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THE FOURIER INTEGRAL FOR A CERTAIN CLASS OF DISTRIBUTIONS

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Summary. The aim of this paper is to derive by elementary means a theorem on the representation of certain distributions in the form of a Fourier integral. The approach chosen was found suitable especially for students of post-graduate courses at technical universities, where it is in some situations necessary to restrict a little the extent of the mathematical theory when concentrating on a technical problem.

Keywords: Fourier integral, distributions.

We shall deal with distributions introduced by Mikusiński – Sikorski (see [1]). By the symbol $f^{(k)}(t)$ (or $f'(t)$ for $k \doteq 1$) we will denote the derivative of the k-th order of a distribution $f(t)$ in the distributive sense, whereas the symbol $\mathrm{d}^k f(t)/\mathrm{d}t^k$ will represent the derivative of the k-th order of a function $f(t)$ in the usual sense. In this connection see Theorem 6.5 in [1].

1. Theorem. *Let a distribution $f(t)$ be defined on the set $\mathbf{E}_1 = (-\infty, \infty)$ and let $x, y \in \mathbf{E}_1$. Further, let $f_1(t)$ be an indefinite integral of the distribution $f(t) . \cos yt$, and let $f_2(t)$ be an indefinite integral of the distribution $f(t) . \sin yt$. Then the distribution*

$$a(t) = \begin{bmatrix} \cos yx \\ -\sin yx \end{bmatrix} . f_1(t + x) + \begin{bmatrix} \sin yx \\ \cos yx \end{bmatrix} . f_2(t + x) \tag{1.1}$$

is an indefinite integral of the distribution $f(t + x) \begin{bmatrix} \cos yt \\ \sin yt \end{bmatrix}$.

Proof. A simple calculation yields

$$[\cos yx . f_1(t + x) + \sin yx . f_2(t + x)]' = \cos yx . f_1'(t + x) + \\ + \sin yx . f_2'(t + x) = \cos yx . \{f(t + x) . \cos y(t + x)\} + \\ + \sin yx . \{f(t + x) . \sin y(t + x)\} = f(t + x) . \cos [yx - y(t + x)] = \\ = f(t + x) . \cos yt .$$

In the second case the proof may be carried out analogously.

2. Obviously the distribution

$$\tilde{a}(t) = a(t - x) = \begin{bmatrix} \cos yx \\ -\sin yx \end{bmatrix} . f_1(t) + \begin{bmatrix} \sin yx \\ \cos yx \end{bmatrix} . f_2(t)$$

generated with the aid of distribution (1.1) is an indefinite integral of the distribution $f(t)\begin{bmatrix} \cos y(t - x) \\ \sin y(t - x) \end{bmatrix}$. If the values $f_1(\pm\infty), f_2(\pm\infty)$ exist, then the values

$$\tilde{a}(\pm\infty) = a(\pm\infty) = \begin{bmatrix} \cos yx \\ -\sin yx \end{bmatrix} . f_1(\pm\infty) + \begin{bmatrix} \sin yx \\ \cos yx \end{bmatrix} . f_2(\pm\infty) \tag{2.1}$$

exist as well.

3. Theorem. *Let the assumptions of Theorem* 1 *be satisfied. Further, let the values* $f_1(\pm\infty), f_2(\pm\infty)$ *exist. Then*

$$\int_{-\infty}^{\infty} f(t + x) \begin{bmatrix} \cos yx \\ \sin yx \end{bmatrix} dx = \int_{-\infty}^{\infty} f(x) \begin{bmatrix} \cos y(x - t) \\ \sin y(x - t) \end{bmatrix} dx . \tag{3.1}$$

Proof. Equality (3.1) follows directly from Theorem 1 and formula (2.1):

$$\int_{-\infty}^{\infty} f(t + x) \begin{bmatrix} \cos yx \\ \sin yx \end{bmatrix} dx = a(\infty) - a(-\infty) = \tilde{a}(\infty) - \tilde{a}(-\infty) =$$

$$= \int_{-\infty}^{\infty} f(x) \begin{bmatrix} \cos y(x - t) \\ \sin y(x - t) \end{bmatrix} dx .$$

4. If the assumptions of Theorem 3 are satisfied, then for any integer k we have by (3.1)

$$\int_{-\infty}^{\infty} f(t + x) \begin{bmatrix} \cos\left(yx + \dfrac{k\pi}{2}\right) \\ \sin\left(yx + \dfrac{k\pi}{2}\right) \end{bmatrix} dx = \tag{4.1}$$

$$= \int_{-\infty}^{\infty} f(x) \begin{bmatrix} \cos\left\{y(x - t) + \dfrac{k\pi}{2}\right\} \\ \sin\left\{y(x - t) + \dfrac{k\pi}{2}\right\} \end{bmatrix} dx .$$

Let $f(t)$ be a distribution with the domain $\mathbf{E}_1$, let $x, y \in \mathbf{E}_1$. Further, let $f_1(t), f_2(t)$ be distributions from Theorem 1. For any integer $k \geqq 1$ we then have

$$\left\{f(t + x) \begin{bmatrix} \cos yt \\ \sin yt \end{bmatrix}\right\}^{(k)} = \sum_{i=0}^{k} \binom{k}{i} f^{(k-i)}(t + x) \begin{bmatrix} \cos^{(i)} yt \\ \sin^{(i)} yt \end{bmatrix} = \tag{4.2}$$

$$= \sum_{i=0}^{k} y^i \binom{k}{i} f^{(k-i)}(t + x) \begin{bmatrix} \cos\left(yt + \dfrac{i\pi}{2}\right) \\ \sin\left(yt + \dfrac{i\pi}{2}\right) \end{bmatrix},$$

i.e. the distribution $\left\{f(t+x)\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix}\right\}^{(k-1)} = a^{(k)}(t)$ [see (1.1)] is an indefinite integral of the distribution on the left-hand side of equality (4.2). If the values $f_1(\pm\infty)$, $f_2(\pm\infty)$ exist, then the values $a(\pm\infty)$ also exist (cf. Sect. 2). By Theorem 18.3 (see [1]) we then have $a^{(k)}(\pm\infty) = 0$. This proves that we have

$$(4.3)\qquad \int_{-\infty}^{\infty}\left\{\sum_{i=0}^{k} y^i \binom{k}{i} f^{(k-i)}(t+x)\begin{bmatrix}\cos\left(yx+\dfrac{i\pi}{2}\right)\\ \sin\left(yx+\dfrac{i\pi}{2}\right)\end{bmatrix}\right\}\mathrm{d}x = 0\,.$$

