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ASYMPTOTIC NORMALITY OF EIGENVALUES OF RANDOM ORDINARY DIFFERENTIAL OPERATORS

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Summary. Boundary value problems for ordinary differential equations with random coefficients are dealt with. The coefficients are assumed to be Gaussian vectorial stationary processes multiplied by intensity functions and converging to the white noise process. A theorem on the limit distribution of the random eigenvalues is presented together with applications in mechanics and dynamics.

Keywords: ordinary differential operators, random coefficient processes, asymptotic normality of eigenvalues.

AMS classification: 60H25, (34B05, 34F05).

1. INTRODUCTION

The aim of this paper is to present some results concerning asymptotic normality of eigenvalues. The theory of random eigenvalue problems (see [12]) has been inspired by certain technical applications, namely in mechanics and dynamics. For illustration we present the *buckling problem of a bar.*

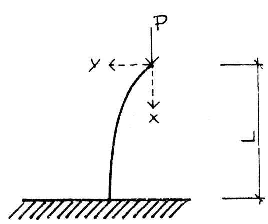

Fig. 1

Consider a vertical bar with clamped lower end and free upper end which is loaded by force P. The bar will stay in the straight position for small values of the force but there exists a critical value of P when this balance is broken and the bar bends.

The differential equation is

(1) $$-y'' = \lambda\left(\frac{1}{\alpha}\right) y\,,$$

with the boundary conditions

(2) $$y(0) = y'(L) = 0\,,$$

where α is the bending stiffness and λ equals to P. (See [2].)

The above mentioned critical value of P corresponds to an eigenvalue of (1), (2). There exists a sequence of simple positive eigenvalues $\{\lambda_g\}_{g=1}^{\infty}$, $\lambda_g \to \infty$ under rather general hypotheses on $\alpha(x)$.

If the shape of the cross-section or the modulus of elasticity varies with position then the term $1/\alpha$ in (1) can be treated as a stochastic process and λ_g as a random variable. Under the assumption that the random deviations are very small the perturbation theory can be used. The random eigenvalues can be expressed in the form of a series of the so called homogeneous terms in the perturbations. Asymptotic normality of λ_g can be established under additional assumptions.

In the paper we deal with eigenvalue problems in the general form:

(3) $$Mu = \lambda Nu\,, \quad U[u] = 0\,,$$

where
M, N are random ordinary differential operators on an interval $\langle 0, L\rangle$, $U[u]$ is an abbreviation for deterministic boundary conditions.

(3) covers a wide range of technical applications (see [2]). A very accomplished theory is presented in [12]. The asymptotic normality of eigenvalues of (3) is derived there under the assumptions that the random coefficients are weakly correlated connected vector processes with sufficiently smooth and uniformly bounded trajectories. Our paper aims at contributing to this theory. We apply the perturbation results from [12]. Instead of weakly correlated processes we work with more common stochastic processes. We obtain analogous results for perturbations being Gaussian stationary vectorial processes near to white noise, which are multiplied by an intensity function.

The application of methods based on the theory of probability in dynamics and mechanics has become more and more common. A distribution of natural frequencies of some engineering structures is studied in [13] from a slightly different point of view, for example.

2. PERTURBATION RESULTS

Consider the deterministic eigenvalue problem

(4) $$M_0 u + M_1 u = \lambda(N_0 u + N_1 u)\,, \quad U_j[u] = 0\,, \quad j = 1, 2, \ldots, 2m\,,$$

where

$$M_k u = \sum_{j=0}^{m} (-1)^j [f_{kj}(x)\, u^{(j)}]^{(j)},$$

$$N_k u = \sum_{j=0}^{m'} (-1)^j [g_{kj}(x)\, u^{(j)}]^{(j)}, \quad k = 0, 1, \quad m > m',$$

$$U_j[u] = \sum_{k=0}^{2m-1} \alpha_{jk}\, u^{(k)}(0) + \beta_{jk}\, u^{(k)}(L),$$

f_{kj}, g_{kj} are sufficiently smooth real functions,
α_{jk}, β_{jk} are real constants.

