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THE STABILITY OF RITZ-VOLTERRA PROJECTION AND ERROR ESTIMATES FOR FINITE ELEMENT METHODS FOR A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

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Summary: In this paper we first study the stability of Ritz-Volterra projection (see below) and its maximum norm estimates, and then we use these results to derive some L^∞ error estimates for finite element methods for parabolic integro-differential equations.

Keywords: Ritz-Volterra projection, stability, finite element, error estimates.

AMS Classification: 65N30.

1. INTRODUCTION

In the study of finite element methods for parabolic integro-differential equations [1, 2, 6], Sobolev equations and the equations of visco-elasticity [6], the following Ritz-Volterra projection has been introduced: For $u(t) \in \mathring{W}_2^1(\Omega)$, $t \in J = (0, T]$, its Ritz-Volterra projection $V_h(t)$: $C(\bar{J}; \mathring{W}_2^1(\Omega)) \to C(\bar{J}; S_h)$ is defined by

$$(1.1) \qquad A(t; V_h u - u, \chi) + \int_0^t B(t, \tau; V_h u(\tau) - u(\tau), \chi) \, d\tau = 0 , \quad \chi \in S_h , \quad t \in \bar{J},$$

where $A(t; \cdot, \cdot)$ and $B(t, \tau; \cdot, \cdot)$ are the bilinear forms associated with the positive symmetric definite elliptic operator $A(t)$ and an arbitrary second order operator $B(t, \tau)$, respectively, with smooth coefficients, $\Omega \subset R^d$ $(d \geqq 1)$ is a bounded domain, and $S_h \subset \mathring{W}_2^1(\Omega)$, with a small parameter $h > 0$, are finite dimensional subspaces. $\|\cdot\|_p = \|\cdot\|_{0,p}$, $\|\cdot\| = \|\cdot\|_{0,2}$ and $\|\cdot\|_{r,p}$ denote the norm on the Sobolev spaces $W_p^r(\Omega)$ for $2 \leqq p \leqq \infty$.

Notice that when $t = 0$, we have $V_h(0) = R_h$, the standard Ritz projection associated with the operator $A(0)$.

It has been proved in [1, 2, 6] that the Ritz-Volterra projection V_h defined by (1.1) exists and is unique, and it also enjoyes the following approximation properties [6]:

for $t \in \bar{J}$,

$$\|D_t^j(V_h u(t) - u(t))\| + h\|D_t^j(V_h u(t) - u(t))\|_{1,2} \leqq Ch^r \sum_{l=0}^{j} |||D_t^l u(t)|||_{r,2} \tag{1.2}$$

$$\text{for} \quad u \in \mathring{W}_2^1 \cap W_2^r, \; j = 0, 1, \quad 1 \leqq r \leqq k,$$

provided that the approximation space S_h satisfies for some $k \geqq 2$ the inequality

$$\inf_{\chi \in S_h} \{\|u - \chi\| + h\|u - \chi\|_{1,2}\} \leqq Ch^s \|u\|_{s,2}, \quad 1 \leqq s \leqq k,$$

where

$$|||u(t)|||_{r,p} = \|u(t)\|_{r,p} + \int_0^t \|u(\tau)\|_{r,p} \, d\tau .$$

Here and in what follows we denote by C the generic constants independent of u and h, if not stated otherwise.

Now we consider the finite element solution for the following parabolic integro-differential equation

$$\begin{aligned} u_t + A(t)\, u + \int_0^t B(t, \tau)\, u(\tau)\, d\tau &= f \quad \text{in} \quad \Omega \times J, \\ u &= 0 \quad \text{on} \quad \partial\Omega \times J, \\ u &= v \quad \text{in} \quad \Omega \times \{0\}, \end{aligned} \tag{1.3}$$

and let $u_h(t)$ be its semi-discrete finite element analogue [1, 6]. By using the Ritz-Volterra projection V_h defined by (1.1) the authors of [6] have shown for smooth data $u(0) = v$ that if $\|u_h(0) - v\| \leqq Ch^r \|v\|_{r,2}$, then

$$\|u(t) - u_h(t)\| \leqq Ch^r \{\|v\|_{r,2} + \int_0^t \|u_t(\tau)\|_{r,2} \, d\tau\}, \tag{1.4}$$

which is the same error as that for parabolic equations [14]. The estimates (1.4) was obtained also by Thomee and Zhang in [13] by employing the standard Ritz projection R_h [10]. A slightly weak error estimates similar to (1.4) has been shown in [1, 2]. We know from [1, 2, 6] that it is easier and more convenient to use the Ritz-Volterra projection V_h than the Ritz projection R_h in the study of finite element methods for problem (1.3), and moreover, this new projection V_h has a variety of other applications [6].