5. Theorem. *Let $k \geqq 0$ be an integer. Further, let the assumptions of Theorem 3 be fulfilled. Then*

$$(5.1)\qquad \int_{-\infty}^{\infty} f^{(k)}(t+x)\begin{bmatrix}\cos yx\\ \sin yx\end{bmatrix}\mathrm{d}x =$$

$$= (-1)^k y^k \int_{-\infty}^{\infty} f(t+x)\begin{bmatrix}\cos\left(yx+\dfrac{k\pi}{2}\right)\\ \sin\left(yx+\dfrac{k\pi}{2}\right)\end{bmatrix}\mathrm{d}x\,.$$

Proof. For $k = 0$, (5.1) is obvious. Let $k \geqq 1$, and let j equal one of the numbers $0, 1, \ldots. k-1$. We have

$$\begin{bmatrix}\cos\left(yx+\dfrac{i+j}{2}\pi\right)\\ \sin\left(yx+\dfrac{i+j}{2}\pi\right)\end{bmatrix} =$$

$$= \begin{bmatrix}\cos\left(yx+\dfrac{i\pi}{2}\right).\cos\dfrac{j\pi}{2} - \sin\left(yx+\dfrac{i\pi}{2}\right).\sin\dfrac{j\pi}{2}\\ \sin\left(yx+\dfrac{i\pi}{2}\right).\cos\dfrac{j\pi}{2} + \cos\left(yx+\dfrac{i\pi}{2}\right).\sin\dfrac{j\pi}{2}\end{bmatrix}.$$

This yields

$$(5.2)\qquad \begin{bmatrix}\cos\left(yx+\dfrac{i\pi}{2}\right)\\ \sin\left(yx+\dfrac{i\pi}{2}\right)\end{bmatrix} = \begin{cases}\begin{bmatrix}\pm\cos\left(yx+\dfrac{i+j}{2}\pi\right)\\ \pm\sin\left(yx+\dfrac{i+j}{2}\pi\right)\end{bmatrix} & \text{for } j \text{ even}\,,\\[2ex] \begin{bmatrix}\pm\sin\left(yx+\dfrac{i+j}{2}\pi\right)\\ \pm\cos\left(yx+\dfrac{i+j}{2}\pi\right)\end{bmatrix} & \text{for } j \text{ odd.}\end{cases}$$

If the number k is replaced by $k - j$ in relation (4.3), then

$$(5.3)\qquad \int_{-\infty}^{\infty}\left\{\sum_{i=0}^{k-j} y^i \binom{k-j}{i} f^{(k-j-i)}(t+x)\begin{bmatrix}\cos\left(yx + \frac{i+j}{2}\pi\right)\\ \sin\left(yx + \frac{i+j}{2}\pi\right)\end{bmatrix}\right\}\mathrm{d}x = 0 .$$

Multiply equality (5.3) by the number $(-1)^j y^j \binom{k}{j}$ and substitute $\binom{k}{j}\binom{k-j}{i} = \binom{k}{i+j}(i+j)!/i!\,j!$. Then

$$(5.4)\qquad \int_{-\infty}^{\infty}\left\{\sum_{i=0}^{k-j}(-1)^j y^{i+j}\binom{k}{i+j}\frac{(i+j)!}{i!\,j!} f^{(k-(i+j))}(t+x)\,.\begin{bmatrix}\cos\left(yx + \frac{i+j}{2}\pi\right)\\ \sin\left(yx + \frac{i+j}{2}\pi\right)\end{bmatrix}\right\}\mathrm{d}x = 0 .$$

For $j = 0, 1, \ldots, k-1$, we obtain from (5.4) k equations and their addition yields

$$(5.5)\qquad \int_{-\infty}^{\infty}\left\{\sum_{j=0}^{k-1}\sum_{i=0}^{k-j}(-1)^j y^{i+j}\binom{k}{i+j}\frac{(i+j)!}{i!\,j!} f^{(k-(i+j))}(t+x)\,.\begin{bmatrix}\cos\left(yx + \frac{i+j}{2}\pi\right)\\ \sin\left(yx + \frac{i+j}{2}\pi\right)\end{bmatrix}\right\}\mathrm{d}x = 0 .$$

The sum under the integral sign in equality (5.5) is denoted by the symbol B. In the sum B we collect into one term all terms for which the expression $i + j$ is constant. Then

$$B = \sum_{\substack{i+j=0\\ j\neq k}}^{k} y^{i+j}\binom{k}{i+j}(i+j)!\,f^{(k-(i+j))}(t+x)\,.\begin{bmatrix}\cos\left(yx + \frac{i+j}{2}\pi\right)\\ \sin\left(yx + \frac{i+j}{2}\pi\right)\end{bmatrix}\cdot\frac{(-1)^j}{i!\,j!}\,.$$

For $i + j = 0$ there exists only one possibility: $i = j = 0$, i.e.

$$(5.6)\qquad B = f^{(k)}(t+x)\begin{bmatrix}\cos yx\\ \sin yx\end{bmatrix} + \tilde{B}\,,$$

where

$$\tilde{B} = \sum_{\substack{i+j=1 \\ j \neq k}}^{k} y^{i+j} \binom{k}{i+j} (i+j)!\, f^{(k-(i+j))}(t+x) \,. \tag{5.7}$$

$$\cdot \begin{bmatrix} \cos\left(yx + \dfrac{i+j}{2}\pi\right) \\ \sin\left(yx + \dfrac{i+j}{2}\pi\right) \end{bmatrix} \cdot \frac{(-1)^j}{i!\,j!} \,.$$

Put $i + j = s(1 \leqq s < k)$ and consider the sum

$$\sum_{i+j=s} \frac{(-1)^j}{i!\,j!} = \sum_{j=0}^{s} \frac{(-1)^j}{j!\,(s-j)!} \,. \tag{5.8}$$

Since $1/j!\,(s-j)! = \binom{s}{j}/s!$ it follows that

$$\sum_{i+j=s} \frac{(-1)^j}{i!\,j!} = \frac{1}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} = \frac{1}{s!} [1 + (-1)]^s = 0 \,. \tag{5.9}$$

For $s = k$ it follows from (5.8) that

$$\sum_{\substack{i+j=k \\ j \neq k}} \frac{(-1)^j}{i!\,j!} = \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} = -\frac{(-1)^k}{k!} \,. \tag{5.10}$$

By (5.9) and (5.10), (5.7) acquires the form

$$\tilde{B} = -(-1)^k y^k f(t+x) \begin{bmatrix} \cos\left(yx + \dfrac{k\pi}{2}\right) \\ \sin\left(yx + \dfrac{k\pi}{2}\right) \end{bmatrix} . \tag{5.11}$$

Substituting (5.11) into (5.6) we get

$$B = f^{(k)}(t+x) \begin{bmatrix} \cos yx \\ \sin yx \end{bmatrix} - (-1)^k y^k f(t+x) \begin{bmatrix} \cos\left(yx + \dfrac{k\pi}{2}\right) \\ \sin\left(yx + \dfrac{k\pi}{2}\right) \end{bmatrix} . \tag{5.12}$$

If the sum B in the integral (5.5) is replaced by (5.12), then

$$\int_{-\infty}^{\infty} \left\{ f^{(k)}(t+x) \begin{bmatrix} \cos yx \\ \sin yx \end{bmatrix} - (-1)^k y^k f(t+x) \,. \right. \tag{5.13}$$

$$\left. \cdot \begin{bmatrix} \cos\left(yx + \dfrac{k\pi}{2}\right) \\ \sin\left(yx + \dfrac{k\pi}{2}\right) \end{bmatrix} \right\} \mathrm{d}x = 0 \,.$$

After adding the integral

$$\int_{-\infty}^{\infty} (-1)^k y^k f(t+x) \begin{bmatrix} \cos\left(yx + \dfrac{k\pi}{2}\right) \\ \sin\left(yx + \dfrac{k\pi}{2}\right) \end{bmatrix} dx$$

to both sides of (5.13) we obtain formula (5.1). This proves the theorem.

