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ON ONE METHOD OF NUMERICAL INTEGRATION

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Summary. The uniform convergence of a sequence of Lienhard approximation of a given continuous function is proved. Further, a method of numerical integration is derived which is based on the Lienhard interpolation method.

Keywords: Lienhard interpolation, numerical integration.

1. In the interval $\langle a, b\rangle$ of finite length $L = b - a > 0$ let us consider a continuous function $y = f(x)$. Let $n \geqq 2$ be a positive integer. We divide the interval $\langle a, b\rangle$ into n equal intervals with dividing points $a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b$. We have $x_j = a + (j - 1)\,h$ for $j = 1, 2, \ldots, n + 1$, where $h = L/n$. We denote $f(x_j) = y_j$ and, further, $P_j = (x_j, y_j)$, $j = 1, 2, \ldots, n + 1$. For every $n \geqq 2$ $y_1 = f(a)$, $y_{n+1} = f(b)$ [see Fig. 1].

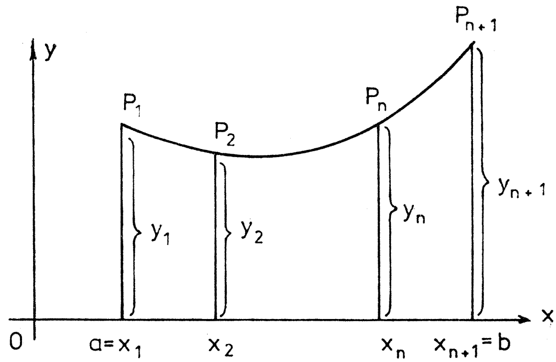

Fig. 1

Using the Lienhard interpolation method [see [1], [2], [3], [4]] we construct an interpolation curve passing through the points $P_1, P_2, \ldots, P_n, P_{n+1}$ whose j-th arc P_jP_{j+1}, $j = 1, 2, \ldots, n$, is parametrized with the aid of the polynomials

(1) $$x = P_0^{(j)}(t) = (1, t, t^2, t^3) \circ \mathbf{C} \circ \begin{bmatrix} x_{j-1} \\ x_j \\ x_{j+1} \\ x_{j+2} \end{bmatrix},$$

(2) $$y = P_1^{(j)}(t) = (1, t, t^2, t^3) \circ \mathbf{C} \circ \begin{bmatrix} y_{j-1} \\ y_j \\ y_{j+1} \\ y_{j+2} \end{bmatrix},$$

where

(3) $$\mathbf{C} = \frac{1}{16} \begin{bmatrix} -1 & 9 & 9 & -1 \\ 1 & -11 & 11 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

and the parameter t varies in the interval $\langle -1, 1 \rangle$. For $j = 1$ and $j = n$ we choose $x_0 = a - h$ and $x_{n+2} = b + h$. Further, it is possible to choose the values y_0 and y_{n+2} [see (2)] more or less arbitrarily. By (1) we have

(4) $$x = P_0^{(j)}(t) = (1, t, t^2, t^3) \circ \mathbf{C} \circ \begin{bmatrix} a + (j-2)h \\ a + (j-1)h \\ a + jh \\ a + (j+1)h \end{bmatrix} = a + jh - \frac{h}{2} + \frac{h}{2} t,$$

where $P_0^{(j)}$ is a function with the domain $\langle -1, 1 \rangle$ and the range $\langle a + (j-1)h, a + jh \rangle$. For the inverse function $[P_0^{(j)}]^{-1} : \langle a + (j-1)h, a + jh \rangle \to \langle -1, 1 \rangle$ we then have

(5) $$t = [P_0^{(j)}]^{-1}(x) = \frac{2}{h}(x - a) - (2j - 1).$$

Substituting (5) into (2) we obtain

(6) $$y = P_1^{(j)} \circ [P_0^{(j)}]^{-1}(x) = p_n^{(j)}(x) =$$
$$= \left(1, \frac{2}{h}(x-a) - (2j-1), \left[\frac{2}{h}(x-a) - (2j-1)\right]^2, \left[\frac{2}{h}(x-a) - (2j-1)\right]^3\right) \circ \mathbf{C} \circ \begin{bmatrix} y_{j-1} \\ y_j \\ y_{j+1} \\ y_{j+2} \end{bmatrix}$$

where $p_n^{(j)} = P_1^{(j)} \circ [P_0^{(j)}]^{-1}$ is a function with the domain $\langle a + (j-1)h, a + jh \rangle$.

For a given number n and given $x \in \langle a, b\rangle$ we determine the number $j = [(x - a)/h] + 1$, where the square bracket denotes the integer part of the corresponding real number. If x runs through the interval $\langle a, b\rangle$ then the number j assumes the values $1, 2, \ldots, n$. We have $j - 1 = [(x - a)/h] \leqq (x - a)/h < [(x - a)/h] + 1 = j$, i.e. $a + (j - 1)h \leqq x < a + jh$. We put $(2/h)(x - a) - (2j - 1) = 2(x - a)/h - 2[(x - a)/h] - 1 = 2\{(x - a)/h - [(x - a)/h]\} - 1 \overset{\text{def.}}{=} \langle(x - a)/h\rangle$. In the interval $\langle a + (j - 1)h, a + jh)$ it is then possible to express (6) in the form

$$(7) \qquad y = p_n^{(j)}(x) = \left(1, \left\langle\frac{x-a}{h}\right\rangle, \left\langle\frac{x-a}{h}\right\rangle^2, \left\langle\frac{x-a}{h}\right\rangle^3\right) \circ \mathbf{C} \circ \begin{bmatrix} y_{j-1} \\ yj \\ y_{j+1} \\ y_{j+2} \end{bmatrix}.$$

Hence, for $j = n$ it follows that

$$(8) \qquad p_n^{(n)}(b) = \lim_{x \to b-} p_n^{(n)}(x) =$$

$$= (1, 1, 1, 1) \circ \mathbf{C} \circ \begin{bmatrix} y_{n-1} \\ y_n \\ y_{n+1} \\ y_{n+2} \end{bmatrix} = \frac{1}{16}(0, 0, 16, 0) \circ \begin{bmatrix} y_{n-1} \\ y_n \\ y_{n+1} \\ y_{n+2} \end{bmatrix} = y_{n+1}.$$

By the symbol p_n we shall denote the function $p_n: \langle a, b\rangle \to \mathbf{R}^1$ having the following properties:

$$(9) \qquad p_n\big|_{\langle a+(j-1)h, a+jh)} = p_n^{(j)} \quad \text{for} \quad j = 1, 2, \ldots, n,$$

$$p_n(b) = y_{n+1}.$$

By the symbol $p_n\big|_I$ we denote the restriction of the function p_n to the interval $I = \langle a + (j - 1)h, a + jh)$. By (8) we have $p_n(b) = p_n^{(n)}(b)$.

2. Consider the function $p_n: \langle a, b\rangle \to \mathbf{R}^1$ [see (9)]. For $x \in \langle a + (j - 1)h, a + jh)$, where $j = [(x - a)/h] + 1$, we have

$$(10) \qquad \frac{\mathrm{d}}{\mathrm{d}x} p_n(x) = \frac{\mathrm{d}}{\mathrm{d}x} p_n^{(j)}(x) = \frac{2}{h}\left(1, \left\langle\frac{x-a}{h}\right\rangle, \left\langle\frac{x-a}{h}\right\rangle^2, \left\langle\frac{x-a}{h}\right\rangle^3\right) \circ$$

$$\circ \frac{1}{16}\begin{bmatrix} 1 & -11 & 11 & -1 \\ 2 & -2 & -2 & 2 \\ -3 & 9 & -9 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} y_{j-1} \\ y_j \\ y_{j+1} \\ y_{j+2} \end{bmatrix},$$

thus

$$(11)\qquad \frac{\mathrm{d}}{\mathrm{d}x}\, p_n(a + (j-1)\,h) =$$

$$= \frac{2}{h}(1, -1, 1, -1) \circ \frac{1}{16}\begin{bmatrix} 1 & -11 & 11 & -1 \\ 2 & -2 & -2 & 2 \\ -3 & 9 & -9 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} y_{j-1} \\ y_j \\ y_{j+1} \\ y_{j+2} \end{bmatrix} =$$

$$= \frac{1}{2h}(y_{j+1} - y_{j-1})\,.$$

For $x \to b-$, (10) implies

$$(12)\qquad \frac{\mathrm{d}}{\mathrm{d}x}\, p_n(b) = \lim_{x \to b-} \frac{\mathrm{d}}{\mathrm{d}x}\, p_n^{(n)}(x) =$$

$$= \frac{2}{h}(1, 1, 1, 1) \circ \frac{1}{16}\begin{bmatrix} 1 & -11 & 11 & -1 \\ 2 & -2 & -2 & 2 \\ -3 & 9 & -9 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} y_{n-1} \\ y_n \\ y_{n+1} \\ y_{n+2} \end{bmatrix} = \frac{1}{2h}(y_{n+2} - y_n)\,.$$