It is well known (see [10]) that if S_h are piecewise polynomial spaces imposed on quasi-uniform triangulations of Ω, the Ritz projection R_h satisfies the stability estimate

$$\|R_h u\|_{1,p} \leqq C\|u\|_{1,p}, \quad 2 \leqq p \leqq \infty . \tag{1.5}$$

More importantly, this stability can be used to derive some optimal error estimates for finite element approximations for elliptic [10] and parabolic equations.

In this paper we study the stability of our Ritz-Volterra projection V_h. Due to the complexity of the problem, the integral term and the corresponding loss of ellipticity, we shall consider only a special case of (1.1). Namely, we assume that $\Omega \subset R^2$,

$$A(t) = -\nabla \cdot a(\cdot, t) \nabla, \quad B(t, \tau) = -\nabla \cdot b(\cdot, t, \tau) \nabla \tag{1.6}$$

where $a(x, t) \geqq a_0 > 0$ and $b = b(x, t, \tau)$ are smooth functions, and ∇ is the gradient operator in R^2. Thus, the Ritz-Volterra projection V_h in (1.1) becomes

$$(a(\cdot, t) \nabla(V_h u(t) - u(t)) + \int_0^t b(\cdot, t, \tau) \nabla(V_h u(\tau) - u(\tau)) \, d\tau, \nabla\chi) = 0 ,$$
$$\chi \in S_h , \quad t \in \bar{J} ,$$

or for short,

$$a(t; V_h u(t) - u(t), \chi) + \int_0^t b(t, \tau; V_h u(\tau) - u(\tau), \chi) \, d\tau = 0 , \tag{1.7}$$
$$\chi \in S_h , \quad t \in \bar{J} ,$$

where $a(t; \cdot, \cdot)$ and $b(t, \tau; \cdot, \cdot)$ are the bilinear forms associated with the above special operators in (1.6).

We shall show in Section 2 the following result for V_h defined in (1.7).

$$\|V_h u(t)\|_{1,p} \leqq C|||u(t)|||_{1,p} , \quad 2 \leqq p \leqq \infty . \tag{1.8}$$

Although (1.7) is a very simple case of (1.1) it still preserves the essential features for the general Ritz-Volterra projection V_h. That is, it is our conjecture that the stability result (1.8) will hold for the general form (1.1).

In Section 2 we state and prove our main theorems. In Section 3 we shall employ the results obtained in Section 2 to derive some optimal error estimates for finite element methods for parabolic integro-different equations.

2. STABILITY OF RITZ-VOLTERRA PROJECTION

Let Ω be a bounded domain in R^2 with smooth boundary $\partial\Omega$. For $k \geqq 2$, $0 < h \leqq 1$, let S_h^k be a one parameter family of finite-dimensional subspaces of $\mathring{W}_2^1(\Omega)$, consisting of piecewise polynomial functions of degree at most $k - 1$, defined on a quasi-uniform partition of Ω. It is required that S_h^k possesses the following approximation property: For all $w \in \mathring{W}_2^1(\Omega) \cap W_p^k(\Omega)$,

$$\inf_{\chi \in S_h{}^k} (\|w - \chi\|_p + h\|w - \chi\|_{1,p}) \leqq Ch^s\|w\|_{s,p} , \quad p \geqq 2 , \quad 1 \leqq s \leqq k . \tag{2.1}$$

Lemma 2.1. *Let P_h: $L^2(\Omega) \to S_h^k$ be the L^2-projection, then*

$$\|P_h w\|_{s,p} \leqq C\|w\|_{s,p} , \quad s = 0, 1 , \quad 2 \leqq p \leqq \infty . \tag{2.2}$$

Proof. See [9]. Q.E.D.