6. Theorem. *Let $k \geqq 0$ be an integer. Further, let the assumptions of Theorem* 3 *be fulfilled. Then*

$$\left(\int_{-\infty}^{\infty} f(t+x) \begin{bmatrix} \cos yx \\ \sin yx \end{bmatrix} dx\right)_t^{(k)} = \int_{-\infty}^{\infty} f^{(k)}(t+x) \begin{bmatrix} \cos yx \\ \sin yx \end{bmatrix} dx . \tag{6.1}$$

Proof. In equality (6.1) the index t indicates that the derivative of the k-th order is understood in the distributive sense of the integral [see (2.1)]

$$\int_{-\infty}^{\infty} f(t+x) \begin{bmatrix} \cos yx \\ \sin yx \end{bmatrix} dx =$$
$$= \begin{bmatrix} \cos yt \\ -\sin yt \end{bmatrix} \cdot \int_{-\infty}^{\infty} f(x) \cos yx \, dx + \begin{bmatrix} \sin yt \\ \cos yt \end{bmatrix} \cdot \int_{-\infty}^{\infty} f(x) \sin yx \, dx ,$$

which is an indefinitely continuously differentiable function of the variable $t \in \mathbf{E}_1$. Integral (6.2) can be viewed as a distribution with the domain $\mathbf{E}_1$. By (6.2) we have

$$\left(\int_{-\infty}^{\infty} f(t+x) \begin{bmatrix} \cos yx \\ \sin yx \end{bmatrix} dx\right)_t^{(k)} =$$
$$= \left(\begin{bmatrix} \cos yt \\ -\sin yt \end{bmatrix} \cdot \int_{-\infty}^{\infty} f(x) \cos yx \, dx + \right.$$
$$\left. + \begin{bmatrix} \sin yt \\ \cos yt \end{bmatrix} \cdot \int_{-\infty}^{\infty} f(x) \sin yx \, dx\right)_t^{(k)} =$$
$$= y^k \begin{bmatrix} \cos\left(yt + \dfrac{k\pi}{2}\right) \\ -\sin\left(yt + \dfrac{k\pi}{2}\right) \end{bmatrix} \cdot \int_{-\infty}^{\infty} f(x) \cos yx \, dx +$$
$$+ y^k \begin{bmatrix} \sin\left(yt + \dfrac{k\pi}{2}\right) \\ \cos\left(yt + \dfrac{k\pi}{2}\right) \end{bmatrix} \cdot \int_{-\infty}^{\infty} f(x) \sin yx \, dx .$$

Hence it follows that

(6.3) $$\left(\int_{-\infty}^{\infty} f(t+x)\begin{bmatrix}\cos yx\\ \sin yx\end{bmatrix}\mathrm{d}x\right)_t^{(k)} =$$

$$= y^k \int_{-\infty}^{\infty} f(x)\begin{bmatrix}\cos\left(yt+\frac{k\pi}{2}\right).\cos yx + \sin\left(yt+\frac{k\pi}{2}\right).\sin yx\\ -\sin\left(yt+\frac{k\pi}{2}\right).\cos yx + \cos\left(yt+\frac{k\pi}{2}\right).\sin yx\end{bmatrix}\mathrm{d}x =$$

$$= y^k \int_{-\infty}^{\infty} f(x)\begin{bmatrix}\cos y(t-x)+\frac{k\pi}{2}\\ -\sin y(t-x)+\frac{k\pi}{2}\end{bmatrix}\mathrm{d}x\,.$$

Since

$$\cos\left\{y(t-x)+\frac{k\pi}{2}\right\} = (-1)^k\cos\left\{y(x-t)+\frac{k\pi}{2}\right\},$$

$$\sin\left\{y(t-x)+\frac{k\pi}{2}\right\} = (-1)^{k+1}\sin\left\{y(x-t)+\frac{k\pi}{2}\right\}$$

it is possible to express (6.3) in the form

(6.4) $$\left(\int_{-\infty}^{\infty} f(t+x)\begin{bmatrix}\cos yx\\ \sin yx\end{bmatrix}\mathrm{d}x\right)_t^{(k)} =$$

$$= (-1)^k y^k \int_{-\infty}^{\infty} f(x)\begin{bmatrix}\cos\left\{y(x-t)+\frac{k\pi}{2}\right\}\\ \sin\left\{y(x-t)+\frac{k\pi}{2}\right\}\end{bmatrix}\mathrm{d}x\,.$$

From equality (6.4) it follows by (4.1) and (5.1) that (6.1) is true.

7. Let the assumptions of Theorem 1 be satisfied. If we put

(7.1) $$b(t) = \begin{bmatrix}\sin yx\\ \cos yx\end{bmatrix}.f_1(t+x) + \begin{bmatrix}-\cos yx\\ \sin yx\end{bmatrix}.f_2(t+x)\,,$$

then we find by a simple calculation that

(7.2) $$b'(t) = f(t+x)\begin{bmatrix}-\sin yt\\ \cos yt\end{bmatrix}.$$

Hence the distribution $b(t)$ is an indefinite integral of the distribution $f(t+x)$ $\begin{bmatrix}-\sin yt\\ \cos yt\end{bmatrix}$. Further, the distribution $\tilde{b}(t) = b(t-x)$ is an indefinite integral of the distribution $f(t)\begin{bmatrix}-\sin y(t-x)\\ \cos y(t-x)\end{bmatrix}$. If the values $f_1(\pm\infty)$, $f_2(\pm\infty)$ exist, then the values

$$(7.3)\qquad \tilde{b}(\pm\infty) = b(\pm\infty) = \begin{bmatrix}\sin yx\\ \cos yx\end{bmatrix}\cdot f_1(\pm\infty) + \begin{bmatrix}-\cos yx\\ \sin yx\end{bmatrix}\cdot f_2(\pm\infty)$$

exist as well.