If the function f has a finite derivative $f'_+(a)$ at the point $x_1 = a$, we put for $j = 1$

$$\frac{\mathrm{d}}{\mathrm{d}x}\, p_n(a) = \frac{1}{2h}(y_2 - y_0) = f'_+(a)$$

in accordance with (11). From here we determine

$$(13)\qquad y_0 = -2h\,f'_+(a) + y_2\,, \quad P_0 = (a - h, y_0)\,.$$

Similarly, if the function f has a finite derivative $f'_-(b)$ at the point $x_{n+1} = b$ we put

$$\frac{\mathrm{d}}{\mathrm{d}x}\, p_n(b) = \frac{1}{2h}(y_{n+2} - y_n) = f'_-(b)$$

in accordance with (12) and determine

$$(14)\qquad y_{n+2} = 2h\,f'_-(b) + y_n\,, \quad P_{n+2} = (b + h, y_{n+2})\,.$$

Example 1. Consider the function $y = f(x) = x^3 - 3x + 2$ in the interval $\langle 0, 10\rangle$. For $n = 5$ we have $h = (b - a)/n = 10/5 = 2$, further for $x = 8{\cdot}3$ we have $j = [(x - a)/h] + 1 = [8{\cdot}3/2] + 1 = 5$. We have $y_4 = 200$, $y_5 = 490$, $y_6 = 972$. We choose, for instance, $y_7 = y_5$. Then we have [see (7)]

$$p_5^{(5)}(x) = (1, x - 9, (x - 9)^2, (x - 9)^3) \circ \mathbf{C} \circ \begin{bmatrix} 200 \\ 490 \\ 972 \\ 490 \end{bmatrix} =$$

$$= 779{\cdot}25 + 313{\cdot}25(x - 9) - 48{\cdot}25(x - 9)^2 - 72{\cdot}25(x - 9)^3$$

in the interval $8 \leqq x \leqq 10$. For $x = 8{\cdot}3$ it follows that $p_5^{(5)}(8{\cdot}3) = 561{\cdot}11425 \approx \approx f(8{\cdot}3) = 548{\cdot}887$, thus $|f(8{\cdot}3) - p_5^{(5)}(8{\cdot}3)| = 12{\cdot}22725$. We have to realize that under the given choice of the value of y_7 the interpolation curve $\{(x, y) \in \mathbf{R}^2 \mid 0 \leqq \leqq x \leqq 10,\ y = p_5(x)\}$ [see (9)] has a horizontal tangent at the point $b = 10$ [see (12)].

Example 2. Consider once more the function $y = f(x) = x^3 - 3x + 2$ in the interval $\langle 0,10 \rangle$. For $n = 40$ we have $h = 0{\cdot}25$, further for $x = 8{\cdot}3$ we have $j = 34$. We find out that $y_{33} = 490$, $y_{34} = 538{\cdot}76563$, $y_{35} = 590{\cdot}625$, $y_{36} = 645{\cdot}67188$. By (7) we have

$$p_{40}^{(34)}(x) = (1, 8x - 67, (8x - 67)^2, (8x - 67)^3) \circ \mathbf{C} \circ \begin{bmatrix} 490 \\ 538{\cdot}76563 \\ 590{\cdot}625 \\ 645{\cdot}67188 \end{bmatrix} =$$

$$= 564{\cdot}30274 + 25{\cdot}92382(8x - 67) + 0{\cdot}39258(8x - 67)^2 + + 0{\cdot}00586(8x - 67)^3$$

in the interval $8{\cdot}25 \leqq x < 8{\cdot}5$. For $x = 8{\cdot}3$ this implies that $p_{40}^{(34)}(8{\cdot}3) = 548{\cdot}889 \approx \approx f(8{\cdot}3) = 548{\cdot}887$, thus $|f(8{\cdot}3) - p_{40}^{(34)}(8{\cdot}3)| = 0{\cdot}002$.

Example 3. Consider again the function $y = f(x) = x^3 - 3x + 2$ in the interval $\langle 0, 10 \rangle$, let $n = 5$, thus $h = 2$. For $x = 8{\cdot}3$ we have $j = 5$, and similarly as in Ex. 1: $y_4 = 200$, $y_5 = 490$, $y_6 = 972$. Since the finite derivative $f'_-(10) = 297$ exists, it is possible to choose $y_7 = 4(297) + y_5 = 1678$ in accordance with (14). By (7) we then have

$$p_5^{(5)}(x) = (1, x - 9, (x - 9)^2, (x - 9)^3) \circ \mathbf{C} \circ \begin{bmatrix} 200 \\ 490 \\ 972 \\ 1678 \end{bmatrix} =$$

$$= 705 + 239(x - 9) + 26(x - 9)^2 + 2(x - 9)^3$$

in the interval $8 \leqq x \leqq 10$. For $x = 8{\cdot}3$ we have

$$p_5^{(5)}(8{\cdot}3) = 549{\cdot}754 \approx f(8{\cdot}3) = 548{\cdot}887\,,$$

thus $|f(8{\cdot}3) - p_5^{(5)}(8{\cdot}3)| = 0{\cdot}867$. The absolute error of the appromation is in this case substantially smaller than in the case of Ex. 1. In the present case the curves $\{(x, y) \in \mathbf{R}^2 \mid 0 \leqq x \leqq 10,\ y = f(x)\}$, $\{(x, y) \in \mathbf{R}^2 \mid 0 \leqq x \leqq 10,\ y = p_5(x)\}$ have coinciding tangents at the point $b = 10$.

3. Let $N \geqq 1$ be a positive integer, $0 < b - a = L < \infty$. If the finite derivative $f'_+(a)$ exists, we put $K_f(a) = f'_+(a)$ or $K_f(a) = 0$. If the right derivative of the function f at the point a does not exist or if $f'_+(a) = \pm\infty$, then we always put $K_f(a) = 0$.

In the interval $\langle a - (L/N), a\rangle$ let us consider the function $g(x) = 2(x - a) K_f(a) + f(2a - x)$; we have $g(a) = f(a)$ and further $g'_-(a) = f'_+(a)$ for $K_f(a) = f'_+(a)$ finite. If the derivative $f'_-(b)$ exists, then we put $K_f(b) = f'_-(b)$ or $K_f(b) = 0$; if the left derivative of the function f at the point b does not exist or if $f'_-(b) = \pm\infty$, then we always put $K_f(b) = 0$. In the interval $\langle b, b + (L/N)\rangle$ let us consider the function $G(x) = 2(x - b) K_f(b) + f(2b - x)$; we have $G(b) = f(b)$ and further $G'_+(b) = f'_-(b)$ for $K_f(b) = f'_-(b)$ finite. Let q_f: $\langle a - (L/N), b + (L/N)\rangle \to \mathbf{R}^1$ be a function defined as follows:

$$g_f(x) = \begin{cases} g(x) & \text{for} \quad x \in \left\langle a - \dfrac{L}{N}, a\right), \\ f(x) & \text{for} \quad x \in \langle a, b\rangle , \\ G(x) & \text{for} \quad x \in \left(b, b + \dfrac{L}{N}\right\rangle . \end{cases}$$

The function q_f is continuous in the interval $\langle a - (L/N), b + (L/N)\rangle$ and we have $q_f(a) = f(a)$, $q_f(b) = f(b)$. Further, we have $q'_f(a) = f'_+(a)$ for $K_f(a) = f'_+(a)$ finite, $q'_f(b) = f'_-(b)$ for $K_f(b) = f'_-(b)$ finite. For the division of the interval $\langle a, b\rangle$ into $n \geqq N$ equal intervals of length $h = L/n$ we then have $q_f(a - h) = g(a - h) = -2h \,.\, K_f(a) + f(a + h) = -2h \,.\, K_f(a) + f(x_2) = -2h \,.\, K_f(a) + y_2$, i.e. $q_f(a - h) = y_0$ for $K_f(a) = f'_+(a)$ finite [see (13)]. For $K_f(a) = 0$ (with the exception of the case $f'_+(a) = 0$) we put

$$y_0 \overset{\text{def.}}{=} q_f(a - h) = y_2 \,. \tag{15}$$

Similarly we have $q_f(b + h) = G(b + h) = 2h \,.\, K_f(b) + f(b - h) = 2h \,.\, K_f(b) + f(x_n) = 2h \,.\, K_f(b) + y_n$, i.e. $q_f(b + h) = y_{n+2}$ for $K_f(b) = f'_-(b)$ finite [see (14)]. For $K_f(b) = 0$ (with the exception of the case $f'_-(b) = 0$) we put

$$y_{n+2} \overset{\text{def.}}{=} q(b + h) = y_n \,. \tag{16}$$

For $x_j = a + (j - 1) h$, $j = 0, 1, 2, \ldots, n + 2$, we thus have $q_f(x_j) = y_j$.