Let $z \in \Omega$ and let $\delta_h^z \in S_h^k$ be the discrete δ-function at z such that

$$(\delta_h^z, \chi) = \chi(z) , \quad \chi \in S_h^k \tag{2.3}$$

Let G^z be the smooth Green's function at z that

$$-\nabla . a \nabla G^z = \delta_h^z \quad \text{in} \quad \Omega , \tag{2.4}$$
$$G^z = 0 \quad \text{no} \quad \delta\Omega .$$

It is obvious that $G^z \in \mathring{W}_2^1(\Omega) \cap W_2^2(\Omega)$ exists and is unique, and it follows by (2.3) that

$$a(t; G^z, w) = P_h w(z)\,, \quad w \in \mathring{W}_2^1(\Omega)\,. \tag{2.5}$$

Let $G_h^z \in S_h^k$ be the Ritz projection of G^z, i.e.,

$$a(t; G^z - G_h^z, \chi) = 0\,, \quad \chi \in S_h^k\,. \tag{2.6}$$

It is well known [12] that

$$\|G^z - G_h^z\|_{1,1} \leqq Ch\left(\log \frac{1}{h}\right)^{k^*}, \quad k^* = \begin{cases} 1 & \text{if} \quad k = 2\,, \\ 0 & \text{if} \quad k \geqq 3\,. \end{cases} \tag{2.7}$$

Define [8]

$$\partial_z G^z = \lim_{\Delta z \to 0,\ \Delta z // L} \frac{G^{z+\Delta z} - G^z}{|\Delta z|}\,,$$

where L is an arbitrary fixed direction. We know from (2.4)–(2.6) and [8] that $\partial_z G^z \in \mathring{W}_2^1(\Omega) \cap W_2^2(\Omega)$ exists and is such that

$$a(t; \partial z G^z, w) = \partial_z w(z)\,, \quad w \in \mathring{W}_2^1(\Omega)\,, \tag{2.8}$$

$$a(t; \partial_z G^z - \partial_z G_h^z, \chi) = 0\,, \quad \chi \in S_h^k\,. \tag{2.9}$$

Let $\phi(x) = (|x - z|^2 + \varrho^2)^{-1}$, with $\varrho = \gamma h$ and γ large enough, be the weight. We define the weighted norms for $\alpha \in R$,

$$\|f\|_{\phi^\alpha} = (\textstyle\int_\Omega \phi^\alpha |f|^2 \, dx)^{1/2}\,,$$

$$\|f\|_{1,\phi^\alpha} = (\textstyle\int_\Omega \phi^\alpha (|f|^2 + |\nabla f|^2) \, dx)^{1/2}\,.$$

It follows from a direct computation that

$$\textstyle\int_\Omega \phi^\alpha(x) \, dx \leqq C(\alpha - 1)^{-1} \varrho^{-2(\alpha-1)}\,, \quad \alpha > 1\,.$$

We now recall the following results concerning the estimates for Green's function G^z and its Ritz projection G_h^z [8, 10].

Lemma 2.1. *Under our assumptions on* S_h^k, *we have*

$$\|\partial_z G^z - \partial_z G_h^z\|_{1,\phi^{-1-\varepsilon}} \leqq Ch^\varepsilon\,, \quad \varepsilon \in (0, 1)\,, \tag{2.10}$$

$$\|\partial_z G^z - \partial_z G_h^z\|_{1,1} + \|G^z\|_{1,1} + \|G_h^z\|_{1,1} + \|G_h^z\| \leqq C\,, \tag{2.11}$$

$$\|\partial_z G^z\|_q \leqq C\,, \quad 1 \leqq q \leqq 3/2\,. \tag{2.12}$$

Proof. (2.10)–(2.11) can be found in [8, 10]. For (2.12), let w satisfy

$$-\nabla \cdot a \nabla w = g\,, \quad x \in \Omega\,, \quad w = 0\,, \quad \text{on} \quad \partial\Omega\,.$$

and

$$\|w\|_{2,p} \leqq C_p \|g\|_p\,, \quad 1 < p < \infty\,.$$

Let $p_0 = 3$, we see from (2.7), stability of P_h and Sobolev imbedding theorem that

$$(\partial_z G^z, g) = a(t; \partial_z G^z, w) = \partial_z P_h w(z) \leqq$$
$$\leqq C\|w\|_{1,\infty} \leqq C\|w\|_{2,3} \leqq C_3\|g\|_3 \leqq C\|g\|_p, \quad 3 \leqq p \leqq \infty .$$

Thus, (2.12) follows. Q.E.D.

We now state and show our main result in this section.