8. Let $k \geqq 0$ be an integer. We shall prove that the relation

$$(8.1)\qquad f^{(k)}(t+x)\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix} = \sum_{j=0}^{k}(-1)^i\binom{k}{i}\left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix}f(t+x)\right\}^{(k-i)}$$

is valid.

Equality (8.1) holds for $k = 0$. Assume that (8.1) holds for a certain integer $k \geqq 0$. We shall prove that (8.1) holds also for the number $k + 1$. By (8.1) we have

$$(8.2)\qquad f^{(k+1)}(t+x)\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix} = (f')^{(k)}(t+x)\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix}f'(t+x)\right\}^{(k-i)}.$$

Further we have

$$(8.3)\qquad \begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix}f'(t+x) = \left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix}f(t+x)\right\}' - \begin{bmatrix}\cos^{(i+1)} yt\\ \sin^{(i+1)} yt\end{bmatrix}f(t+x).$$

Substituting (8.3) into (8.2) we obtain

$$\begin{aligned}
f^{(k+1)}(t+x)\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix} &= \\
&= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix}f(t+x)\right\}^{(k+1-i)} - \\
&- \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left\{\begin{bmatrix}\cos^{(i+1)} yt\\ \sin^{(i+1)} yt\end{bmatrix}f(t+x)\right\}^{(k-i)} = \\
&= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix}f(t+x)\right\}^{(k+1-i)} - \\
&- \sum_{i=1}^{k+1}(-1)^{i-1}\binom{k}{i-1}\left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix}f(t+x)\right\}^{(k+1-i)} = \\
&= \left\{\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix}f(t+x)\right]^{(k+1)} + \\
&+ \sum_{i=1}^{k}(-1)^i\binom{k}{i}\left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix}f(t+x)\right\}^{(k+1+i)} + \\
&+ \sum_{i=1}^{k}(-1)^i\binom{k}{i-1}\left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix}f(t+x)\right\}^{(k+1-i)} + \\
&+ (-1)^{k+1}\begin{bmatrix}\cos^{(k+1)} yt\\ \sin^{(k+1)} yt\end{bmatrix}f(t+x) =
\end{aligned}$$

$$= \left\{\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix} f(t+x)\right\}^{(k+1)} + \sum_{i=1}^{k} (-1)^i \left[\binom{k}{i} + \right.$$

$$\left. + \binom{k}{i-1}\right] \left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix} f(t+x)\right]^{(k+1-i)} +$$

$$+ (-1)^{k+1} \begin{bmatrix}\cos^{(k+1)} yt\\ \sin^{(k+1)} yt\end{bmatrix} f(t+x) =$$

$$= \left\{\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix} f(t+x)\right\}^{(k+1)} +$$

$$+ \sum_{i=1}^{k} (-1)^i \binom{k+1}{i} \left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix} f(t+x)\right\}^{(k+1-i)} +$$

$$+ (-1)^{k+1} \begin{bmatrix}\cos^{(k+1)} yt\\ \sin^{(k+1)} yt\end{bmatrix} f(t+x) =$$

$$= \sum_{i=0}^{k+1} (-1)^i \left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix} f(t+x)\right\}^{(k+1-i)}.$$

This yields that formula (8.1) is also valid for the number $k + 1$.

9. Let $k \geqq 0$ be an integer, and the assumptions of Theorem 3 be satisfied. By (8.1) we have

$$(9.1) \qquad f^{(k)}(t+x)\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left\{\begin{bmatrix}\cos^{(i)} yt\\ \sin^{(i)} yt\end{bmatrix} f(t+x)\right\}^{(k-i)} =$$

$$= \sum_{i=0}^{k} y^i (-1)^i \binom{k}{i} \left\{\begin{bmatrix}\cos\left(yt + \dfrac{i\pi}{2}\right)\\ \sin\left(yt + \dfrac{i\pi}{2}\right)\end{bmatrix} f(t+x)\right\}^{(k-i)} =$$

$$= \sum_{i=0}^{k} y^i (-1)^i \binom{k}{i} \left\{ f(t+x)\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix} \cos\frac{i\pi}{2} + \right.$$

$$\left. + f(t+x)\begin{bmatrix}-\sin yt\\ \cos yt\end{bmatrix} \sin\frac{i\pi}{2}\right\}^{(k-i)}.$$

Since $a'(t) = f(t+x)\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix}$ (see Theorem 1) and (7.2) holds, (9.1) can be written in the form

$$(9.2) \qquad f^{(k)}(t+x)\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix} =$$

$$= \sum_{i=0}^{k} y^i (-1)^i \binom{k}{i} \left\{ a'(t)\cos\frac{i\pi}{2} + b'(t)\sin\frac{i\pi}{2}\right\}^{(k-i)} =$$

$$= \left[\sum_{i=0}^{k} y^i (-1)^i \binom{k}{i} \left\{ a(t)\cos\frac{i\pi}{2} + b(t)\sin\frac{i\pi}{2}\right\}^{(k-i)}\right]' = c'(t).$$

Relation (9.2) implies that the distribution $c(t)$ is an indefinite integral of the distribution $f^{(k)}(t+x)\begin{bmatrix}\cos yt\\ \sin yt\end{bmatrix}$. Then the distribution

$$(9.3)\qquad \tilde{c}(t) = c(t-x) = \sum_{i=0}^{k} y^i(-1)^i \binom{k}{i}\left\{\tilde{a}(t)\cos\frac{i\pi}{2} + \tilde{b}(t)\sin\frac{i\pi}{2}\right\}^{(k-i)}$$

(cf. Sect. 2 and 7) is an indefinite integral of the distribution $f^{(k)}(t)\begin{bmatrix}\cos y(t-x)\\ \sin y(t-x)\end{bmatrix}$. Since the values $f_1(\pm\infty), f_2(\pm\infty)$ exist, it follows from (9.3), (2.1), (7.3) and Theorem 18.3 (see [1]) that the values

$$(9.4)\qquad \tilde{c}(\pm\infty) = c(\pm\infty) = y^k(-1)^k\left\{a(\pm\infty)\cos\frac{k\pi}{2} + b(\pm\infty)\sin\frac{k\pi}{2}\right\}$$

exist as well.