Since the function q_f is continuous in the interval $\langle a - (L/N), b + (L/N)\rangle$, it is uniformly continuous in this interval. Consequently, to a given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all points $x', x'' \in \langle a - (L/N), b + (L/N)\rangle$ whose distance $|x' - x''|$ is less than δ we have $|q_f(x') - q_f(x'')| < \varepsilon$. We put $n_0 = \max\{N, [3L/\delta] + 1\}$. For $n > n_0$ and $z \in \langle a, b)$, $|x_{k-1} - z| < \delta$ holds for the respective $j = [(z - a)/h] + 1$ and for $k = j, j + 1, j + 2, j + 3$, thus $|y_{k-1} - f(z)| < \varepsilon$ for these numbers k. Consequently, $y_{k-1} = f(z) + \Delta_{k-1}$ holds for these numbers k, where

$$|\Delta_{k-1}| < \varepsilon \,. \tag{17}$$

Further, for $x \in \langle a + (j-1)h, a + jh)$ we have, by (7), (9), the following equalities:

$$p_n(x) = p_n^{(j)}(x) = \left(1, \left\langle\frac{x-a}{h}\right\rangle, \left\langle\frac{x-a}{h}\right\rangle^2, \left\langle\frac{x-a}{h}\right\rangle^3\right) \circ$$

$$\circ \frac{1}{16}\begin{bmatrix} -1 & 9 & 9 & -1 \\ 1 & -11 & 11 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix} \circ \begin{bmatrix} f(z) + \Delta_{j-1} \\ f(z) + \Delta_j \\ f(z) + \Delta_{j+1} \\ f(z) + \Delta_{j+2} \end{bmatrix} =$$

$$= \left(1, \left\langle\frac{x-a}{h}\right\rangle, \left\langle\frac{x-a}{h}\right\rangle^2, \left\langle\frac{x-a}{h}\right\rangle^3\right) \circ$$

$$\circ \frac{1}{16}\begin{bmatrix} 16f(z) - \Delta_{j-1} + 9\Delta_j + 9\Delta_{j+1} - \Delta_{j+2} \\ \Delta_{j-1} - 11\Delta_j + 11\Delta_{j+1} - \Delta_{j+2} \\ \Delta_{j-1} - \Delta_j - \Delta_{j+1} + \Delta_{j+2} \\ -\Delta_{j-1} + 3\Delta_j - 3\Delta_{j+1} + \Delta_{j+2} \end{bmatrix}.$$

Thus, for $x = z$ we have

(18) $$p_n(z) - f(z) = \frac{1}{16}\left\{-\Delta_{j-1} + 9\Delta_j + 9\Delta_{j+1} - \Delta_{j+2} + \right.$$
$$+ \left\langle\frac{z-a}{h}\right\rangle [\Delta_{j-1} - 11\Delta_j + 11\Delta_{j+1} - \Delta_{j+2}] +$$
$$+ \left\langle\frac{z-a}{h}\right\rangle^2 [\Delta_{j-1} - \Delta_j - \Delta_{j+1} + \Delta_{j+2}] +$$
$$\left. + \left\langle\frac{z-a}{h}\right\rangle^3 [-\Delta_{j-1} + 3\Delta_j - 3\Delta_{j+1} + \Delta_{j+2}]\right\}.$$

Since $-1 \leqq \langle(z-a)/h\rangle < 1$, i.e.

(19) $$\left|\left\langle\frac{z-a}{h}\right\rangle\right| < 1,$$

we obtain by (17), (18), (19) the relation

(20) $$|p_n(z) - f(z)| < \frac{56}{16}\varepsilon = 3\cdot5\varepsilon$$

which holds for all $n > n_0$ and arbitrary $z \in \langle a, b)$. Since for every n we have $p_n(b) = y_{n+1} = f(x_{n+1}) = f(b)$ [see (9)], (20) holds for all $n > n_0$ and arbitrary $z \in \langle a, b\rangle$. This proves that

(21) $$\lim_{n\to\infty} p_n(x) = f(x) \quad \text{uniformly in the interval} \quad \langle a, b\rangle.$$

Example 4. Consider the function $f(x) = 0{\cdot}001x^3 + x$ in the interval $\langle 0, 10\rangle$. By (15) we construct, for $N = 5$, $K_f(0) = f'_+(0) = 1$, $K_f(10) = f'_-(10) = 1{\cdot}3$, the function

$$q_f(x) = \begin{cases} -0{\cdot}001x^3 + x & \text{for} \quad x \in \langle -2, 0) , \\ 0{\cdot}001x^3 + x & \text{for} \quad x \in \langle 0, 10\rangle , \\ -0{\cdot}001x^3 + 0{\cdot}06x^2 + 0{\cdot}4x + 2 & \text{for} \quad x \in (10, 12\rangle . \end{cases}$$

For $x, x_0 \in \langle -2, 0\rangle$ we have $|q_f(x) - q_f(x_0)| = |x - x_0| \, |0{\cdot}001[x^2 + xx_0 + x_0^2] - 1| \leqq |x - x_0|$; for $|x - x_0| < \delta_1 = \varepsilon/2$ we then have $|q_f(x) - q_f(x_0)| < \varepsilon/2$. Further, for $x, x_0 \in \langle 0, 10\rangle$ we have $|q_f(x) - q_f(x_0)| = |x - x_0| \, |0{\cdot}001|[x^2 + xx_0 + x_0^2] + 1| \leqq (1{\cdot}3)\, |x - x_0|$; for $|x - x_0| < \delta_2 = (\varepsilon/2)(1{\cdot}3)^{-1}$ we then have $|q_f(x) - q_f(x_0)| < \varepsilon/2$. Finally, for $x, x_0 \in \langle 10, 12\rangle$ we have $|q_f(x) - q_f(x_0)| = |x - x_0| \, |0{\cdot}001[x^2 + xx_0 + x_0^2] - 0{\cdot}06[x + x_0] - 0{\cdot}4| \leqq (1{\cdot}408)\, |x - x_0|$; for $|x - x_0| < \delta_3 = (\varepsilon/2)(1{\cdot}408)^{-1}$ we then have $|q_f(x) - q_f(x_0)| < \varepsilon/2$. If we put $\delta = \min\{\delta_1, \delta_2, \delta_3\} = (\varepsilon/2)(1{\cdot}408)^{-1}$, then for arbitrary points $x', x'' \in \langle -2, 12\rangle$ whose distance $|x' - x''|$ is less than δ we have $|q_f(x') - q_f(x'')| < 2(\varepsilon/2) = \varepsilon$. By (20)

$$|p_n(x) - f(x)| < 3{\cdot}5\varepsilon \tag{22}$$

holds for all $n > n_0 = \max\{5, [3L/\delta] + 1\} = \max\{5, [295{\cdot}68/\varepsilon] + 1\}$ and arbitrary $x \in \langle 0, 10\rangle$. For instance, for $\varepsilon = 0{\cdot}7$ we have $n_0 = 423$. Since the estimate (20) which is based on (17) and (19) is in a sense "rough", we may expect inequality (22) to hold for all $x \in \langle 0, 10\rangle$ for substantially smaller n_0.

A continuous extension q_f of the function f from the interval $\langle a, b\rangle$ to an interval $\langle a - (L/N), b + (L/N)\rangle$, which we have constructed above, can obviously be replaced by any other continuous extension. This fact is mentioned in the program No. 1 as a possibility to choose the values of "derivatives" at limit points of the interval $\langle a, b\rangle$, by using any method, directly from the keyboard.

4. By (7) we have

$$p_n^{(j)}(x) = \left(1, \left\langle\frac{x - a}{h}\right\rangle, \left\langle\frac{x - a}{h}\right\rangle^2, \left\langle\frac{x - a}{h}\right\rangle^3\right) \circ \mathbf{C} \circ \begin{bmatrix} y_{j-1} \\ y_j \\ y_{j+1} \\ y_{j+2} \end{bmatrix} \tag{23}$$

in the interval $\langle x_j, x_j + h) \equiv \langle a + (j - 1)\, h, a + jh)$. Then

$$\int_{x_j}^{x_j+h} p_n^{(j)}(x)\, dx = \left(\int_{x_j}^{x_j+h} dx , \int_{x_j}^{x_j+h} \left\langle\frac{x - a}{h}\right\rangle dx , \int_{x_j}^{x_j+h} \left\langle\frac{x - a}{h}\right\rangle^2 dx , \right. \tag{24}$$

$$\left. \int_{x_j}^{x_j+h} \left\langle\frac{x - a}{h}\right\rangle^3 dx\right) \circ \mathbf{C} \circ \begin{bmatrix} y_{j-1} \\ y_j \\ y_{j+1} \\ y_{j+2} \end{bmatrix} .$$

