$, $\hat{d} = 0{\cdot}0409$. The moment estimate of d used as starting value in the method of scoring is $d_0 = \tilde{d} = 0{\cdot}0352$.

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