10. Theorem. *Let $k \geqq 0$ be an integer. Further, let the assumptions of Theorem* 3 *be fulfilled. Then*

$$(10.1)\qquad \int_{-\infty}^{\infty} f^{(k)}(t+x)\begin{bmatrix}\cos yx\\ \sin yx\end{bmatrix}\mathrm{d}x = \int_{-\infty}^{\infty} f^{(k)}(x)\begin{bmatrix}\cos y(x-t)\\ \sin y(x-t)\end{bmatrix}\mathrm{d}x\,.$$

Proof. Formula (10.1) holds for $k = 0$ [see (3.1)]. For an integer $k \geqq 1$ we have, by Sect. 9 and formula (9.4),

$$\int_{-\infty}^{\infty} f^{(k)}(t+x)\begin{bmatrix}\cos yx\\ \sin yx\end{bmatrix}\mathrm{d}x = c(\infty) - c(-\infty) = \tilde{c}(\infty) - \tilde{c}(-\infty) =$$
$$= \int_{-\infty}^{\infty} f^{(k)}(x)\begin{bmatrix}\cos y(x-t)\\ \sin y(x-t)\end{bmatrix}\mathrm{d}x\,.$$

11. Equality (5.1) easily follows from relation (9.4) [see (2.1), (7.3) and (4.1)]:

$$\int_{-\infty}^{\infty} f^{(k)}(t+x)\begin{bmatrix}\cos yx\\ \sin yx\end{bmatrix}\mathrm{d}x = c(\infty) - c(-\infty) =$$
$$= (-1)^k y^k\left\{\cos\frac{k\pi}{2}\begin{bmatrix}\cos yt\\ -\sin yt\end{bmatrix}\int_{-\infty}^{\infty} f(x)\cos yx\,\mathrm{d}x\,+\right.$$
$$+\cos\frac{k\pi}{2}\begin{bmatrix}\sin yt\\ \cos yt\end{bmatrix}\int_{-\infty}^{\infty} f(x)\sin yx\,\mathrm{d}x\,+$$
$$+\sin\frac{k\pi}{2}\begin{bmatrix}\sin yt\\ \cos yt\end{bmatrix}\int_{-\infty}^{\infty} f(x)\cos yx\,\mathrm{d}x\,+$$
$$+\sin\frac{k\pi}{2}\begin{bmatrix}-\cos yt\\ \sin yt\end{bmatrix}\int_{-\infty}^{\infty} f(x)\sin yx\,\mathrm{d}x =$$

$$= (-1)^k y^k \int_{-\infty}^{\infty} f(x) \begin{bmatrix} \cos y(x-t) . \cos \frac{k\pi}{2} - \sin y(x-t) . \sin \frac{k\pi}{2} \\ \sin y(x-t) . \cos \frac{k\pi}{2} + \cos y(x-t) . \sin \frac{k\pi}{2} \end{bmatrix} dx =$$

$$= (-1)^k y^k \int_{-\infty}^{\infty} f(x) \begin{bmatrix} \cos y(x-t) + \frac{k\pi}{2} \\ \sin y(x-t) + \frac{k\pi}{2} \end{bmatrix} dx =$$

$$= (-1)^k y^k \int_{-\infty}^{\infty} f(t+x) \begin{bmatrix} \cos \left(yx + \frac{k\pi}{2}\right) \\ \sin \left(yx + \frac{k\pi}{2}\right) \end{bmatrix} dx .$$

The following theorem is a direct consequence of Theorems 6 and 10.

12. Theorem. *Let $f(t)$ be a distribution with the domain $\mathbf{E}_1$, and let $y \in \mathbf{E}_1$. Let $f_1(t)$ be an indefinite integral of the distribution $f(t) \cos yt$, and let $f_2(t)$ be an indefinite integral of $f(t) \sin yt$. Let the values $f_1(\pm\infty)$, $f_2(\pm\infty)$ exist. Then for every integer $k \geqq 0$ we have*

(12.1) $$\left(\int_{-\infty}^{\infty} f(x) \begin{bmatrix} \cos y(x-t) \\ \sin y(x-t) \end{bmatrix} dx\right)_t^{(k)} = \int_{-\infty}^{\infty} f^{(k)}(x) \begin{bmatrix} \cos y(x-t) \\ \sin y(x-t) \end{bmatrix} dx .$$

13. Example. Let us consider Dirac's $\delta(t)$ "function". It is a distribution with the domain $\mathbf{E}_1$. For any number $y \in \mathbf{E}_1$ we have: $\delta(t) \cos yt = \delta(t) \cos 0 = \delta(t)$, $\delta(t)$ . . $\sin yt = \delta(t) \sin 0 = 0$ (the zero distribution). Let

(13.1) $$H(t) = \begin{cases} 1 & \text{for} \quad t \geqq 0 \\ 0 & \text{for} \quad t < 0 \end{cases}$$

(Heaviside's function). Further, let $m(t) = m \in \mathbf{E}_1$ be a constant function with the domain $\mathbf{E}_1$. If we put $f_1(t) = H(t)$, then $f_1(t)$ is an indefinite integral of the distribution $\delta(t) \cos yt$. If we put $f_2(t) = m(t)$, then $f_2'(t) = 0 = \delta(t) \sin yt$, i.e. the distribution $f_2(t)$ is an indefinite integral of the (zero) distribution $\delta(t) \sin yt$. The values $f_1(\infty) = 1$, $f_1(-\infty) = 0$, $f_2(\pm\infty) = m$ exist. Since all assumptions of Theorem 12 are satisfied, (12.1) yields

$$\left(\int_{-\infty}^{\infty} \delta(x) \begin{bmatrix} \cos y(x-t) \\ \sin y(x-t) \end{bmatrix} dx\right)_t^{(k)} = \int_{-\infty}^{\infty} \delta^{(k)}(x) \begin{bmatrix} \cos y(x-t) \\ \sin y(x-t) \end{bmatrix} dx .$$

14. Let $f(t) \in L(-\infty, \infty)$. Then

(14.1) $$F_A(t) = \frac{1}{\pi} \int_0^A \left(\int_{-\infty}^{\infty} f(x) \cos y(x-t)\, dx\right) dy$$

is a convergent Lebesgue integral. By the symbol A we understand a continuous variable which assumes positive and finite values.

The following two theorems are true.

15. Theorem. *Let $k \geqq 0$ be an integer, $f(t) \in L(-\infty, \infty)$. Then for every number $t \in \mathbf{E}_1$ the integral*

$$(15.1)\qquad {}^kF_A(t) = \frac{(-1)^k}{\pi} \int_0^A \left(\int_{-\infty}^{\infty} f(x)\, y^k \cos\left[y(x-t) + \frac{k\pi}{2} \right] \mathrm{d}x \right) \mathrm{d}y$$

converges as the Lebesgue integral.

Proof. For $k = 0$ we have ${}^0F_A(t) \equiv F_A(t)$ [see (14.1)]. Since the estimate

$$\iint_{\substack{x \in \mathbf{E}_1 \\ y \in \langle 0, A \rangle}} \left| f(x)\, y^k \cos\left[y(x-t) + \frac{k\pi}{2} \right] \right| \mathrm{d}x\, \mathrm{d}y \leqq$$
$$\leqq \frac{A^{k+1}}{k+1} \int_{-\infty}^{\infty} |f(x)|\, \mathrm{d}x < \infty$$

holds, the theorem is proved.