Theorem 2.1. *Assume that $u \in L^1(J; \mathring{W}^1_p(\Omega))$. Then the following stability estimate for our Ritz-Volterra projection V_h holds,*

$$\|V_h u(t)\|_{1,p} \leqq C|||u(t)|||_{1,p}, \quad t \in \bar{J}, \quad 2 \leqq p \leqq \infty . \tag{2.13}$$

Remark. When $t = 0$, (1.2) is just the stability estimate (1.5) obtained by Rannacher and Scott [10] for Ritz projection R_h.

Proof. It has been shown by an argument of duality in [6] that

$$\|V_h u - u\|_{1,p} \leqq C_p|||u|||_{1,p}, \quad 2 \leqq p < \infty .$$

Thus, the case of $2 \leqq p \leqq 3$ follows.

For $3 \leqq p < \infty$, let $\eta = u(t) - V_h u(t)$, then we see from the definition of V_h and Green's functions that

$$\partial_z P_h \eta(z, t) = a(t; \eta, \partial_z G^z) + \int_0^t b(t, \tau; \eta(\tau), \partial_z G^z)\, d\tau -$$
$$- \int_0^t b(t, \tau; \eta(\tau), \partial_z G^z)\, d\tau = a(t; u, \partial_z G^z - \partial_z G^z_h) +$$
$$+ \int_0^t b(t, \tau; \eta(\tau), \partial_z G^z - \partial_z G^z_h)\, d\tau - \int_0^t b(t, \tau; \eta(\tau), \partial_z G^z)\, d\tau$$
$$= a(t; u, \partial_z G^z - \partial_z G^z_h) + \int_0^t b(t, \tau; u(\tau) - P_h u(\tau), \partial_z G^z - \partial_z G^z_h)\, d\tau +$$
$$+ \int_0^t b(t, \tau; P_h \eta(\tau)), \partial_z G^z - \partial_z G^z_h)\, d\tau - \int_0^t b(t, \tau; \eta(\tau), \partial_z G^z)\, d\tau =$$
$$= I_1 + \int_0^t (I_2 + I_3 + I_4)\, d\tau .$$

We see from Lemma 2.1 and Hölder inequality [10] that for I_1,

$$|I_1| \leqq C(\int_\Omega \phi^{1+\varepsilon}\, dx)^{(p-2)/2p} (\int_\Omega \phi^{1+\varepsilon}(|u|^p + |\nabla u|^p)\, dx)^{1/p} \|\partial_z G^z -$$
$$- \partial_z G^z_h\|_{1,\phi^{-1-\varepsilon}} \leqq Ch^{2\varepsilon/p}(\int_\Omega \phi^{1+\varepsilon}(|u|^p + |\nabla u|^p)\, dx)^{1/p} . \tag{2.14}$$

Similarly, we have

$$|I_2| \leqq Ch^{2\varepsilon/p}(\int_\Omega \phi^{1+\varepsilon}(|u(\tau) - P_h u(\tau)|^p + |\nabla(u(\tau) - P_h u(\tau))|^p)\, dx)^{1/p} , \tag{2.15}$$

$$|I_3| \leqq Ch^{2\varepsilon/p}(\int_\Omega \phi^{1+\varepsilon}(|P_h \eta(\tau)|^p + |\nabla P_h \eta(\tau)|^p)\, dx)^{1/p} . \tag{2.16}$$

We can write I_4 as

$$I_4 = -b(t, \tau; u(\tau) - P_h u(\tau), \partial_z G^z) - b(t, \tau; P_h \eta(\tau), \partial_z G^z) =$$
$$= -M_1 - M_2 .$$

Thus, it follows from the structure of the two operators in (1.6) and by integration by parts that

$$|M_2| = \left|(a(\cdot, t)\,\nabla\left[\left(\frac{b(\cdot, t, \tau)}{a(\cdot, t)}\right) P_h\,\eta(\tau)\right], \nabla\,\partial_z G^z) -\right.$$

$$\left. - \left(a(\cdot, t)\,P_h\,\eta(\tau)\,\nabla\left(\frac{b(\cdot, t, \tau)}{a(\cdot, t)}\right),\quad \nabla\,\partial_z G^z\right)\right| =$$

$$= \left|\partial_z P_h\left[\left(\frac{b(z, t, \tau)}{a(z, t)}\right) P_h\,\eta(z, \tau)\right] +\right.$$

$$\left. + \left(\nabla\,.\,a(\cdot, t)\,P_h\,\eta(\tau)\,\nabla\left(\frac{b(\cdot, t, \tau)}{a(\cdot, t)}\right), \partial_z G^z\right)\right| \leqq$$