The principle of the perturbation theory is to express the eigenvalues and eigenfunctions of (4) in terms of the perturbations f_{1j}, g_{1j} and various characteristics of the so called unperturbed problem

$$(5) \qquad M_0 u = \mu N_0 u, \quad U_j[u] = 0, \quad j = 1, 2, \ldots, 2m.$$

This is possible under proper conditions. The most important one is the uniform smallness of the coefficients f_{1j}, g_{1j}.

Theorem 1. *Assume the operators M_k, N_k in (4) to be positive, let the order of M_1 be less than the order of M_0. Furthermore, let the equations*

$$(6) \qquad \sum_{j=0}^{m} \sum_{t=0}^{j-1} (-1)^{j+t} [f_{kj}(x)\, u^{(j)}]^{(j-t-1)}\, v^{(t)}|_0^L = 0,$$

$$\sum_{j=0}^{m'} \sum_{t=0}^{j-1} (-1)^{j+t} [g_{kj}(x)\, u^{(j)}]^{(j-t-1)}|_0^L = 0$$

hold for all admissible functions u, v and for $k = 1, 2$.

Assume that the unperturbed problem (5) possesses a discrete spectrum, let μ denote some simple eigenvalue of (5) and $w(x)$ the normalized eigenfunction associated with μ.

There exists a constant $\tilde{\eta} > 0$ depending only on the problem (5) such that for every $\eta \in (0, \tilde{\eta})$ the following statement holds.

If $|f_{1j}(x)| \leqq \eta$, $|g_{1j}(x)| \leqq \eta$ for every j and x then there exist terms λ_k, $u_k(x)$ $k = 0, 1, 2, \ldots$ such that the series

$$\lambda = \sum_{k=0}^{\infty} \lambda_k, \quad u(x) = \sum_{k=0}^{\infty} u_k(x)$$

converge and determine a solution of (4).

In particular,

$$\lambda_0 = \mu, \quad u_0(x) = w(x),$$

$$(7) \qquad \lambda_1 = \sum_{j=0}^{m-1} \int_0^L (u_0^{(j)}(x))^2 (f_{1j}(x) - \mu g_{1j}(x))\, dx,$$

$$(8) \qquad |\lambda - (\lambda_0 + \lambda_1)| \leqq C\eta^2,$$

where we set $g_{1j}(x) = 0$ *for* $j > m'$ *and* C *is a constant depending only on the problem* (5).

For the proof see [12].

Some comments to this theorem:

The *admissible functions* are the elements of $C^{2m}\langle 0, L\rangle$ satisfying the boundary conditions. The *positiveness* means $(M_k u, u) > 0$, $(N_k u, u) > 0$ for $k = 1, 2$ and for all admissible u, v, not equal to zero.

The conditions (6) enable us to derive the equations for all admissible u, v:

$$(9) \qquad (M_k u, v) = \sum_{j=0}^{m} (f_{kj} u^{(j)}, v^{(j)}) ,$$

$$(N_k u, v) = \sum_{j=0}^{m} (g_{kj} u^{(j)}, v^{(j)}) .$$

Consequently, the conditions (6) are in fact sufficient for the symmetry of M_k, N_k. In any case (6) holds when the boundary conditions in (4) are

$$u(0) = u'(0) = \ldots = u^{(m-1)}(0) = 0 ,$$

$$u(L) = u'(L) = \ldots = u^{(m-1)}(L) = 0 .$$

Analogous perturbation results can be stated in a more general situation for operators in Hilbert or Banach spaces (see [3]). (7) follows from the general expression for λ_1

$$\lambda_1 = (M_0 u_0, u_0) - \mu(N_0 u_0, u_0)$$

and from (9).

3. THE REPRESENTATION OF A VECTORIAL GAUSSIAN STATIONARY PROCESS

We will deal with problem (4), where M_0, N_0, U are deterministic and M_1, N_1 are operators the coefficients of which are centralized stochastic processes derived from Gaussian stationary vectorial processes with rational spectral density. The following theorem gives us the representation of such processes by means of stochastic integrals. This representation is used in the proof of *Theorem* 3. The one-dimensional case of this representation is dealt e.g. in [4].