The proof of the following theorem is given for the reader, who does not know the Lebesgue's theorem on the derivative of the integral with respect to the parameter (for the unbounded domain of interation); in the oposite, reader can prove this theorem by a simple derivative of the integral (14.1) after integral sign with respect to t.

16. Theorem. *Let $f(t) \in L(-\infty, \infty)$. Then for any integer $k \geqq 1$ the derivative of the k-th order of the function (14.1) exists at any point $t \in \mathbf{E}_1$ and we have*

$$(16.1)\qquad \frac{\mathrm{d}^k}{\mathrm{d}t^k} F_A(t) = {}^kF_A(t)$$

[cf. (15.1)].

Proof. By the symbol $V(k)$ we denote the following assertion (which depends on the integer $k \geqq 1$): At any point $t \in \mathbf{E}_1$ the derivative $\mathrm{d}^k F_A(t)/\mathrm{d}t^k = {}^kF_A(t)$ exists.

Let us consider the function

$$(16.2)\qquad I(y, t) = \textstyle\int_{-\infty}^{\infty} f(x) \cos y(x-t)\, \mathrm{d}x =$$
$$= \cos yt \,.\, \textstyle\int_{-\infty}^{\infty} f(x) \cos yx\, \mathrm{d}x + \sin yt \,.\, \int_{-\infty}^{\infty} f(x) \sin yx\, \mathrm{d}x$$

of the variables $y, t \in \mathbf{E}_1$. A simple calculation yields

$$(16.3)\qquad \frac{\partial^k}{\partial t^k} I(y, t) = y^k \cos\left(yt + \frac{k\pi}{2} \right) . \int_{-\infty}^{\infty} f(x) \cos yx\, \mathrm{d}x + y^k \sin\left(yt + \frac{k\pi}{2} \right) .$$
$$. \int_{-\infty}^{\infty} f(x) \sin yx\, \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\, y^k \cos\left[y(x-t) - \frac{k\pi}{2} \right] \mathrm{d}x =$$
$$= (-1)^k \int_{-\infty}^{\infty} f(x)\, y^k \cos\left[y(x-t) + \frac{k\pi}{2} \right] \mathrm{d}x \,.$$

We know (see Theorem 15 for $k = 0$) that for every number $t \in \mathbf{E}_1$ the integral $F_A(t) = (1/\pi) \int_0^A I(y, t)\, dy$ is a convergent Lebesgue integral. Further, for any number $t \in \mathbf{E}_1$ we have [see (16.3) for $k = 1$]

$$\left|\frac{\partial}{\partial t} I(y, t)\right| \leqq 2y \int_{-\infty}^{\infty} |f(x)|\, dx = \tilde{I}(y) \tag{16.4}$$

where $\tilde{I}(y)$ is a continuous and bounded function on the interval $\langle 0, A \rangle$, thus integrable over this interval. A familiar theorem from theory of integral implies [see (16.3), (15.1) for $k = 1$] that

$$\begin{aligned} \frac{d}{dt} F_A(t) &= \frac{1}{\pi} \int_0^A \frac{\partial}{\partial t} I(y, t)\, dy = \\ &= -\frac{1}{\pi} \int_0^A \left(\int_{-\infty}^{\infty} f(x)\, y \cos\left[y(x - t) + \frac{\pi}{2} \right] dx \right) dy = {}^1F_A(t) \end{aligned} \tag{16.5}$$

at every point $t \in \mathbf{E}_1$. Assertion $V(1)$ follows from (16.5).

Let us suppose that assertion $V(k)$ holds for a certain integer $k \geqq 1$:

$$\frac{d^k}{dt^k} F_A(t) = {}^kF_A(t) \tag{16.6}$$

at any point $t \in \mathbf{E}_1$. By Theorem 15 the integral

$${}^kF_A(t) = \frac{1}{\pi} \int_0^A \frac{\partial^k}{\partial t^k} I(y, t)\, dy \tag{16.7}$$

[see (16.3), (15.1)] is a convergent Lebesgue integral for every number $t \in \mathbf{E}_1$. Further [see (16.3) where the number k is replaced by $k + 1$], for any number $t \in \mathbf{E}_1$ we have the estimate

$$\left| \frac{\partial}{\partial t} \left(\frac{\partial^k}{\partial t^k} I(y, t) \right) \right| \leqq 2y^{k+1} \int_{-\infty}^{\infty} |f(x)|\, dx = y^k\, \tilde{I}(y)$$

where $y^k\, \tilde{I}(y)$ is a continuous and bounded function on $\langle 0, A \rangle$, thus integrable over this interval. By (16.6), (16.3), (15.1) and a familiar theorem from the theory of integral (where the number k is replaced by $k + 1$)

$$\begin{aligned} \frac{d}{dt}\, {}^kF_A(t) &= \frac{d}{dt} \left(\frac{d^k}{dt^k} F_A(t) \right) = \frac{d^{k+1}}{dt^{k+1}} F_A(t) = \\ &= \frac{1}{\pi} \int_0^A \frac{\partial}{\partial t} \left(\frac{\partial^k}{\partial t^k} I(y, t) \right) dy = \frac{1}{\pi} \int_0^A \frac{\partial^{k+1}}{\partial t^{k+1}} I(y, t)\, dy = {}^{k+1}F_A(t) \end{aligned} \tag{16.8}$$

holds for the integral (16.7) at every point $t \in \mathbf{E}_1$. The assertion $V(k + 1)$ holds by (16.8). Thus it is proved that the assertion $V(k)$ holds for all integer $k \geqq 1$.

A direct consequence of Theorem 16 is the following theorem.

17. Theorem. *Let* $f(t) \in L(-\infty, \infty)$. *The function* $F_A(t)$ *of the variable* $t \in \mathbf{E}_1$ [see (14.1)] *is infinitely continuously differentiable in the set* $\mathbf{E}_1$.

18. Let the interval (a, b) $(-\infty \leqq a < b \leqq \infty)$ be a part of the interval (α, β) $(-\infty \leqq \alpha < \beta \leqq \infty)$. Every distribution $f(t)$ with the domain (α, β) can be interpreted as a distribution with the domain (a, b) (see [1], page 30). We write $f(t) = g(t)$ if and only if the distributions $f(t)$, $g(t)$ are defined in the same interval and are equal. We write $f(t) = g(t)$ in the interval $a < t < b$ if and only if the interval (a, b) is a part of the domains of the distributions $f(t)$, $g(t)$ and the "interpreted" distributions $f(t)$, $g(t)$ coincide on the interval (a, b). For instance

$$\delta(t) = 0 \quad \text{on the interval} \quad -\infty < t < 0\,,$$
$$\delta(t) = 0 \quad \text{on the interval} \quad 0 < t < \infty\,.$$

19. Theorem. *Let* $f(t) \in L(-\infty, \infty)$. *Further, let* $f(t)$ *be continuous on the interval* $\langle a, b\rangle$ $(-\infty < a < b < \infty)$ *and have finite variation in this interval. For the interpreted distributions* $F_A(t)$ [see (14.1)], $f(t)$ *in the interval* (a, b) *we then have*

$$\lim_{A\to\infty} F_A(t) = f(t) \tag{19.1}$$

in the distributive sense.