$$\leqq \left|\partial_z P_h\left[\left(\frac{b(z, t, \tau)}{a(z, t)}\right) P_h\,\eta(z, \tau)\right]\right| + C\|P_h\,\eta(\tau)\|_{1,p}\|\partial_z G^z\|_q \leqq$$

$$\leqq \left|\partial_z P_h\left[\left(\frac{b(z, t, \tau)}{a(z, t)}\right) P_h\,\eta(z, \tau)\right]\right| + C\|P_h\,\eta(\tau)\|_{1,p},$$

where we have used (2.12) for $1 \leqq q \leqq 3/2$ since $p \geqq 3$ and $p^{-1} + q^{-1} = 1$. Also, for the same reason we have

$$|M_1| \leqq \left|\partial_z P_h\left[\left(\frac{b(z, t, \tau)}{a(z, t)}\right)(u(z, \tau) - P_h\,u(z, \tau))\right]\right| + C\|u(\tau)\|_{1,p}.$$

Thus, we obtain from (2.14)–(2.16)

$$\|I_1\|_p \leqq Ch^{2\varepsilon/p}(\max_{x\in\Omega} \textstyle\int_\Omega \phi^{1+\varepsilon}\,\mathrm{d}z)^{1/p}\,\|u\|_{1,p} \leqq 1C\|u\|_{1,p},$$

$$\|I_2\|_p \leqq C\|u - P_h u\|_{1,p} \leqq C\|u\|_{1,p},$$

$$\|I_3\|_p \leqq C\|P_h\eta\|_{1,p},$$

and by estimates for $M_i's$, we have for I_4,

$$\|I_4\|_p \leqq C\|P_h\eta\|_{1,p} + C\|u\|_{1,p}.$$

Notice that if

$$H(x) = N(x) + \textstyle\int_0^t K(x, \tau)\,\mathrm{d}\tau,$$

then

$$\|H\|_p \leqq \|N\|_p + \textstyle\int_0^t \|K(\tau)\|_p\,\mathrm{d}\tau, \quad 2 \leqq p \leqq \infty.$$

Thus, we see from the estimates for $I_i's$ that

$$\|P_h\eta\|_{1,p} \leqq C|\!|\!|u(t)|\!|\!|_{1,p} + C\textstyle\int_0^t \|P_h\eta\|_{1,p}\,\mathrm{d}\tau, \quad 3 \leqq p < \infty.$$

Notice that the above inequality also holds for $p = \infty$ by using (2.7) [10]. Thus, Gronwall's lemma implies

$$\|P_h\eta\|_{1,p} \leqq C|\!|\!|u(t)|\!|\!|_{1,p}$$

and

(2.17) $$\|V_h u\|_{1,p} \leqq \|P_h \eta\|_{1,p} + \|P_h u\|_{1,p} \leqq C |||u(t)|||_{1,p} .$$

Hence, Theorem 2.1 follows. Q.E.D.

As a direct application of Theorem 2.1 we show the following result.

Corollary. *For any function* $u \in L^1(J; \mathring{W}^1_p \cap W^k_p)$ *we have*

(2.18) $$\|u(t) - V_h u(t)\|_{1,p} \leqq Ch^{k-1} |||u(t)|||_{k,p} , \quad 2 \leqq p \leqq \infty ,$$

(2.19) $$\|u(t) - V_h u(t)\|_p \leqq C_p h^k |||u(t)|||_{k,p} , \quad 2 \leqq p < \infty .$$

Remark. (2.19) has been shown in [6] by a different method and (2.18) is an improvement of the estimates obtained in [6].

Proof. Let I_h be the interpolant operator on S^k_h. We apply Theorem 2.1 for $u - I_h u$ and observe that $V_h \equiv \text{id}$ on S^k_h to obtain

(2.20) $$\|V_h u(t) - I_h u(t)\|_{1,p} \leqq C |||u(t) - I_h u(t)|||_{1,p} , \quad 2 \leqq p \leqq \infty .$$

Then, (2.18) follows from the approximation properties of the interpolant operator I_h.