The real stationary process is called *symmetric*, if the correlation function is a symmetric matrix for arbitrary t.

Theorem 2. *Let* $\mathbf{X}(t) = (X_1(t), \ldots, X_n(t))^T$ *denote a vectorial real symmetric Gaussian stationary process with continuous trajectories and with rational spectral density* $\mathbf{f}(\lambda)$.

Then

(i) $\mathbf{f}(\lambda)$ *is a real symmetric even positive-semidefinite matrix function. It has a constant rank* $m \leqq n$ *with possible exception of a finite set* $\{\lambda_j, j = 1, 2, \ldots, k\}$.

(ii) $\mathbf{f}(\lambda)$ *can be decomposed by*

$$(10) \qquad \mathbf{f}(\lambda) = \frac{1}{2\pi} \mathbf{B}(i\lambda)\, \mathbf{B}^T(-i\lambda)\,,$$

where $\mathbf{B}_{n\times m}(s)$ *is a rational matrix function analytic in* $\{s \in \mathbf{C};\ \operatorname{Re} s \geqq 0\}$ *and real for real* s.

(iii) *If* $\mathbf{H}(t)$, $t \geqq 0$ *denotes the inverse Laplace transform of* $\mathbf{B}(s)$, *i.e.*

$$(11) \qquad \mathbf{B}(s) = \int_0^\infty \exp(-st)\, \mathbf{H}(t)\, \mathrm{d}t\,,$$

then the stochastic process

$$\tilde{\mathbf{X}}(t) = (\tilde{X}_1(t), \ldots, \tilde{X}_n(t))^T\,,$$

$$\tilde{X}_j(t) = \sum_{k=1}^{m} \int_{-\infty}^{t} H_{jk}(t-s)\, \mathrm{d}W_s^{(k)}$$

has continuous trajectories and the same correlation function as the process $\mathbf{X}(t)$.
Here $(W^{(1)}, \ldots, W^{(m)})^T$ denotes the standard m-dimensional Wiener process.

The proof is a simple modification of results presented e.g. in [11], [4]. When dealing with distribution problems for Gaussian processes we can identify $\mathbf{X}(t)$ and $\tilde{\mathbf{X}}(t)$.

4. THE LIMIT THEOREM

We will consider the random problem in the form (4) with the following notation.

$$M_0 u = \sum_{j=0}^{m} (-1)^j \,[f_j(t)\, u^{(j)}]^{(j)}\,, \quad f_m(t) \neq 0 \quad \text{for} \quad t \in \langle 0, L\rangle\,,$$

$$N_0 u = \sum_{j=0}^{n} (-1)^j \,[g_j(t)\, u^{(j)}]^{(j)}\,, \quad m > n\,.$$

$U_j[u]$ are the same as in Section 2.

Here f_j, g_j are sufficiently smooth deterministic real functions, so that M_0, N_0 are well defined.

The operators M_1, N_1 are assumed to be random:

$$M_1 u = \sum_{j=0}^{m-1} (-1)^j \,[X_j(t)\, u^{(j)}]^{(j)}\,, \quad N_1 u = \sum_{j=0}^{n} (-1)^j \,[X_{m+j}(t)\, u^{(j)}]^{(j)}\,,$$

where

$$\mathbf{X}(t) = (X_0(t), \ldots, X_{m-1}(t)\,,\ \ X_m(t), \ldots, X_{m+n}(t))^T$$

is a vectorial stochastic process with sufficiently smooth trajectories that depends on a real parameter. The details will be described later.