Proof. It can be proved (see [2], e.g.) that $\lim_{A\to\infty} F_A(t) = f(t)$ uniformly with respect to t in the interval (a, b). Then (19.1) holds in the distributive sense by Theorem 10.2 (see [1]).

20. Theorem. *Let* $-\infty \leqq a < b < \infty$. *Further let* $f(t) \in L(a, b)$. *Then we have*

$$\int_a^b f(t)\,\mathrm{d}t = \mathrm{D}\int_a^b f(t)\,\mathrm{d}t\,. \tag{20.1}$$

By the symbol D *we indicate that the integral on the right-hand side of equality* (20.1) *is to be understood in the distributive sense.*

Proof. Put $f(t) = 0$ for $t \in \mathbf{E}_1 - (a, b)$. Further let

$$G(t) = \begin{cases} 0 & \text{for} \quad t \leqq a \quad (\text{provided } a \in \mathbf{E}_1)\,, \\ \int_a^t f(x)\,\mathrm{d}x & \text{for} \quad t \in (a, b)\,, \\ \int_a^b f(x)\,\mathrm{d}x & \text{for} \quad t \geqq b\,. \end{cases} \tag{20.2}$$

The distribution $G(t)$ is an indefinite integral of the distribution $f(t)$ with the domain $\mathbf{E}_1$.

I. Let $a \in \mathbf{E}_1$. The function $G(t)$ is locally integrable and continuous at the points a, b. By Theorem 16.3 (see [1]) there exist values of the distribution $G(t)$ at the points a, b and they are equal to the number $G(a)$, $G(b)$, respectively. We have

$$\mathrm{D}\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a) = \int_a^b f(x)\,\mathrm{d}x - 0 = \int_a^b f(x)\,\mathrm{d}x\,.$$

II. Let $a = -\infty$. Since the distribution $G(t)$ is a continuous function and $\lim_{t \to -\infty} G(t) = 0$ holds (in the usual sense), by Theorem 18.2 (see [1]) the value of the distribution $G(t)$ at the point $-\infty$ exists and we have $G(-\infty) = 0$. Hence

$$\mathrm{D} \int_a^b f(x)\, \mathrm{d}x = G(b) - G(-\infty) = G(b) = \int_a^b f(x)\, \mathrm{d}x\,.$$

In both cases (20.1) is true.

Similarly the following theorem holds.

21. Theorem. *Let* $-\infty < a < b \leqq \infty$. *Further, let* $f(t) \in L(a, b)$. *Then*

(21.1) $$\int_a^b f(t)\, \mathrm{d}t = \mathrm{D} \int_a^b f(t)\, \mathrm{d}t\,.$$

The following theorem is proved similarly as Theorem 20 and 21.

22. Theorem. *Let* $f(t) \in L(-\infty, \infty)$. *Then we have*

(22.1) $$\int_{-\infty}^{\infty} f(t)\, \mathrm{d}t = \mathrm{D} \int_{-\infty}^{\infty} f(t)\, \mathrm{d}t\,.$$

Proof. Let $c \in \mathbf{E}_1$. We may assume that $f(c) = 0$. Let us consider the functions

$$f_1(t) = \begin{cases} f(t) & \text{for } t < c\,, \\ 0 & \text{for } t \in \mathbf{E}_1 - (-\infty, c) \end{cases}$$

and

$$f_2(t) = \begin{cases} 0 & \text{for } t \in \mathbf{E}_1 - (c, \infty)\,, \\ f(t) & \text{for } t > c\,. \end{cases}$$

Then $f(t) = f_1(t) + f_2(t)$ holds in the set $\mathbf{E}_1$. To the function $f_1(t)$ we assign the function [cf. (20.2)]

(22.2) $$G_1(t) = \begin{cases} \int_{-\infty}^{t} f_1(x)\, \mathrm{d}x & \text{for } t \in (-\infty, c)\,, \\ \int_{-\infty}^{c} f_1(x)\, \mathrm{d}x & \text{for } t \geqq c \end{cases}$$

and to the function $f_2(t)$ we assign the function

(22.3) $$G_2(t) = \begin{cases} -\int_t^{\infty} f_2(x)\, \mathrm{d}x & \text{for } t \in (c, \infty)\,, \\ -\int_c^{\infty} f_2(x)\, \mathrm{d}x & \text{for } t \leqq c\,. \end{cases}$$

Then

(22.4) $$G(t) = G_1(t) + G_2(t) = \begin{cases} \int_{-\infty}^{t} f_1(x)\, \mathrm{d}x - \int_c^{\infty} f_2(x)\, \mathrm{d}x & \text{for } t < c\,, \\ \int_{-\infty}^{c} f_1(x)\, \mathrm{d}x - \int_c^{\infty} f_2(x)\, \mathrm{d}x & \text{for } t = c\,, \\ \int_{-\infty}^{c} f_1(x)\, \mathrm{d}x - \int_t^{\infty} f_2(x)\, \mathrm{d}x & \text{for } t > c\,. \end{cases}$$

The distribution $G(t)$ is an indefinite integral of the distribution $f(t)$ with the domain $\mathbf{E}_1$. By (20.1) (where we put $a = -\infty$, $b = c$), (21.1) (where we put $a = c$, $b = \infty$)

and (22.2)–(22.4) we then have

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = \int_{-\infty}^{c} f_1(t)\,\mathrm{d}t + \int_c^{\infty} f_2(t)\,\mathrm{d}t =$$
$$= \mathrm{D}\int_{-\infty}^{c} f_1(t)\,\mathrm{d}t + \mathrm{D}\int_c^{\infty} f_2(t)\,\mathrm{d}t = G_1(c) - G_1(-\infty) + G_2(\infty) -$$
$$- G_2(c) = G(\infty) - G(-\infty) = \mathrm{D}\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t\,,$$

i.e. (22.1) holds.

23. Let $f(t) \in L(-\infty, \infty)$. For the inner integral in relation (14.1) which is a convergent Lebesgue integral, the following holds by Theorem 22: $\int_{-\infty}^{\infty} = \mathrm{D}\int_{-\infty}^{\infty}$ (we do not write out the corresponding integrands in full). Further, the outer integral ${}^0F_A(t) \equiv F_A(t) = (1/\pi)\int_0^A$ in this relation is a convergent Lebesgue integral (the integrand is again omitted here), i.e. by Theorem 20 and 21 we have $\int_0^A = \mathrm{D}\int_0^A$. Both the inner and outer integrals in relation (14.1) can be understood either in the usual or in the distributive sense. The same holds for the integral ${}^kF_A(t)$ [see (15.1)].