To prove (2.19), let $p \in [2, \infty)$, $q = p/(p-1) \in (1, 2]$ and $w \in \mathring{W}^1_q \cap W^2_q$ be such that

(2.21) $$Aw = g = \operatorname{sgn}(u - V_h u) |u - V_h u|^{p-1} \quad \text{in} \quad \Omega ,$$

and

(2.22) $$\|w\|_{2,q} \leqq C_p \|g\|_q \leqq C_p \|u - V_h u\|_p^{p-1} .$$

Thus, by (2.21), (2.22) and Hölder's inequality we have

(2.23) $$\begin{aligned} \|u - V_h u\|_p^p &= a(t; u - V_h u, w - I_h w) + a(t; u - V_h u, I_h w) \leqq \\ &\leqq C \|u - V_h u\|_{1,p} \|w - I_h w\|_{1,q} + a(t; u - V_h u, I_h w) \end{aligned}$$

and by (1.8)

$$\begin{aligned} a(t; u - V_h u, I_h w) &= - \int_0^t b(t, \tau; u(\tau) - V_h u(\tau), I_h w - w) \, d\tau - \\ &- \int_0^t b(t, \tau; u(\tau) - V_h u(\tau), w) \, d\tau = \\ &= - \int_0^t b(t, \tau; u(\tau) - V_h u(\tau), I_h w - w) \, d\tau + \\ &+ \int_0^t (u(\tau) - V_h u(\tau), B(t, \tau) w) \, d\tau \leqq \\ &\leqq C \int_0^t \|u - V_h u\|_{1,p} \, d\tau (\|w - I_h w\|_{1,q} + \|w\|_{2,q}) , \end{aligned}$$

so that we see from (2.22)–(2.23) that

(2.24) $$\|u - V_h u\|_p \leqq C_p h^k |||u(t)|||_{k,p} + C_p \int_0^t \|u - V_h u\|_p \, d\tau .$$

Hence, the proof is complete by Gronwall's lemma.

We now consider the case of $p = \infty$, the maximum norm estimates, and show:

Theorem 2.2. *Under the assumptions of Theorem* 2.1, *we have*

$$(2.25) \qquad \|u(t) - V_h u(t)\|_{s,\infty} \leqq Ch^{k-s}\left(\log\frac{1}{h}\right)^{(1-s)k^*} |||u(t)|||_{k,\infty},$$

$$s = 0, 1, \quad k^* = \begin{cases} 1 & \text{if } k = 2, \\ 0 & \text{if } k \geqq 3. \end{cases}$$

Proof. For $s = 1$, this is a special case of (2.18) with $p = \infty$.
For $s = 0$, we have as shown in Theorem 2.1,

$$P_h \eta(z, t) = a(t; \eta, G^z - G^z_h) + \textstyle\int_0^t b(t, \tau; \eta, G^z - G^z_h)\, d\tau -$$
$$- \textstyle\int_0^t b(t, \tau; u(\tau) - P_h u(\tau)), G^z)\, d\tau - \int_0^t b(t, \tau; P_h \eta(\tau), G^z)\, d\tau =$$
$$= J_1 + J_2 + J_3 + J_4 .$$

From (2.7) and Theorem 2.1 we obtain

$$|J_1 + J_2| \leqq C|||\eta|||_{1,\infty} \|G^z - G^z_h\|_{1,1} \leqq Ch^k(\log(1/h)^{k^*} |||u(t)|||_{k,\infty},$$

and for J_3 we see from the stability of P_h that

$$J_3 = \textstyle\int_0^t \left(a(\cdot, t)\, \nabla\left(\frac{b(\cdot, t, \tau)}{a(\cdot, t)}\right)(u(\tau) - P_h u(\tau)), \nabla G^z\right) d\tau -$$
$$- \textstyle\int_0^t \left(a(\cdot, t)\,(u(\tau) - P_h u(\tau))\, \nabla\left(\frac{b(\cdot, t, \tau)}{a(\cdot, t)}\right), \nabla G^z\right) d\tau =$$
$$= \textstyle\int_0^t P_h \left[\left(\frac{b(z, t, \tau)}{a(z, t)}\right)(u(z, \tau) - P_h u(z, \tau))\right] d\tau +$$
$$+ \textstyle\int_0^t \left(a(\cdot, t)\,(u(\tau) - P_h u(\tau))\, \nabla\left(\frac{b(\cdot, t, \tau)}{a(\cdot, t)}\right), \nabla G^z\right) d\tau \leqq$$
$$\leqq C \textstyle\int_0^t \|u - P_h u\|_{0,\infty}\, d\tau + C \int_0^t \|u - P_h u\|_{0,\infty}\, d\tau \|G^z\|_{1,1} \leqq$$
$$\leqq Ch^k \textstyle\int_0^t \|u\|_{k,\infty}\, d\tau .$$