We have

$$(25)\qquad \int_{x_j}^{x_j+h} \mathrm{d}x = h\,,$$

$$\int_{x_j}^{x_j+h} \left\langle \frac{x-a}{h} \right\rangle \mathrm{d}x = \int_{x_j}^{x_j+h} \left[2\,\frac{x-a}{h} - (2j-1) \right] \mathrm{d}x =$$

$$= \frac{h}{2} \int_{-1}^{1} u \,\mathrm{d}u = 0\,,$$

$$\int_{x_j}^{x_j+h} \left\langle \frac{x-a}{h} \right\rangle^2 \mathrm{d}x = \int_{x_j}^{x_j+h} \left[2\,\frac{x-a}{h} - (2j-1) \right]^2 \mathrm{d}x =$$

$$= \frac{h}{2} \int_{-1}^{1} u^2 \,\mathrm{d}u = \frac{h}{3}\,,$$

$$\int_{x_j}^{x_j+h} \left\langle \frac{x-a}{h} \right\rangle^3 \mathrm{d}x = \int_{x_j}^{x_j+h} \left[2\,\frac{x-a}{h} - (2j-2) \right]^3 \mathrm{d}x =$$

$$= \frac{h}{2} \int_{-1}^{1} u^3 \,\mathrm{d}u = 0\,,$$

whence, substituting (25) into (24), we obtain

$$(26)\qquad \int_{x_j}^{x_j+h} p_n^{(j)}(x)\,\mathrm{d}x = \frac{h}{3}(3, 0, 1, 0) \circ \frac{1}{16} \begin{bmatrix} -1 & 9 & 9 & -1 \\ 1 & -11 & 11 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix} \circ \begin{bmatrix} y_{j-1} \\ y_j \\ y_{j+1} \\ y_{j+2} \end{bmatrix} =$$

$$= \frac{h}{24}(-1, 13, 13, -1) \circ \begin{bmatrix} y_{j-1} \\ y_j \\ y_{j+1} \\ y_{j+2} \end{bmatrix} =$$

$$= \frac{h}{24}(-y_{j-1} + 13y_j + 13y_{j+1} - y_{j+2})\,.$$

By (9), (26) we then have, for $n \geqq 4$,

$$(27)\qquad \int_a^b p_n(x)\,\mathrm{d}x = \sum_{j=1}^{n} \int_{x_j}^{x_j+h} p_n^{(j)}(x)\,\mathrm{d}x =$$

$$= \frac{h}{24}\left[-(y_0 + y_{n+2}) + 12(y_1 + y_{n+1}) + 25(y_2 + y_n) + \right.$$

$$\left. + 24(y_3 + y_4 + \ldots + y_{n-1})\right] = \mathscr{L}_n\,.$$

Since by (21) $\lim_{n\to\infty} p_n(x) = f(x)$ holds uniformly in the interval $\langle a, b\rangle$ we have

(28) $$\lim_{n\to\infty} \int_a^b p_n(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} p_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x\,,$$

which means

(29) $$\int_a^b f(x)\,\mathrm{d}x \approx \mathscr{L}_n\,.$$

Example 5. We are to determine the approximate value of the integral $I = \int_0^6 [1/(1 + x^2)]\,\mathrm{d}x = \operatorname{arctg} 6 = 1{\cdot}40564765$ if we put $n = 4$ in relation (29), that is $h = 1{\cdot}5$. We have $y_1 = 1$, $y_2 = 0{\cdot}30769231$, $y_3 = 0{\cdot}1$, $y_4 = 0{\cdot}04705882$, $y_5 = 0{\cdot}02702703$, further for $K_f(0) = f'_+(0) = 0$ we have $y_0 = y_2 = 0{\cdot}30769231$ [see (13)], while for $K_f(6) = f'_-(6) = -0{\cdot}00876552$ we have $y_6 = 2h\,.\,K_f(6) + y_4 = 0{\cdot}02076225$ [see (14)]. Consequently, by (27), (29)

$$I \approx \mathscr{L}_4 = \frac{1{\cdot}5}{24}\,[-(0{\cdot}30769231 + 0{\cdot}02076225) +$$
$$+ 12(1 + 0{\cdot}02702703) + 25(0{\cdot}30769231 + 0{\cdot}04705882) +$$
$$+ 24(0{\cdot}1)] = 1{\cdot}4504050$$

holds, thus $|I - \mathscr{L}_4| = 0{\cdot}04839285$. Applying the Simpson formula $\mathscr{S}_{2n}$ for $n = 2$ we obtain

$$I \approx \mathscr{S}_4 = \frac{1{\cdot}5}{3}\,[y_1 + y_5 + 2y_3 + 4(y_2 + y_4)] = 1{\cdot}32301578\,,$$

so that $|I - \mathscr{S}_4| = 0{\cdot}08263188$. We see that in this particular case formula $\mathscr{L}_4$ yields a better approximation of the integral I than formula $\mathscr{S}_4$.

Example 6. We are to determine the approximate value of the integral $I = \int_0^{1.2} \operatorname{tg} x\,\mathrm{d}x = 1{\cdot}01512328$ if we put $n = 6$ in relation (29), thus $h = 0{\cdot}2$. We have $y_1 = 0$, $y_2 = 0{\cdot}20271004$, $y_3 = 0{\cdot}42279322$, $y_4 = 0{\cdot}68413681$, $y_5 = 1{\cdot}02963856$, $y_6 = 1{\cdot}55740773$, $y_7 = 2{\cdot}57215162$, further for $K_f(0) = f'_+(0) = 1$ we have $y_0 = -2h\,.\,K_f(0) + y_2 = -0{\cdot}19728996$ [see (13)], for $K_f(1{\cdot}2) = f'_-(1{\cdot}2) = 7{\cdot}61596397$ we have $y_8 = 2h\,.\,K_f(1{\cdot}2) + y_6 = 4{\cdot}603793317$ [see (14)]. Consequently, by (27), (29) we have $I \approx \mathscr{L}_6 = 1{\cdot}01449922$, so that $|I - \mathscr{L}_6| = 0{\cdot}00062406$. Applying the Simpson formula $\mathscr{S}_{2n}$ for $n = 3$ we obtain $I \approx \mathscr{S}_6 = 1{\cdot}01693557$, thus $|I - \mathscr{S}_6| = 0{\cdot}00181229$. Even for this case formula $\mathscr{L}_6$ yields a better aproximation of the integral I than formula $\mathscr{S}_6$.

If we choose, e.g., $n = 24$ and determine the value y_0 or y_{26} by (13) or (14), respectively, we find out that $|I - \mathscr{L}_{24}| = 0{\cdot}00000271$ while $|I - \mathscr{S}_{24}| = 0{\cdot}00001052$. Thus we obtain a better approximation of the integral I in the first case again.

Example 7. We are to determine the approximate value of the integral $I = \int_0^2 (4 - x^2)^{3/2} \, dx = 9{\cdot}42477796$ if we put $n = 4$ in formula (29), thus $h = 0{\cdot}5$. We have $y_1 = 8$, $y_2 = 7{\cdot}26184377$, $y_3 = 5{\cdot}19615242$, $y_4 = 2{\cdot}31503239$, $y_5 = 0$, further for $K_f = 0 = f'_+(0) = 0$ we have $y_0 = y_2$ [see (13)], for $K_f(2) = f'_-(2) = 0$ we have $y_6 = y_4$ [see (14)]. Consequently, by (27), (29) we then have $I \approx \mathscr{L}_4 = 9{\cdot}38651429$, i.e. $|I - \mathscr{L}_4| = 0{\cdot}03826367$. Applying the Simpson formula $\mathscr{S}_{2n}$ for $n = 2$ we obtain $I \approx \mathscr{S}_4 = 9{\cdot}44996825$, thus $|I - \mathscr{S}_4| = 0{\cdot}02519029$. In this case formula $\mathscr{S}_4$ yields a better approximation of the integral I then formula $\mathscr{L}_4$.

Note that in the case when $y_0 = y_2$ and $y_{n+2} = y_n$, $\mathscr{L}_n$ is equal to $\mathscr{T}_{1,n} = (h/2)[y_1 + y_n + 2(y_2 + y_3 + \ldots + y_{n-1})]$. We know that the first trapezoidal method for the approximate computation of the integral $\int_a^b f(x) \, dx$ leads to the formula $\mathscr{T}_{1,n}$.

Example 8. We are to determine the approximate value of the integral $I = \int_{-3}^6 \sqrt{(36 - x^2)} \, dx = 45{\cdot}49334048$ if we put $n = 18$ in relation (29), thus $h = 0{\cdot}5$. We find out that for $K_f(-3) = f'_+(-3) = 0{\cdot}57735027$ we have $y_0 = -2h \,.\, K_f(-3) + y_2 = 4{\cdot}87700579$ [see (13)]. Since $f'_-(6) = -\infty$, we have $K_f(6) = 0$ in the relation at the beginning of Section 3, thus $y_{20} = y_{18} = 2{\cdot}39791576$ by (16). By (27), (29) we then have $I \approx \mathscr{L}_{18} = 45{\cdot}23938825$, i.e. $|I - \mathscr{L}_{18}| = 0{\cdot}25395223$ while $|I - \mathscr{S}_{18}| = 0{\cdot}09981280$. Consequently, formula $\mathscr{S}_{18}$ yields a better approximation of the integral I than formula $\mathscr{L}_{18}$.