We suppose the operators M_0, N_0 to be positive, the same must hold for $M_0 + M_1$, $N_0 + N_1$ a.s. Analogously to (6) the following equations are assumed to hold for all admissible u, v.

$$\sum_{j=0}^{m} \sum_{k=0}^{j-1} (-1)^{j+k} \left[f_j(t)\, u^{(j)}\right]^{(j-k-1)} v^{(k)}\Big|_0^L = 0\,, \tag{12}$$

$$\sum_{j=0}^{n} \sum_{k=0}^{j-1} (-1)^{j+k} \left[g_j(t)\, u^{(j)}\right]^{(j-k-1)} v^{(k)}\Big|_0^L = 0\,,$$

$$\sum_{j=0}^{m-1} \sum_{k=0}^{j-1} (-1)^{j+k} \left[X_j(t)\, u^{(j)}\right]^{(j-k-1)} v^{(k)}\Big|_0^L = 0 \quad \text{a.s.}, \tag{13}$$

$$\sum_{j=0}^{n} \sum_{k=0}^{j-1} (-1)^{j+k} \left[X_{m+j}(t)\, u^{(j)}\right]^{(j-k-1)} v^{(k)}\Big|_0^L = 0 \quad \text{a.s.}$$

The order of M_1 is supposed to be less than that of M_0 so that the perturbation theory, namely *Theorem* 1 can be used. Conditions (12), (13) are commented in Section 2. When the boundary conditions include

$$u(0) = u'(0) = \ldots = u^{(m-2)}(0) = 0\,,$$

$$u(L) = u'(L) = \ldots = u^{(m-2)}(L) = 0\,,$$

then (13) is trivially fulfilled.

We will describe in a more detailed way the supposed nature of $\mathbf{X}(t)$. Let each of its components be in the form

$$X_j(t) = \sqrt{(\varepsilon)}\, \varphi_j(t)\, \tilde{X}_{j,a}(t)\,, \quad j = 0, 1, \ldots, m + n \tag{14}$$

Here a means a real parameter, $\varepsilon = \varepsilon(a)$ is a positive function of a. $\varphi_j(t)$ is any real deterministic sufficiently smooth function. We suppose $\tilde{\mathbf{X}}_a(t)$ to be a vectorial real symmetric Gaussian stationary process with sufficiently smooth components and with rational spectral density depending on a. By sufficient smoothness we again mean the possession of so many derivatives that M_1, N_1 are well defined.

We will study the limit distribution of the eigenvalues of (4) for $a \to \infty$ assuming that

$$\tilde{\mathbf{X}}_a(t) = (\tilde{X}_{0,a}(t), \ldots, \tilde{X}_{m+n,a}(t))^T$$

converges to a vectorial white noise and $\varepsilon(a)$ converges sufficiently quickly to zero so that *Theorem* 1 can be used. The functions φ_j express a possible nonhomogeneity of the perturbations.

Let $\mathbf{R}_a(t)$ be the matrix correlation function of $\tilde{\mathbf{X}}_a(t)$, its elements being denoted by $R_a^{jk}(t)$, $j, k = 0, 1, \ldots, m + n$.

$\mathbf{f}_a(\lambda)$ denotes the spectral matrix density of $\tilde{\mathbf{X}}_a(t)$ with elements $f_a^{jk}(\lambda)$. $\mathbf{f}_a(\lambda)$ is supposed to be rational, and according to *Theorem* 2 it is a real symmetric even matrix function with a constant rank r. In particular, the decomposition (10) holds.

$\mathbf{B}_a(s)$ from (10) is a matrix function with elements $B_a^{jk}(s)$, where $j = 0, 1, \ldots, m + n$; $k = 1, 2, \ldots, r$. We introduce their decomposition into partial fractions

$$B_a^{jk}(s) = \sum_{v=1}^{V_{jk}} \sum_{z=1}^{Z_{jkv}} \frac{A_{jkvz}(a)}{(s + c_{jkv}(a))^z} + \tag{15}$$

$$+ \sum_{e=1}^{E_{jk}} \sum_{p=1}^{P_{jke}} \frac{B_{jkep}(a)}{(s + d_{jke}(a))^p} + \frac{\overline{B_{jkep}(a)}}{(s + \overline{d_{jke}(a)})^p},$$

where $A_{jkvz}(a)$, $c_{jkv}(a)$ are real and $B_{jkep}(a)$, $d_{jke}(a)$ are complex numbers.