Similarly, we have

$$|J_4| \leqq C \textstyle\int_0^t \|P_h \eta\|_{0,\infty}\, d\tau .$$

Collecting the above estimates for J_i's we obtain

$$\|P_h \eta\|_{0,\infty} \leqq Ch^k(\log(1/h))^{k^*} |||u(t)|||_{k,\infty} + C \textstyle\int_0^t \|P_h \eta\|_{0,\infty}\, d\tau .$$

Thus, Gronwall's lemma implies

$$(2.26) \qquad \|P_h \eta\|_{0,\infty} \leqq Ch^k(\log(1/h)^{k^*} |||u(t)|||_{k,\infty} .$$

Hence, Theorem 2.2 follows from the inequality

$$\|V_h u - u\|_{0,\infty} \leqq \|P_h(V_h u - u)\|_{0,\infty} + \|P_h u - u\|_{0,\infty}$$

and (2.26) Q.E.D.

3. AN APPLICATION TO PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

In this section we consider some L^∞ error estimates for finite element methods for the parabolic integro-differential equation (1.3). As before we assume that the operators $A(t)$ and $B(t, \tau)$ are the special forms in (1.6).

Let $u_h(t)\colon \bar{J} \to S_h^k$ be the finite element solution of problem (1.3) defined by

$$(u_{h,t}, \chi) + a(t; u_h, \chi) + \int_0^t b(t, \tau; u_h(\tau), \chi)\, d\tau = (f, \chi), \quad \chi \in S_h^k,$$
$$u_h(0) = v_h \in S_h^k.$$

It has been shown in [6] that finite element approximations of parabolic integro-differential equations have "weak" L^∞ error estimates. That is, for any $\varepsilon > 0$ there exists a $C(\varepsilon, u) >$ such that

$$\|u(t) - u_h(t)\|_{L^\infty(\Omega)} \leqq C(\varepsilon, u)\, h^{k-\varepsilon}, \tag{3.1}$$

which is not optimal. Here we shall show the following result assuming sufficient regularity of the solution u at $t = 0$.

Theorem 3.1. *For $k = 2$, we assume that $u \in L^1(J; \mathring{W}_\infty^1 \cap W_\infty^2)$, $u_t \in L^2(J, W_2^2)$ and $v_h = V_h(0)\, v = R_h(0)\, v$. Then we have*

$$\|u(t) - u_h(t)\|_{0,\infty} \leqq Ch^2\{\log(1/h)\, (\|v\|_{2,\infty} + |||u(t)|||_{2,\infty}) + \log(1/h) \left(\int_0^t \|u_t\|_{2,2}^2\, d\tau\right)^{1/2}\}. \tag{3.2}$$

For $k \geqq 3$, we assume that $u \in L^1(J; \mathring{W}_\infty^1 \cap W_\infty^k)$, $u_t \in L^2(J; W_2^k)$ and $v_h = V_h(0)\, v = R_h(0)\, v$, $u_{tt} \in L^2(J; W_2^k)$, we have

$$\|u(t) - u_h(t)\|_{0,\infty} \leqq Ch^k\{\|v\|_{k,\infty} + |||u(t)|||_{k,\infty} + \|u_t(0)\|_{k,2} + \int_0^t \|u_{tt}\|_{k,2}\, d\tau\}. \tag{3.3}$$

Proof. As usual we write the error $e(t) = u(t) - u_h(t) = (u - V_h u) + (V_h u - u_h) = \eta + \theta$. Thus, we see from Theorem 2.2 that we need to estimate θ only.

We first show the case of $k = 2$. Since $v_h = V_h(0)\, v = R_h(0)\, v$, then $\theta(0) = 0$. It has been shown in [6] that

$$\|\theta\|_{1,2} \leqq Ch^2(\|v\|_{2,2} + (\int_0^t \|u_t\|_{2,2}^2\, d\tau)^{1/2}). \tag{3.4}$$

Thus, (3.2) follows from the "weak" Sobolev inequality on S_h^k [11],

$$\|\theta\|_{0,\infty} \leqq C(\log(1/h))^{1/2}\, \|\theta\|_{1,2}$$

and the triangle inequality.