5. In the interval $\langle a, b\rangle$ of the length $0 < b - a = L < \infty$ we consider a function f which possesses in the interval continuous derivatives of at fourth order (at the endpoints of the interval the derivatives are understood to be onesided). When deriving the formula which yields the approximate value of the integral $\int_a^b f(x) \, dx$, we replace the function f in accordance with (26) in every partial interval $\langle x_j, x_j + h\rangle \equiv \langle a + (j - 1) h, \; a + jh\rangle$, $j = 1, 2, \ldots, n$ $(n \geqq 4)$, by the constant

$$c_j = \frac{1}{24}[-y_{j-1} + 13y_j + 13y_{j+1} - y_{j+2}] ;$$

for $1 \leqq k \leqq n + 1$ y_k denotes the value of the function f at the point $x_k = a + (k - 1) h$. By (13) we further have $y_0 = -2h \,.\, f'(a) + y_2$, by (14) we have $y_{n+2} = 2h \,.\, f'(b) + y_n$. If

$$\left| \int_{x_j}^{x_j + h} [f(x) - c_j] \, dx \right| \leqq A$$

holds for $j = 1, n$, and

$$\left| \int_{x_j}^{x_j + h} [f(x) - c_j] \, dx \right| \leqq B$$

for $1 < j < n$, then

$$(30)\qquad \left|\int_a^b f(x)\,\mathrm{d}x - \mathscr{L}_n\right| \leqq 2A + (n-2)\,B\,.$$

We express the function f and the constant c_j with the aid of the Taylor formula with centre at the point x_j. Assume that $|f'''(x)| \leqq A_3$, $|f^{\mathrm{IV}}(x)| \leqq B_4$ for $x \in \langle a, b\rangle$. Integrating from x_j to $x_j + h$ we obtain an expansion in the powers of the increment h. This expansion is terminated with the first nonzero summand written down in the form of the remainder and estimated by the brackets A_3 and B_4. We have

$$c_1 = \frac{1}{24}\left[2h\,f'(a) - f(a+h) + 13 f(a) + 13 f(a+h) - f(a+2h)\right] =$$

$$= \frac{h}{12} f'(a) + \frac{13}{24} f(a) +$$

$$+ \frac{1}{2}\left[f(a) + h\,f'(a) + \frac{h^2}{2} f''(a) + \frac{h^3}{6} f'''(a) + \frac{h^4}{24} f^{\mathrm{IV}}(a) + \ldots\right] -$$

$$- \frac{1}{24}\left[f(a) + 2h\,f'(a) + \frac{4h^2}{2} f''(a) + \frac{8h^3}{6} f'''(a) + \frac{16h^4}{24} f^{\mathrm{IV}}(a) + \ldots\right] =$$

$$= f(a) + \frac{h}{2} f'(a) + \frac{h^2}{6} f''(a) + \frac{h^3}{36} f'''(a) - \frac{h^4}{144} f^{\mathrm{IV}}(a) + \ldots,$$

thus

$$(31)\qquad \left|\int_a^{a+h} [f(x) - c_1]\,\mathrm{d}x\right| =$$

$$= \left|\int_a^{a+h}\left[f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2} f''(a) + \ldots - f(a) -\right.\right.$$

$$\left.\left. - \frac{h}{2} f'(a) - \frac{h^2}{6} f''(a) - \ldots\right]\mathrm{d}x\right| =$$

$$= \left|\frac{h^2}{2} f'(a) + \frac{h^3}{6} f''(a) + \frac{h^4}{24} f'''(a) + \ldots - \frac{h^2}{2} f'(a) -\right.$$

$$\left. - \frac{h^3}{6} f''(a) - \frac{h^4}{36} f'''(a) + \ldots\right| =$$

$$= \left|\frac{h^4}{72} f'''(a) + \ldots\right| = \frac{h^4}{72}\left|f'''(\xi_1)\right| \leqq \frac{h^4}{72} A_3\,.$$

Further, we have

$$c_n = \frac{1}{24}\left[-f(b-2h) + 13f(b-h) + 13f(b) - 2hf'(b) - f(b-h)\right] =$$

$$= -\frac{h}{12}f'(b) + \frac{13}{24}f(b) + \frac{1}{2}\left[f(b) - hf'(b) + \frac{h^2}{2}f''(b) - \right.$$

$$\left. - \frac{h^3}{6}f'''(b) + \frac{h^4}{24}f^{\mathrm{IV}}(b) - \dots\right] -$$

$$- \frac{1}{24}\left[f(b) - 2hf'(b) + \frac{4h^2}{2}f''(b) - \frac{8h^3}{6}f'''(b) + \frac{16h^4}{24}f^{\mathrm{IV}}(b) - \dots\right] =$$

$$= f(b) - \frac{h}{2}f'(b) + \frac{h^2}{6}f''(b) - \frac{h^3}{36}f'''(b) - \frac{h^4}{144}f^{\mathrm{IV}}(b) + \dots,$$

thus

(32) $$\left|\int_{b-h}^{b}[f(x) - c_n]\,\mathrm{d}x\right| =$$

$$= \left|\int_{b-h}^{b}\left[f(b) + (x-b)f'(b) + \frac{(x-b)^2}{4}f''(b) + \dots - f(b) + \right.\right.$$

$$\left.\left. + \frac{h}{2}f'(b) - \frac{h^2}{6}f''(b) + \dots\right]\mathrm{d}x\right| =$$

$$= \left|-\frac{h^2}{2}f'(b) + \frac{h^3}{6}f''(b) - \frac{h^4}{24}f'''(b) + \dots + \frac{h^2}{2}f'(b) - \right.$$

$$\left. - \frac{h^3}{6}f''(b) + \frac{h^4}{36}f'''(b) + \dots\right| =$$

$$= \left|-\frac{h^4}{72}f'''(b) + \dots\right| = \frac{h^4}{72}\left|f'''(\xi_n)\right| \leqq \frac{h^4}{72}A_3\,.$$

Finally, for $1 < j < n$ we have

(33) $$c_j = \frac{1}{24}\left[-f(x_j - h) + 13f(x_j) + 13f(x_j + h) - f(x_j + 2h)\right] =$$

$$= \frac{13}{24}f(x_j) - \frac{1}{24}\left[f(x_j) - hf'(x_j) + \frac{h^2}{2}f''(x_j) - \frac{h^3}{6}f'''(x_j) + \right.$$

$$\left. + \frac{h^4}{24}f^{\mathrm{IV}}(x_j) - \dots\right] +$$

$$+ \frac{13}{24}\left[f(x_j) + hf'(x_j) + \frac{h^2}{2}f''(x_j) + \frac{h^3}{6}f'''(x_j) + \frac{h^4}{24}f^{\mathrm{IV}}(x_j) + \dots\right] -$$

$$- \frac{1}{24}\left[f(x_j) + 2h\,f'(x_j) + \frac{4h^2}{2} f''(x_j) + \frac{8h^3}{6} f'''(x_j) + \right.$$

$$\left. + \frac{16h^4}{24} f^{\mathrm{IV}}(x_j) + \ldots \right] =$$

$$= f(x_j) + \frac{h}{2} f'(x_j) + \frac{h^2}{6} f''(x_j) + \frac{h^3}{24} f'''(x_j) - \frac{h^4}{144} f^{\mathrm{IV}}(x_j) + \ldots,$$

thus

$$(34) \qquad \left| \int_{x_j}^{x_j+h} [f(x) - c_j]\, \mathrm{d}x \right| =$$

$$= \left| \int_{x_j}^{x_j+h} \left[f(x_j) + (x - x_j) f'(x_j) + \frac{(x - x_j)^2}{2} f''(x_j) + \ldots - f(x_j) - \right.\right.$$

$$\left.\left. - \frac{h}{2} f'(x_j) - \frac{h^2}{6} f''(x_j) - \ldots \right] \mathrm{d}x \right| =$$

$$= \left| \frac{h^2}{2} f'(x_j) + \frac{h^3}{6} f''(x_j) + \frac{h^4}{24} f'''(x_j) + \frac{h^5}{120} f^{\mathrm{IV}}(x_j) + \ldots - \right.$$

$$\left. - \frac{h^2}{2} f'(x_j) - \frac{h^3}{6} f''(x_j) + \frac{h^4}{24} f'''(x_j) + \frac{h^5}{144} f^{\mathrm{IV}}(x_j) + \ldots \right| =$$

$$= \left| \frac{11h^5}{720} f^{\mathrm{IV}}(x_j) + \ldots \right| = \frac{11h^5}{720} \left| f^{\mathrm{IV}}(\xi_j) \right| \leqq \frac{11h^5}{720} B_4 .$$

By (30), (31), (32), (33), (34) we then have

$$(35) \qquad \left| \int_a^b f(x)\, \mathrm{d}x - \mathscr{L}_n \right| \leqq \frac{h^4}{36} A_3 + \frac{11(n-2)}{720} B_4 .$$

If we put $K = \max\{A_3, B_4\}$, it is possible to estimate the lefthand side of inequality (35) as follows:

$$(36) \qquad \left| \int_a^b f(x)\, \mathrm{d}x - \mathscr{L}_n \right| \leqq \frac{L^4 K}{36 n^4} (1 + L) .$$

Consequently, the precision of formula $\mathscr{L}_n$ is proportional to the number n^4 similarly as in the case of the Simpson formula. Therefore the precision increases rapidly with the number of partial intervals.