According to *Theorem* 2 we suppose $c_{jkv}(a) > 0$ and $d_{jke}(a) = \varrho_{jke}(a) + i\,\sigma_{jke}(a)$ with positive $\varrho_{jke}(a)$, $\sigma_{jke}(a)$.

We define the matrix $\mathbf{H}_a(t)$ by the relation

$$\mathbf{B}_a(s) = \int_0^\infty \mathbf{H}_a(t) \exp(-st)\,\mathrm{d}t\,.$$

Here we can give the explicit formula

$$H_a^{jk}(t) = \sum_{v=1}^{V_{jk}} \left[\sum_{z=1}^{Z_{jkv}} A_{jkvz}(a) \frac{t^{z-1}}{(z-1)!} \right] \exp(-c_{jkv}(a)\,t) + \tag{16}$$

$$+ \sum_{e=1}^{E_{jk}} \sum_{p=1}^{P_{jke}} \frac{t^{p-1}}{(p-1)!} \left[B_{jkep}(a) \exp(-d_{jke}(a)\,t) + \right.$$

$$\left. + \overline{B_{jkep}(a)} \exp(\overline{-d_{jke}(a)\,t}) \right].$$

Finally we obtain the representation

$$\tilde{X}_{j,a}(t) = \sum_{k=1}^{r} \int_{-\infty}^{t} H_a^{jk}(t-s)\,\mathrm{d}W_s^{(k)} \tag{17}$$

as in *Theorem* 2.

Theorem 3. *Let the assumptions and notation from this section hold. Let the deterministic problem* (5) *have a discrete spectrum. Let* μ *be any simple eigenvalue and* $w(t)$ *the corresponding normalized eigenfunction.*

Let the matrices $\mathbf{R}_a(t)$ *fulfil the conditions*

$$\lim_{a\to\infty} \int_{-\Delta}^{\Delta} \mathbf{R}_a(t)\,\mathrm{d}t = \mathbf{R} \quad \textit{for arbitrary} \quad \Delta > 0\,, \tag{18}$$

where $\mathbf{R} = \{R_{jk}\}_{j,k=0}^{m+n}$ *is a constant matrix,*

$$\lim_{a\to\infty} \int_{\mathbf{R}\setminus\langle-\Delta,\Delta\rangle} |R_a^{jk}(t)|\,\mathrm{d}t = 0 \quad \textit{for arbitrary} \quad j, k, \Delta > 0, \tag{19}$$

$$\int_{-\infty}^{\infty} |R_a^{jk}(t)|\,\mathrm{d}t \leqq K < \infty \quad \textit{for arbitrary} \quad j, k\,, \tag{20}$$

where K *is a constant independent of* a.

Let the terms from the decomposition (15) *fulfil*

(21) $$\lim_{a\to\infty} c_{jkv}(a) = \infty\,, \quad \lim_{a\to\infty} \varrho_{jke}(a) = \infty \quad \textit{for arbitrary} \quad j, k, v, e\,.$$

Let there exist a positive constant q such that for arbitrary j, k, v, z, e, p and for sufficiently large a we have

(22) $$\varepsilon(a) \leqq \frac{c_{jkv}^{1-q}(a)}{A_{jkvz}^2(a)}\,, \quad \varepsilon(a) \leqq \frac{\varrho_{jke}^{1-q}(a)}{|B_{jkep}(a)|^2}\,,$$

(23) $$\varepsilon(a) \leqq \frac{c_{jkv}^{2-q}(a)}{A_{jkvz}^4(a)}\,, \quad \varepsilon(a) \leqq \frac{\varrho_{jke}^{2-q}(a)}{|B_{jkep}(a)|^4}\,.$$

Then, as $a \to \infty$, there exists a random eigenvalue $\lambda(a)$ of (4) *with probability tending to* 1 *such that*

$$(\lambda(a) - \mu)/\sqrt{(\varepsilon(a))}$$

converges in distribution to a centralized Gaussian random variable with variance