The following theorem is true.

24. Theorem. *Let $f(t) \in L(-\infty, \infty)$. Further, let this function be continuous on the interval $\langle a, b\rangle$ $(-\infty < a < b < \infty)$ and have finite variation in this interval. Let $k \geqq 0$ be an arbitrary integer. For the interpreted distributions $F_A(t)$* [see (14.1)] *and $f(t)$ on the interval (a, b) we then have*

$$\lim_{A\to\infty} F_A^{(k)}(t) = \lim_{A\to\infty} \frac{1}{\pi}\int_0^A \left(\int_{-\infty}^{\infty} f(x)\cos y(x-t)\,\mathrm{d}x\right)_t^{(k)} \mathrm{d}y = f^{(k)}(t) \tag{24.1}$$

in the distributive sense (see Sect. 23).

Proof. By Theorem 19, $\lim_{A\to\infty} F_A(t) = f(t)$ holds (in the distributive sense). By Theorem VII (see [1]) we then have (in the distributive sense)

$$\lim_{A\to\infty} F_A^{(k)}(t) = f^{(k)}(t)\,. \tag{24.2}$$

By Theorem 17, the function $F_A(t)$ is infinitely continuously differentiable on the set $\mathbf{E}_1$, and by Theorem 6.5 (see [1]) we have (under the corresponding interpretations in the interval $a < t < b$)

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\int_{-\infty}^{\infty} f(x)\cos y(x-t)\,\mathrm{d}x\right) = \left(\int_{-\infty}^{\infty} f(x)\cos y(x-t)\,\mathrm{d}x\right)_t^{(k)},$$

i.e. [see (24.3)]

$$F_A^{(k)}(t) = \frac{1}{\pi}\int_0^A \left(\int_{-\infty}^{\infty} f(x)\cos y(x-t)\,\mathrm{d}x\right)_t^{(k)} \mathrm{d}y\,. \tag{24.4}$$

Then (24.2) and (24.4) imply (24.1). The proof is complete.

25. Let $f(t)$ be a distribution with the domain $\mathbf{E}_1$. By Theorem VII (see [1]) there exists an integer $k \geqq 0$ and a function $\tilde{f}(t)$ continuous in the set $\mathbf{E}_1$ such that $f(t) = \tilde{f}^{(k)}(t)$.

26. Definition. *We say that a distribution $f(t)$ with the domain $\mathbf{E}_1$ is an element of a class $\mathscr{K}$, if there exist an interval $\langle a, b\rangle$ $(-\infty < a < b < \infty)$, an integer $k \geqq 0$ and a function $\tilde{f}(t)$ continuous in the set $\mathbf{E}_1$ with the following properties:*

1. $f(t) = \tilde{f}^{(k)}(t)$,
2. $\tilde{f}(t) \in L(-\infty, \infty)$,
3. *the function $\tilde{f}(t)$ has finite variation in the interval $\langle a, b\rangle$.*

27. The class $\mathscr{K}$ is not empty. For instance, every function $f(t)$ continuous in the set $\mathbf{E}_1$, belonging to the class $L(-\infty, \infty)$ and having finite variation in an interval $\langle a, b\rangle$ $(-\infty < a < b < \infty)$ is, as a distribution, an element of the class $\mathscr{K}$. It is easy to show that this class contains also some discontinuous functions, e.g. $H(t+1) - 2H(t) + H(t-1)$ for $t \in \mathbf{E}_1$ [see (13.1)] belongs as a distribution to the class $\mathscr{K}$.

In conclusion we prove the following theorem:

28. Theorem. *Let a distribution $f(t)$ with the domain $\mathbf{E}_1$ be an element of the class $\mathscr{K}$* (see Definition 26). *For an interpreted distribution $f(t)$ in the interval (a, b) we have*

$$f(t) = \lim_{A\to\infty} \frac{1}{\pi} \int_0^A \left(\int_{-\infty}^{\infty} f(x) \cos y(x - t)\, dx\right) dy \tag{28.1}$$

(the limit and integrals in equality (28.1) are understood in the distributive sense).

Proof. By Definition 26 there exist an interval $\langle a, b\rangle$ $(-\infty < a < b < \infty)$, an integer $k \geqq 0$ and a function $\tilde{f}(t)$ continuous in the set $\mathbf{E}_1$ so that 1, 2 and 3 hold true. For the interpreted distribution $\tilde{f}(t)$ in the interval (a, b) we then have, by (24.1),

$$\tilde{f}^{(k)}(t) = \lim_{A\to\infty} \frac{1}{\pi} \int_0^A \left(\int_{-\infty}^{\infty} \tilde{f}(x) \cos y(x - t)\, dx\right)_t^{(k)} dy\,. \tag{28.2}$$

Since the distribution $\tilde{f}(t)$ with the domain $\mathbf{E}_1$ satisfies the assumptions of Theorem 12, it follows by (12.1), property 1 from Definition 26, and (28.2) that for the interpreted distribution $f(t)$ in the interval (a, b) we have

$$\tilde{f}^{(k)}(t) = \lim_{A\to\infty} \frac{1}{\pi} \int_0^A \left(\int_{-\infty}^{\infty} \tilde{f}(x) \cos y(x - t)\, dx\right)_t^{(k)} dy =$$

$$= \lim_{A\to\infty} \frac{1}{\pi} \int_0^A \left(\int_{-\infty}^{\infty} \tilde{f}^{(k)}(x) \cos y(x - t)\, dx\right) dy =$$

$$= \lim_{A\to\infty} \frac{1}{\pi} \int_0^A \left(\int_{-\infty}^{\infty} f(x) \cos y(x - t)\, dx\right) dy = f(t)\,,$$

i.e. (28.1) holds true.

References

[1] *J. Mikusiński, R. Sikorski:* The elementary theory of distributions (I). Rozprawy matematyczne, Warszawa 1957.

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Souhrn

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V práci uvažované distribuce se chápou ve smyslu Mikusińského-Sikorského. Předmětem je třída $\mathscr{K}$ distribucí $f(t)$ s definičním oborem $\mathbf{E}_1$, k nimž existuje interval $\langle a, b\rangle$ $(-\infty < a < b < \infty)$, celé číslo $k \geqq 0$ a funkce $\tilde{f}(t)$ spojitá v množině $\mathbf{E}_1$ tak, že platí: 1. $f(t) = \tilde{f}^{(k)}(t)$ (v distributivním smyslu), 2. $\tilde{f}(t) \in L(-\infty, \infty)$, 3. $\tilde{f}(t)$ má konečnou variaci v intervalu $\langle a, b\rangle$. V práci je dokázáno, že pro každou distribuci z třídy $\mathscr{K}$, interpretovanou v intervalu (a, b), platí reprezentace Fourierovým integrálem.

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