Example 9. We are to determine the value of the complete elliptic integral of the second kind $\mathrm{E}(1/\sqrt{2}) = \int_0^{\pi/2} \sqrt{[1 - (1/2)\sin^2 x]}\, \mathrm{d}x$ with precision up to 0·001. For $x \in \langle 0, \pi/2 \rangle$ we have $y = f(x) = \sqrt{[1 - (1/2)\sin^2 x]} \geqq 1/\sqrt{2}$. Differentiating the equality $y^2 = 1 - (1/2)\sin^2 x$ we find out easily that $|f'''(x)| \leqq A_3 = 3$,

$|f^{IV}(x)| \leqq B_4 = 12$ hold for $x \in \langle 0, \pi/2 \rangle$. By (35) we thus have

$$\left|E\left(\frac{1}{\sqrt{2}}\right) - \mathscr{L}_n\right| \leqq \frac{\left(\frac{\pi}{2}\right)^4}{36n^4} \cdot 3 + \frac{11(n-2)\left(\frac{\pi}{2}\right)^5}{720n^5} \cdot 12 < \frac{7}{12n^4} + \frac{11}{6}\frac{n-2}{n^5},$$

since $(\pi/2)^4 < 7$, $(\pi/2)^5 < 10$. If we choose $n = 7$, then $|E(1/\sqrt{2}) - \mathscr{L}_7| < 0{\cdot}00079$. We have $y_1 = 1$, $y_2 = 0{\cdot}98754$, $y_3 = 0{\cdot}95177$, $y_4 = 0{\cdot}89756$, $y_5 = 0{\cdot}83328$, $y_6 = 0{\cdot}77079$, $y_7 = 0{\cdot}72440$, $y_8 = 0{\cdot}70711$. Since $K_f(0) = f'_+(0) = 0$, $K_f(\pi/2) = f'_-(\pi/2) = 0$, we have $y_0 = y_2$ by (13) and $y_9 = y_7$ by (14). Then $\mathscr{L}_7 = \mathscr{T}_{1,7} = 1{\cdot}35063\ldots$ [see Ex. 7], thus $1{\cdot}34984 < E(1/\sqrt{2}) < 1{\cdot}35142$. Thus we find out that $E(1/\sqrt{2}) = 1{\cdot}3506_{\pm 0.001}$. In the obtained result $\mathscr{L}_7 = 1{\cdot}35063\ldots$, all the decimal places are actually valid.

6. Let $0 < b - a = L < \infty$, let $N \geqq 4$ be a fixed chosen positive integer. In the interval $\langle a - (L/N), b + (L/N) \rangle$ let us consider a function f which possesses continuous derivatives of at least fourth order. We assume that $|f^{IV}(x)| \leqq \tilde{B}_4$ in the interval. When deriving the formula which yields the approximate value of the integral $\int_a^b f(x)\,dx$, we use (26) and replace the function f in every partial interval $\langle x_j, x_j + h \rangle \equiv \langle a + (j-1)h, a + jh \rangle$, $j = 1, 2, \ldots, n$ $(n \geqq N)$, again by the constant

$$c_j = \frac{1}{24}\left[-y_{j-1} + 13y_j + 13y_{j+1} - y_{j+2}\right].$$

Here, for $0 \leqq k \leqq n + 2$ y_k denotes the value of the function f at the point $x_k = a + (k-1)h$. Similarly as above [cf. (27)] we denote

$$\tilde{\mathscr{L}}_n = \frac{h}{24}\left[-(y_0 + y_{n+2}) + 12(y_1 + y_{n+1}) + 25(y_2 + y_n) + 24(y_3 + y_4 + \ldots + y_{n-1})\right]. \tag{37}$$

Then (33) is valid even for $1 \leqq j \leqq n$, thus [cf. (34)]

$$\left|\int_{x_j}^{x_j+h} [f(x) - c_j]\,dx\right| = \left|\frac{11h^5}{720} f^{IV}(x_j) + \ldots\right| = \frac{11h^5}{720}|f^{IV}(\xi_j)| \leqq \frac{11h^5}{720}\tilde{B}_4.$$

Hence we obtain

$$\left|\int_a^b f(x)\,dx - \tilde{\mathscr{L}}_n\right| \leqq \frac{11nh^5}{720}\tilde{B}_4 = \frac{11L^5}{720n^4}\tilde{B}_4, \tag{38}$$

thus

$$\int_a^b f(x)\,dx \approx \tilde{\mathscr{L}}_n. \tag{39}$$

Example 10. We are to determine the approximate value of the integral $I = \int_0^6 [1/(1 + x^2)]\, dx = \operatorname{arctg} 6 = 1{\cdot}40564765$ if we put $n = 4$ in relation (39) We find out that $y_0 = 0{\cdot}30769231$, $y_1 = 1$, $y_2 = 0{\cdot}30769231$, $y_3 = 0{\cdot}1$, $y_4 = 0{\cdot}04705882$, $y_5 = 0{\cdot}02702703$, $y_6 = 0{\cdot}01746725$, thus by (37), (39) we have $I \approx \tilde{\mathscr{L}}_4 = 1{\cdot}45424644$, i.e. $|I - \tilde{\mathscr{L}}_4| = 0{\cdot}04859879$. Comparing with Ex. 5 we see that in this case formula $\tilde{\mathscr{L}}_4$ yields a worse approximation of the integral I than formula $\mathscr{L}_4$ and a better approximation than formula $\mathscr{S}_4$.

Example 11. We are to determine the approximate value of the integral $I = \int_{-2}^{8} (x^3 - 3x + 2)\, dx = 950$ if we put $n = 5$. We find out that $y_0 = -50$, $y_1 = 0$, $y_2 = 2$, $y_3 = 4$, $y_4 = 54$, $y_5 = 200$, $y_6 = 490$, $y_7 = 972$, thus we have $I \approx \tilde{\mathscr{L}}_5 = 950$ in accordance with (37), (39). We have obtained the exact value of the integral I. Since $\tilde{B}_4 = 0$, $I = \tilde{\mathscr{L}}_n$ follows from (38) for $n \geqq 4$.

7. Let $0 < b - a = L < \infty$, $m > 2$ integer. In the interval $\langle a, b \rangle$ let us consider a continuous vector function $\boldsymbol{f} = (f_1, f_2, \ldots, f_{m-1}) : \langle a, b \rangle \to \mathbf{R}^{m-1}$ and, further, the continuous curve $k = \{(x, y_1, y_2, \ldots, y_{m-1}) \in \mathbf{R}^m \mid a \leqq x \leqq b,\ y_i = f_i(x),\ i = 1, 2, \ldots, m - 1\}$. We denote $\tilde{y} = (y_1, y_2, \ldots, y_{m-1})$, $(x, y_1, y_2, \ldots, y_{m-1}) = (x, \tilde{y})$. Let $n \geqq 2$ be a positive integer. We divide the interval $\langle a, b \rangle$ into n equally large intervals with the dividing points $a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b$; we have $x_j = a + (j - 1)\, h$ for $j = 1, 2, \ldots, n + 1$, where $h = L/n$. For $j = 1, 2, \ldots, n + 1$ we put $f(x_j) = \tilde{y}_j = (y_{1,j}, y_{2,j}, \ldots, y_{m-1,j})$ and denote by P_j the point $(x_j, y_{1,j}, y_{2,j}, \ldots, y_{m-1,j}) = (x_j, \tilde{y}_j) \in k$. For every $n \geqq 2$ we have $P_1 = (a, f(a))$, $P_{n+1} = (b, f(b))$.

Using the Lienhard interpolation method we construct an interpolation curve passing through the points $P_1, P_2, \ldots, P_n, P_{n+1}$. The j-th arc P_jP_{j+1}, $j = 1, 2, \ldots, n$, of this curve is parametrized by polynomials in the real variable $t \in \langle -1, 1 \rangle$

$$(40) \qquad x = P_0^{(j)}(t) = (1, t, t^2, t^3) \circ \mathbf{C} \circ \begin{bmatrix} x_{j-1} \\ x_j \\ x_{j+1} \\ x_{j+2} \end{bmatrix},$$

$$(41) \qquad y_i = P_i^{(j)}(t) = (1, t, t^2, t^3) \circ \mathbf{C} \circ \begin{bmatrix} y_{i,j-1} \\ y_{i,j} \\ y_{i,j+1} \\ y_{i,j+2} \end{bmatrix}, \quad i = 1, 2, \ldots, m - 1,$$

where $\mathbf{C}$ is the matrix (3) while $P_0^{(j)}$ is the polynomial (4). For $j = 1$ and $j = n$ the values $y_{i,0}$, and $y_{i,n+2}$, respectively, may be chosen more or less arbitrary [cf. Section 1]. By (6), (7), in the interval $\langle a + (j - 1)\, h, a + jh \rangle$ we similarly have the relation

$$(42)\qquad y_i = P_i^{(j)} \circ [P_0^{(j)}]^{-1}(x) = p_{i,n}^{(j)}(x) =$$

$$= \left(1, \left\langle\frac{x-a}{h}\right\rangle, \left\langle\frac{x-a}{h}\right\rangle^2, \left\langle\frac{x-a}{h}\right\rangle^3\right) \circ \mathbf{C} \circ \begin{bmatrix} y_{i,j-1} \\ y_{i,j} \\ y_{i,j+1} \\ y_{i,j+2} \end{bmatrix},$$