(24) $$\sigma^2 = \sum_{j,k=0}^{m-1} \int_0^L (w^{(j)}(t)\, w^{(k)}(t))^2 \left[A^{jk} - \mu A^{m+j,k} - \right.$$
$$\left. - \mu A^{j,m+k} + \mu^2 A^{m+j,m+k}\right] \mathrm{d}t\,.$$

$A^{jk}(t)$ is an abbreviation for the function $R^{jk}\, \varphi_j(t)\, \varphi_k(t)$ and we set $R^{jk} = 0$, $\varphi_j(t) = 0$ for $j, k > m + n$.

Remarks

a) Conditions (18), (19), (20) express the convergence of the process $\breve{\mathbf{X}}_a(t)$ to the vectorial white noise.

b) Conditions (22), (23) ensure sufficiently fast convergence of $\varepsilon(a)$ to zero and the possibility of using the perturbation theory.

c) The existence of a small neighbourhood of μ including exactly one simple eigenvalue λ of (4) has been proved in a more general situation than in our theorem (see [7]–[10], or [3]).

d) We can compare *Theorem* 3 with *Theorem* 2.14 in [12]. Similar results are obtained there under different assumptions on the process $\mathbf{X}(t)$, namely it is supposed to be a weakly correlated connected vector process with reciprocal intensities $a^{jk}(t)$. The functions $A^{jk}(t)$ play the role of the intensities in our theorem.

Outline of proof. The detailed proof of *Theorem* 3 is rather extensive, we present therefore only a scheme of it.

(i) For a sufficiently large a and an arbitrary $\Delta > 0$ the estimate

$$(25)\qquad P[\max_{\substack{j=0,1,\ldots,m+n\\ t\in\langle 0,L\rangle}} |X_j(t)| > \Delta] \leqq$$
$$\leqq \sum_{j=0}^{m+n} \sum_{k=1}^{r} \left\{ \sum_{v=1}^{V_{jk}} \sum_{z=1}^{Z_{jkv}} K_{jkvz} c_{jkv}(a) \exp\left(- \frac{\tilde{K}_{jkvz} \Delta^2 c_{jkv}(a)}{\varepsilon A_{jkvz}^2(a)} \right) + \right.$$
$$\left. + \sum_{e=1}^{E_{jk}} \sum_{p=1}^{P_{jke}} L_{jkep}\, \varrho_{jke}(a) \exp\left(- \frac{\tilde{L}_{jkep}\, \Delta^2 \varrho_{jke}(a)}{\varepsilon |B_{jkep}(a)|^2} \right) \right\},$$

where K_{jkvz}, $\tilde{K}_{jkvz}$, L_{jkep}, $\tilde{L}_{jkep}$ are positive constants, is derived using the representation of $X_a(t)$ by means of (17). We note that for deriving (25) only the first from the assumptions (22), (23) is needed.

From (25) it immediately follows that

$$\max_{\substack{j=0,1,\ldots,m+n\\ t\in\langle 0,L\rangle}} |X_j(t)|$$

converges to zero in probability. Due to this convergence we can apply *Theorem* 1. Moreover, with probability tending to 1 we can use the expansion for λ, particularly the expression (7) for λ_1 and the estimate (8).

In the introduced notation we have

$$\lambda = \mu + \sum_{i=1}^{\infty} \lambda_i\,,$$

$$(26)\qquad \lambda_1 = \sum_{j=0}^{m-1} \int_0^L (w^{(j)}(t))^2 (X_j(t) - \mu X_{m+j}(t))\, \mathrm{d}t\,.$$

(ii) The random variable λ_1 is well defined. It can be easily shown that λ_1 has centralized normal distribution. The variance of $\lambda_1(a)/\sqrt{(\varepsilon(a))}$ converges to σ^2 given in (24).