Now for the case of $k \geqq 3$, we see that θ satisfies

$$a(t; \theta, \chi) + \int_0^t b(t, \tau; \theta(\tau), \chi) = -(e_t, \chi), \quad \chi \in S_h^k .$$

Letting $\chi = G_h^z$, it follows

$$\theta(z, t) = a(t; \theta, G_h^z) = -(e_t, G_h^z) - \int_0^t b(t, \tau; \theta(\tau), G_h^z)\, d\tau = K_1 + K_2 ,$$

and as before by Lemma 2.1 we write K_2 as

$$K_2 = - \int_0^t a\left(t; \frac{b(\cdot, t, \tau)}{a(\cdot, t)} \theta(\tau), G_h^h\right) d\tau +$$

$$+ \int_0^t \left(a(\cdot, t)\, \theta(\tau) \nabla \left(\frac{b(\cdot, t, \tau)}{a(\cdot, t)}\right), \nabla G_h^z\right) d\tau \leqq$$

$$\leqq -\int_0^t a\left(t; \frac{b(\cdot, t, \tau)}{a(\cdot, t)} \theta(\tau), G_h^h - G^z\right) d\tau -$$

$$- \int_0^t a\left(t; \frac{b(\cdot, t, \tau)}{a(\cdot, t)} \theta(\tau), G^z\right) d\tau + C \int_0^t \|\theta\|_{0,\infty}\, d\tau \|G^z\|_{1,1} \leqq$$

$$\leqq -\int_0^t a\left(t; \frac{b(\cdot, t, \tau)}{a(\cdot, t)} \theta(\tau), G_h^h - G^z\right) d\tau -$$

$$- \int_0^t P_h\left[\frac{b(z, t, \tau)}{a(z, t)} \theta(z, \tau)\right] d\tau + C \int_0^t \|\theta\|_{0,\infty}\, d\tau \|G^z\|_{1,1} \leqq$$

$$\leqq - \int_0^t P_h\left[\frac{b(z, t, \tau)}{a(z, t)} \theta(z, \tau)\right] d\tau +$$

$$+ C \int_0^t \|\theta\|_{1,\infty}\, d\tau\, \|G_h^z - G^z\|_{1,1} + C \int_0^t \|\theta\|_{0,\infty}\, d\tau .$$

By the inverse assumption (quasi-uniformity), stability of P_h, (2.7) and Lemma 2.1, we obtain

$$K_2 \leqq C \int_0^t \|\theta\|_{0,\infty}\, d\tau$$

and

$$K_1 \leqq \|e_t\|\, \|G_h^z\| \leqq C\|e_t\| .$$

Thus, we have

$$\|\theta\|_{0,\infty} \leqq C\|e_t\| + C \int_0^t \|\theta\|_{0,\infty}$$

and Gronwall's lemma implies

$$\|\theta\|_{0,\infty} \leqq C(\|e_t\| + \int_0^t \|e_t\|\, d\tau) .$$

However, we have from [6] that

$$\|e_t\| \leqq \|\eta_t\| + \|\theta_t\| \leqq$$

$$\leqq Ch^k\{\mathrm{f}|||u|||_{k,2} + |||u_t|||_{k,2} + \|v\|_{k,2} + \|u_t(0)\|_{k,2} + \int_0^t \|u_{tt}\|_{k,2}\, d\tau\} .$$

Hence, we have

$$(3.5)\qquad \|\theta\|_{0,\infty} \leqq Ch^k\{|||u|||_{k,2} + |||u_t|||_{k,2} + \|v\|_{k,2} + \|u_t(0)\|_{k,2} + \int_0^t \|u_{tt}\|_{k,2}\, d\tau\}$$

so that (3.3) follows from (3.5), Theorem 2.2 and the triangle inequality.

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Souhrn

STABILITA RITZ-VOLTERROVY PROJEKCE A ODHADY CHYBY PRO METODU KONEČNÝCH PRVKŮ PRO JEDNU TŘÍDU INTEGRO-DIFERENCIÁLNÍCH ROVNIC PARABOLICKÉHO TYPU

YANPING LIN, TIE ZHANG

V článku se nejdříve studuje stabilita Ritz-Volterrovy projekce a její odhady v maximální normě. Pomocí dosažených výsledků se odhadují L_∞-odhady chyb pro metodu konečných prvků pro parabolické integrodiferenciální rovnice.

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