$$i = 1, 2, \ldots, m-1 .$$

For $j = n$ this implies

$$(43)\qquad p_{i,n}^{(n)}(b) = \lim_{x \to b-} p_{i,n}^{(n)}(x) =$$

$$= (1, 1, 1, 1) \circ \mathbf{C} \circ \begin{bmatrix} y_{i,n-1} \\ y_{i,n} \\ y_{i,n+1} \\ y_{i,n+2} \end{bmatrix} = \frac{1}{16}(0, 0, 16, 0) \circ \begin{bmatrix} y_{i,n-1} \\ y_{i,n} \\ y_{i,n+1} \\ y_{i,n+2} \end{bmatrix} = y_{i,n+1},$$

$$i = 1, 2, \ldots, m-1 .$$

For $i = 1, 2, \ldots, m-1$ we denote by $p_{i,n}$ the function $p_{i,n}\colon \langle a, b\rangle \to \mathbf{R}^1$ which possesses the following properties:

$$(44)\qquad p_{i,n}\big|_{\langle a+(j-1)h,\, a+jh)} = p_{i,n}^{(j)} \quad \text{for} \quad j = 1, 2, \ldots, n,$$

$$p_{i,n}(b) = y_{i,n+1} .$$

By (43) we thus have $p_{i,n}(b) = p_{i,n}^{(n)}(b)$ for $i = 1, 2, \ldots, m-1$. By $\boldsymbol{p}_n^{(j)}$ and $\boldsymbol{p}_n$ we denote the vector functions $(p_{1,n}^{(j)}, p_{2,n}^{(j)}, \ldots, p_{m-1,n}^{(j)}) : \langle a, b\rangle \to \mathbf{R}^{m-1}$ and $(p_{1,n}, p_{2,n}, \ldots, p_{m-1,n}) : \langle a, b\rangle \to \mathbf{R}^{m-1}$, respectively. Consequently, we have $\boldsymbol{p}_n(b) = \boldsymbol{p}_n^{(n)}(b)$.

For $i = 1, 2, \ldots, m-1$ let the numbers $K_{f_i}(a)$ and $K_{f_i}(b)$ have the same meaning as in Section 3. We put $K_{\boldsymbol{f}}(a) = (K_{f_1}(a), K_{f_2}(a), \ldots, K_{f_{m-1}}(a))$ and $K_{\boldsymbol{f}}(b) = (K_{f_1}(b), K_{f_2}(b), \ldots, K_{f_{m-1}}(b))$, further we put [cf. (13), (14)]

$$(45)\qquad \tilde{y}_0 = -2h \,.\, K_{\boldsymbol{f}}(a) + \tilde{y}_2, \quad P_0 = (a-h, \tilde{y}_0),$$

$$(46)\qquad \tilde{y}_{n+2} = 2h \,.\, K_{\boldsymbol{f}}(b) + \tilde{y}_n, \quad P_{n+2} = (b+h, \tilde{y}_{n+2}).$$

Example 12. In the space $\mathbf{R}^3$ let us consider the curve $k = \{(x, y_1, y_2) \in \mathbf{R}^3 \mid -8 \leqq x \leqq 12,\ y_1 = f_1(x) = 5\cos x,\ y_2 = f_2(x) = 10\sin x\}$. For $n = 40$ we have $h = (b-a)/n = 20/40 = 0{\cdot}5$, further for $x = 11{\cdot}6$ we have $j = [(x-a)/h] + 1 = [19{\cdot}6/0{\cdot}5] + 1 = 40$. We have $\tilde{y}_{39} = (0{\cdot}02212849, -9{\cdot}99990207)$, $\tilde{y}_{40} = (2{\cdot}41652379, -8{\cdot}75452175)$, $\tilde{y}_{41} = (4{\cdot}21926979, -5{\cdot}36572918)$, further $K_{\boldsymbol{f}}(12) =$

$= (2{\cdot}68286459,\ 8{\cdot}43853959)$. By (46) we then have $\tilde{y}_{42} = K_f(12) + \tilde{y}_{40} =$ $= (5{\cdot}09938838,\ -0{\cdot}31598216)$. In the interval $11{\cdot}5 \leqq x \leqq 12$ we have, by (42),

$$p_{1,40}^{(40)}(x) = (1,\ 4x - 47,\ (4x - 47)^2,\ (4x - 47)^3) \circ \mathbf{C} \circ \begin{bmatrix} 0{\cdot}02212849 \\ 2{\cdot}41652379 \\ 4{\cdot}21926979 \\ 2{\cdot}68286459 \end{bmatrix} =$$

$$= 3{\cdot}56357182 + 1{\cdot}07309187(4x - 47) - 0{\cdot}24567503(4x - 47)^2 -$$
$$- 0{\cdot}17171887(4x - 47)^3\,,$$

$$p_{2,40}^{(40)}(x) = (1,\ 4x - 47,\ (4x - 47)^2,\ (4x - 47)^3) \circ \mathbf{C} \circ \begin{bmatrix} -9{\cdot}99990207 \\ -8{\cdot}75452175 \\ -5{\cdot}36572918 \\ -0{\cdot}31598216 \end{bmatrix} =$$

$$= -7{\cdot}29789838 + 1{\cdot}72454989(4x - 47) +$$
$$+ 0{\cdot}23777292(4x - 47)^2 - 0{\cdot}03015361(4x - 47)^3\,.$$

We have $\boldsymbol{p}_{40}^{(40)}(11{\cdot}6) = (2{\cdot}86836496,\ -8{\cdot}24051688)$, $\boldsymbol{f}(11{\cdot}6) = (2{\cdot}84144815,$ $-8{\cdot}22828595)$, thus the Euclidean distance is $\|\boldsymbol{f}(11{\cdot}6) - \boldsymbol{p}_{40}^{(40)}(11{\cdot}6)\| = 0{\cdot}02956536$.

By Section 3 [see (21)] we have, for $i = 1, 2, \ldots, m - 1$,

$$\lim_{n \to \infty} p_{i,n}(x) = f_i(x) \quad \text{uniformly in the interval} \quad \langle a, b \rangle\,,$$

i.e. for a given $\varepsilon > 0$ there exists an index $n_{i,0}$ such that for every $n > n_{i,0}$

$$(47) \qquad |p_{i,n}(x) - f_i(x)| < \frac{\varepsilon}{\sqrt{(m - 1)}}$$

holds independently of $x \in \langle a, b \rangle$. If we put $n_0 = \max\{n_{1,0}, n_{2,0}, \ldots, n_{m-1,0}\}$ then by (47) we have, for all $n > n_0$,

$$\|\boldsymbol{p}_n(x) - \boldsymbol{f}(x)\| = \sqrt{\sum_{i=1}^{m-1} [p_{i,n}(x) - f_i(x)]^2} < \varepsilon$$

independently of $x \in \langle a, b \rangle$, i.e.

$$(48) \qquad \lim_{n \to \infty} \boldsymbol{p}_n(x) = \boldsymbol{f}(x) \quad \text{uniformly in the interval} \quad \langle a, b \rangle\,.$$

Thus, if we construct the Euclidean neighbourhood of diameter 2ε around the curve k, then for all sufficiently large n the Lienhard approximations $k_n =$ $= \{(x, y_1, y_2, \ldots, y_{m-1}) \in \mathbf{R}^m |\ a \leqq x \leqq b,\ y_i = p_{i,n}(x),\ i = 1, 2, \ldots, m - 1\}$ lie inside this neighbourhood.

8. In the conclusion we present two programs in BASIC. The first constructs the approximating Lienhard's curve of a continuous function for a chosen partition of the interval $\langle a, b \rangle$. The functional values of the derivatives at the limit points are determined either from the given function of from the keyboard. The approximation curves can be "confronted" with an ε-neighbourhood of a considered function graph. The latter program calculates a numerical value of the definite integral of the function from a to b using both Lienhard's and Simpson's method and considering an even number of subintervals and an arbitrary precision.