To prove this we have

$$E(\lambda_1/\sqrt{\varepsilon})^2 = \sum_{j,k=0}^{m-1}\sum \int_0^L \int_0^L (w^{(j)}(t)\, w^{(k)}(s))^2 \times$$
$$\times [\varphi_j(t)\, \varphi_k(s)\, R_a^{jk}(t-s) - \mu \varphi_{m+j}(t)\, \varphi_k(s)\, R_a^{m+j,k}(t-s) -$$
$$- \mu \varphi_j(t)\, \varphi_{m+k}(s)\, R_a^{j,m+k}(t-s) +$$
$$+ \mu^2 \varphi_{m+j}(t)\, \varphi_{m+k}(s)\, R_a^{m+j,m+k}(t-s)]\, \mathrm{d}t\, \mathrm{d}s\,.$$

Next we use the easily verifiable fact that under assumptions (18), (19), (20) an arbitrary continuous function $f(t, s)$ on $\langle 0, L\rangle^2$ satisfies

$$\int_0^L \int_0^L f(t,s)\, R_a^{jk}(t-s)\, \mathrm{d}t\, \mathrm{d}s \to R^{jk} \int_0^L f(t,t)\, \mathrm{d}t\,.$$

Since $\lambda_1/\sqrt{\varepsilon}$ is normal we conclude it converges in distribution to the variable defined in *Theorem* 3.

(iii) Finally we establish that the last term in the decomposition

$$(\lambda - \mu)/\sqrt{\varepsilon} = \lambda_1/\sqrt{\varepsilon} + (\lambda - \mu - \lambda_1)/\sqrt{\varepsilon}$$

converges to zero in probability. This follows from the estimate

$$P[|\lambda - \mu - \lambda_1|/\sqrt{\varepsilon} > \Delta] \leqq$$
$$\leqq P[\max_{\substack{j=0,1,\dots,m+n \\ t\in\langle 0,L\rangle}} |X_j(t)| > \tilde{\eta}] + P[\max_{\substack{j=0,1,\dots,m+n \\ t\in\langle 0,L\rangle}} |X_j(t)| > C^{-1/2}\varepsilon^{1/4}\Delta^{1/2}],$$

where $\tilde{\eta}$ is introduced in *Theorem* 1. The first term on the right converges to zero according to (i). The second term can be estimated by means of (25). The resulting terms converge to zero in virtue of (23).

5. EXAMPLES FOR THE ONE-DIMENSIONAL CASE

In this section we present two special versions of *Theorem* 3 when the process $\tilde{X}_a(t)$ is one-dimensional with some commonly used correlation function. Let $X(t)$ be the unique random coefficient in (4). We suppose for the sake of simplicity that it appears in the zero term of M_1 or N_1.

Let conditions (12) be fulfilled, while conditions (13) can be omitted here.

Let

$$X(t) = \sqrt{(\varepsilon)}\, \varphi(t)\, \tilde{X}_a(t), \tag{27}$$

where $\varphi(t)$ is a continuous function on $\langle 0, L\rangle$. As concerns $\tilde{X}_a(t)$ we will consider two cases.

A) $\tilde{X}_a(t)$ is a real Gaussian stationary process with continuous trajectories and with the correlation function

$$r_a(t) = r_a(0) \exp(-a|t|).$$

The spectral density of $\tilde{X}_a(t)$ is then

$$f_a(\lambda) = r_a(0)\, a/\pi(\lambda^2 + a^2).$$

The density can be decomposed as in (10) and the representation

$$\tilde{X}_a(t) = \int_{-\infty}^{t} \sqrt{(2r_a(0)\, a)} \exp(-a(t-s))\, \mathrm{d}W_s \quad \text{(see [4] or [5])}$$

can be derived.

Assumptions (18), (19), (20) of *Theorem* 3 are satisfied if $2r_a(0)/a \to R$. (21) is obvious. (22) and (23) hold if

$$\varepsilon(a) \leqq a^{-2-h}, \tag{28}$$

where h is any positive constant. *Theorem* 3 is then valid and the limit variance of $(\lambda - \mu)/\sqrt{\varepsilon}$ equals

$$R \int_0^L \varphi^2(t)\, w^4(t)\, \mathrm{d}t \quad \text{or} \quad R\mu^2 \int_0^L \varphi^2(t)\, w^4(t)\, \mathrm{d}t$$

if $X(t)$ appears in the zero term of M_1 or N_1, respectively.