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Program No. 1

```
10 REM *** BEGIN ***
20 GOSUB 710:REM INITIALIZATION
30 GOSUB 130:REM SCREEN DISPLAY
40 GOSUB 210:REM DRAW FUNCTION
50 GOSUB 290:REM END-POINTS
60 GOSUB 390:REM LIENHARD
65 V$=INPUT$(1)
70 IF V$="Q" OR V$="q" THEN GOTO 120
80 IF V$="G" OR V$="g" THEN GOSUB 210
90 IF V$="R" OR V$="r" THEN RUN
100 IF V$=" " THEN GOTO 70
110 GOTO 580
120 CLS:SCREEN 0:SYSTEM
130 REM *** SCREEN DISPLAY ***
135 SCREEN 9
137 VIEW PRINT 1TO7
140 CLS:VIEW
150 X1=A-DELTA:X2=B+DELTA
160 Y1=MIN:Y2=MAX
170 WINDOW (X1,Y1)-(X2,Y2)
180 REM IF X1*X2<0 THEN DRAW 0,MIN,1:DRAW 0,MAX,2
185 IF X1*X2<0 THEN LINE (0,MIN)-(0,MAX),2
190 REM IF MIN*MAX<0 THEN DRAW X1,0,1:DRAW X2,0,2
195 IF MIN*MAX<0 THEN LINE (X1,0)-(X2,0),2
197 PR=0
200 RETURN
210 REM *** DRAW FUNCTION PATH ***
212 INPUT"EPS:";EPS
214 FORK=1TO3
216 IF K=2 THEN BAR=7 ELSE BAR=3
220 DY=-(MAX-MIN)/1000:YP=MIN:IF MIN*MAX<0 THEN
    YP=0:DY=-DY
```

```
255 PSET (A,FNC(A)+(K-2)*EPS)
260 FORI=1TON*M:X=A+I*GAMA:Y=FNC(X)+(K-2)*EPS
262 LINE -(X,Y), BAR:NEXTI
270 REM LINE -(B,YP):YP=YP-B*DY:PSET (B,YP):YP=YP+B*DY
272 NEXTK
275 X$=INPUT$(1)
280 RETURN
290 REM *** SET END-POINT DERIVATIVES ***
295 LOCATE 1,1
300 PRINT"End-Point derivatives will be given"
310 PRINT"1-BY SETTING" : PRINT"2-FROM KEYBOARD"
315 V$=INPUT$(1)
320 IF V$="1" THEN DA=FND(A):DB=FND(B):GOTO 370
330 IF V$="2" THEN GOTO 360
340 IF V$=" " THEN GOTO 320
350 GOSUB 580
360 INPUT"Leftmost derivative=";DA : INPUT"Rightmost derivative=";DB
370 Y(1)=Y(3)-2*DA*DELTA:Y(N+3)=Y(N+1)+2*DB*DELTA
380 RETURN
390 REM *** LIENHARD ***
392 FORKK=1TO4
394 IF KK>1 THEN N=2*N
396 GOSUB 740:GOSUB 370
400 FORI=1TON
410 GOSUB 550-GOSUB 450
420 NEXTI
422 NEXTKK
430 IF PR=1 THEN LINE -(B,YP),2 ELSE PSET (B,YP),2:PR=0
440 RETURN
450 FORJ=1TO4:Y0(J)=Y(I-1+J):NEXTJ
455 PR=0
460 RO=GAMA*2/DELTA:IF PR=1 THEN LINE -(X(I),Y(I+1)) ELSE
    PSET (X(I),Y(I+1))
465 PR=1
470 FORL=1TOM+1:XL=(L-1)*RO-1:X0(1)=1
480 FORJ=2TO4:X0(J)=X0(J-1)*XL:NEXTJ
490 SK=0:XX=X(I)+(L-1)*GAMA
500 FORK=1TO4:SJ=0
510 FORJ=1TO4:SJ=SJ+X0(J)*C(J,K):NEXTJ
520 SK=SK+SJ*Y0(K):NEXTK
530 LINE -(XX,SK),KK:PR=1:NEXTL
540 RETURN
```

```
550 LINE (X(I),YP)-(X(I),YP):P=1:YY=FNC(X(I))
560 LINE -(X(I),YY),2:PR=1
570 RETURN
580 REM *** MENU ***
590 REM CLS 2
595 LOCATE 1,1:PRINT"
597 LOCATE 1,1
600 PRINT" *** FUNCTION MENU *** "
610 PRINT" R - NEW GRAPH"
620 PRINT" G - PLOT FUNCTION"
630 PRINT" L - NEW DATA"
640 PRINT" Q - QUIT"
645 B$=INPUT$(1)
650 IF B$="R" OR B$="r" THEN RUN
660 IF B$="G" OR B$="g" THEN:GOSUB 185:GOSUB 210
665 IF B$="Q" OR B$="q" THEN GOTO 120
670 IF B$="L" OR B$="l" THEN CLS:VIEW PRINT 1TO13:LIST 850-950
675 VIEW PRINT 1TO7
690 GOTO 645
700 GOTO 580
710 REM *** MATRIX ***
720 GOSUB 870
725 DIM X(500+3):DIM Y(500+3)
730 DIM C(4,4):DIM X0(4):DIM Y0(4)
740 DELTA=(B-A)/N:GAMA=DELTA/M
750 FORI=2TON+2:X(I-1)=A+(I-2)*DELTA
760 Y(I)=FNC(X(I-1))
770 NEXTI
780 C(1,1)=-1/16:C(1,2)=9/16:C(1,3)=9/16:C(1,4)=-1/16
790 C(2,1)=1/16:C(2,2)=-11/16:C(2,3)=11/16:C(2,4)=-1/16
800 C(3,1)=1/16:C(3,2)=-1/16:C(3,3)=-1/16:C(3,4)=1/16
810 C(4,1)=-1/16:C(4,2)=3/16:C(4,3)=-3/16:C(4,4)=1/16
820 RETURN
830 REM
840 REM
850 REM ****************************
860 REM *        DATA SETTING       *
870 REM ****************************
880 A=0
890 B=3.1415926
900 N=2
910 DEFFNC(X)=SIN(X)+COS(X)
```

```
920 DEFFND(X)=COS(X)-SIN(X)
930 MAX=2
940 MIN=-2
950 M=20
960 RETURN
970 REM ***************************
980 REM               END

 10 REM *** INTEGRAL CALCULATION USING METHODS L, S ***
 20 REM *** WITH PRESCRIBING ACCURACY ***
 30 DEFFNC(X) = 1/(1 + X*X)
 40 INPUT"INPUT VALUES OF D1, D2"; D1, D2
 50 INPUT"INPUT VALUES OF N, A, B, E"; N, A, B, E
 60 REM *** D1 DERIVATIVE AT A ***
 70 REM *** D2 DERIVATIVE AT B ***
 80 REM *** N≧4 NUMBER OF DIVISION POINTS (EVEN) ***
 90 REM *** A, B FUNCTION INTERVAL ***
100 REM *** E PRESCRIBING ACCURACY ***
110 GOSUB230
120 N=2*N
130 L1=L
140 GOSUB230
150 IFABS(L-L1)>E THEN120
160 PRINT"L";L
170 GOSUB230
180 S1=S
190 GOSUB230
200 IFABS(S-S1)>E THEN120
210 PRINT"S";S
220 GOTO40
230 H=(B-A)/N
240 R=FNC(A)
250 T=FNC(B)
260 P=2*H*D1+13*R
270 Q=12*FNC(A+H)-FNC(A+2*H)
280 U=-2*H*D2+13*T
290 V=12*FNC(B-H)-FNC(B-2*H)
300 Z=P+Q
310 W=U+V
320 FORI=0TON-3
330 Z=Z-FNC(A+I*H)+13*FNC(A+(I+1)*H)
340 NEXTI
```

Program No. 2

```
350 FORJ=0TON-3
360 W=W+13*FNC(A+(J+2)*H)-FNC(A+(J+3)*H)
370 NEXTJ
380 L=Z+W
390 L=(L*H)/24
400 FORK=2TON-2 STEP2
410 R=R+2*FNC(A+K*H)
420 NEXTK
430 FORM=1TON-1 STEP2
440 T=T+4*FNC(A+M*H)
450 NEXTM
460 S=R+T
470 S=(S*H)/3
480 RETURN
490 END
```

Literature

[1] *H. Lienhard:* Interpolation von Funktionswerten bei numerischen Bahnsteuerungen. Undated publication of CONTRAVES AG, Zürich.

[2] *J. Matušů:* The Lienhard interpolation method and some of its generalization (in Czech). Act. Polytechnica — Práce ČVUT, Prague 3 (IV, 2), 1978.

[3] *J. Matušů, J. Novák:* Constructions of interpolation curves from given supporting elements (I). Aplikace matematiky 30 (1985), 4, Prague.

[4] *J. Matušů, J. Novák:* Constructions of interpolation curves from given supporting elements (II). Aplikace matematiky 31 (1986), 2, Prague.

Souhrn

O JEDNÉ METODĚ NUMERICKÉ INTEGRACE

JOSEF MATUŠŮ, GEJZA DOHNAL, MARTIN MATUŠŮ

Předmětem článku je důkaz stejnoměrné konvergence posloupnosti lienhardovských aproximací dané spojité funkce v intervalu $\langle a, b\rangle$. Na tomto principu je dále odvozena metoda numerické integrace včetně odhadu chyby. Závěrem jsou připojeny dva programy v jazyku BASIC. Pomocí prvního lze kreslit aproximující Lienhardovy křivky při zvoleném dělení intervalu $\langle a, b\rangle$ uvažované spojité funkce. Hodnoty derivací v krajních bodech intervalu se určují buď ze zadání funkce nebo libovolným způsobem z klávesnice. Vykreslené aproximace lze „zkontrolovat" pomocí ε-ho okolí grafu uvažované funkce. Pomocí druhého programu lze počítat hodnotu integrálu dané spojité funkce od a do b Lienhardovou a Simpsonovou metodou, a to při zvoleném dělení intervalu $\langle a, b\rangle$ na sudý počet dílků a při zvolené přesnosti.

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