B) $\tilde{X}_a(t)$ is a process as in *A*) but with the correlation function

$$r_a(t) = r_a(0) \exp(-a|t|) \cos bt \, .$$

Here,

$$f_a(\lambda) = \frac{r_a(0)\, a(\lambda^2 + a^2 + b^2)}{(\lambda^2 - (b - ia)^2)(\lambda^2 - (b + ia)^2)},$$

$$\tilde{X}_a(t) = \int_{-\infty}^{t} h_a(t - s)\, dW_s \, ,$$

where

$$h_a(u) = \sqrt{(2\, r_a(0)\, a)} \exp(-au) \left[\cos bu + \frac{1}{\sqrt{(a^2 + b^2)} + a} \sin bu\right]$$

(see [4] or [5]).

We suppose $b = b(a)$ in order to have only one parameter. (18), (19), (20) are valid if

$$\frac{2a\, r_a(0)}{a^2 + b^2} \to R \, , \quad b \leqq Ka \quad (K \text{ fixed}) \, .$$

(21) holds, and a sufficient condition for (22), (23) is (28) again. We have the same expressions for the limit variance as in *A*).

6. APPLICATIONS

In this section we indicate two applications of the results in the buckling problems.

A) *The buckling problem of a bar*

We have described this problem in *Introduction* – see Figure 1. Taking into account the random fluctuations in the shape of the cross-section and in the quality of the material we set in (1)

$$\frac{1}{\alpha(t)} = g(t) + X(t) \, ,$$

where $g(t)$ is a deterministic function and $X(t)$ is a stochastic process in the form (27). *Theorem* 3 is applicable if $X(t)$ satisfies (18)–(23) under rather general hypotheses on $g(t)$.

B) *The buckling problem of a supported stanchion*

Consider a stanchion in a horizontal position (for example a rail) lying on a resilient subsoil, which is loaded by a force *P*.

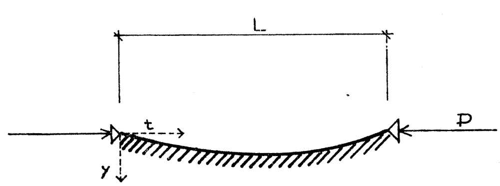

Fig. 2

Winkler's hypothesis is often made. The substance of it is that the reaction of the subsoil is in proportion to the deviation y. The constant of this proportionality is the coefficient of Winkler's subsoil $\varkappa$.

The differential equation for this problem is derived in [1]:

$$(\alpha y'')'' + \varkappa y = -\lambda y'' ,$$

$$y(0) = y(L) = 0 ,$$

$$y''(0) = y''(L) = 0 .$$

α means the bending stiffness (it is considered to be deterministic here), and the eigenvalues λ are equal to the critical values of P at which the stanchion bends.

Admitting small perturbations of the coefficient $\varkappa$, we can assume

$$\varkappa(t) = f(t) + X(t) ,$$

where $f(t)$ is a deterministic function and $X(t)$ is a stochastic process again in the form (27). *Theorem* 3 is applicable under the hypotheses stated in A).

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Souhrn

ASYMPTOTICKÁ NORMALITA VLASTNÍCH ČÍSEL NÁHODNÝCH OBYČEJNÝCH DIFERENCIÁLNÍCH OPERÁTORŮ

MARTIN HÁLA

Autor se zabývá okrajovými úlohami pro obyčejné diferenciální rovnice s náhodnými koeficienty. Tyto koeficienty jsou považovány za Gaussovské vektorové stacionární procesy, vynásobené intenzitami a konvergující k bílému šumu. Autor předkládá větu týkající se limitního rozdělení náhodných vlastních čísel společně s aplikacemi v teoretické mechanice a dynamice.

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