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GLOBAL SOLUTION TO THE ISOTHERMAL COMPRESSIBLE BIPOLAR FLUID IN A FINITE CHANNEL WITH NONZERO INPUT AND OUTPUT

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Summary. The paper contains the proof of global existence of weak solutions viscous compressible isothermal bipolar fluid of initial boundary value in a finite channel.

Keywords: Viscous compressible bipolar fluid, initial boundary value problem, global existence of weak solutions.

AMS classification: 35Q20, 76N10.

1. INTRODUCTION

This article is inspired by the paper by J. Nečas, A. Novotný, M. Šilhavý [7] concerning the global solution to the isothermal compressible bipolar fluid where Orlicz spaces were used for describing finite entropy and theorems of the compensated compactness type. I follow in this paper the ideas of M. Feistauer, J. Nečas, V. Šverák [1], J. Nečas, A. Novotný, M. Šilhavý [7], M. Padula [9].

The main step in this work is the study of a bipolar fluid in a finite channel. Higher stress tensor implies the use of higher derivations of the velocity field. The existence of a global Hopf solution, under general initial data $(0, t_0) \times \boldsymbol{\Omega}$ with t_0 arbitrary and $\boldsymbol{\Omega} \subseteq \boldsymbol{R}^N$, $N = 2$ or 3 is proved.

In the present case, only one new stress tensor is needed, such that the momentum equations are of the 4th order. So we come to a bipolar fluid. The corresponding stress strain relations are supposed to be linear.

We suppose that density ϱ on input is $\varrho_0 > 0$, the velocity $v = v^0$ on input and output, where v_0 is extended to the entire $\boldsymbol{Q}_t$.

2. FORMULATION OF THE PROBLEM

We consider the classical state equation

$$(2.1) \qquad p = R\varrho T$$

where p, ϱ, T are pressure, density and temperature, respectively and R is the universal gas constant.

The isothermal process means that

$$p = \varrho\lambda\,, \quad \lambda = \text{const} > 0\,. \tag{2.2}$$

As usual we denote by v the velocity vector. The continuity equation assumes its stadard form

$$\frac{\partial\varrho}{\partial t} + \frac{\partial(\varrho v_i)}{\partial x_i} = 0\,. \tag{2.3}$$

A standard symmetric stress tensor τ_{ij} is considered such that

$$\tau_{ij} = -p\delta_{ij} + \tau_{ij}^d\,. \tag{2.4}$$

The general linear form for τ_{ij}^d, with coefficients depending on the temperature T only and therefore constant in our case, provided τ_{ij}^d are symmetric, is

$$\tau_{ij}^d = \gamma\frac{\partial v_l}{\partial x_l}\,\delta_{ij} + 2\mu e_{ij} - \gamma_1\Delta\frac{\partial v_l}{\partial x_l}\,\delta_{ij} - 2\mu_1\Delta e_{ij} + \gamma_2\frac{\partial^2}{\partial x_i\,\partial x_j}\left(\frac{\partial v_l}{\partial x_l}\right), \tag{2.5}$$

see [3].

We shall suppose that $\gamma \geqq -\frac{2}{3}\mu$, $\mu > 0$, $\gamma_1 > -\frac{2}{3}\mu_1$, $\mu_1 > 0$, $\gamma_2 = 0$, $2e_{ij} = \partial v_i/\partial x_j + \partial v_j/\partial x_i$. We consider further a 3rd order stress tensor τ_{ijk}^d. For it we require symmetry in i, j, then its general form according to [3] is

$$\tau_{ijk}^d = 2\mu_1\frac{\partial e_{ij}}{\partial x_k} + \gamma_1\delta_{ij}\frac{\partial e_{ll}}{\partial x_k} + \gamma_3\delta_{ij}\Delta v_k + \gamma_4\delta_{ik}\Delta v_j + \gamma_4\delta_{jk}\Delta v_i + \tag{2.6}$$
$$+ \gamma_5\delta_{ik}\frac{\partial e_{ll}}{\partial x_j} + \gamma_5\delta_{jk}\frac{\partial e_{ll}}{\partial x_i} + \gamma_6\frac{\partial^2 v_k}{\partial x_i\,\partial x_j} + \gamma_7\frac{\partial^2 v_i}{\partial x_j\,\partial x_k} + \gamma_7\frac{\partial^2 v_j}{\partial x_i\,\partial x_k}\,.$$

We shall restrict ourselves to the case $\gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = 0$. The Clausius-Duhem inequality (see [3])

$$\tau_{ij}^d e_{ij} + \tau_{ijk}^d\frac{\partial^2 v_i}{\partial x_j\,\partial x_i} + \frac{\partial\tau_{ijk}^d}{\partial x_k}\frac{\partial v_i}{\partial x_k} \geqq 0 \tag{2.7}$$

is satisfied for (2.6), (2.5).

Let $\mathbf{\Omega} \subseteq \mathbf{R}^N$, $N = 2$ or 3 be a bounded domain with a smooth, infinitely differentiable boundary and let $\mathbf{Q}_{t_0} = (0, t_0) \times \mathbf{\Omega}$ be the time-space cylinder.

The momentum equations combined with (2.3) yield

$$\frac{\partial(\varrho v_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\varrho v_i v_j + \delta_{ij}p - \tau_{ij}^d\right) = 0\,. \tag{2.8}$$

In addition to initial conditions for v and ϱ, we suppose that $\mathbf{\Omega}$ is a finite channel, where we have the following conditions: for the input, output and on the upper and lower sides.

We suppose that the velocity v need not be equal to zero in two parts of $\partial\mathbf{\Omega}$:

$\mathbf{\Gamma}_{inp} = \{x \in \partial\mathbf{\Omega};\ v\nu < 0\}$,

$\mathbf{\Gamma}_{out} = \{x \in \partial\mathbf{\Omega};\ v\nu > 0\}$; and we denote

$\mathbf{\Gamma}_{upp+low} = \{x \in \partial\mathbf{\Omega};\ \partial\mathbf{\Omega} \setminus [\mathbf{\Gamma}_{inp} \cup \mathbf{\Gamma}_{out}]\}$,

where ν is the outer normal.

Conditions for the velocity are:

(2.9) $$v = v^0 \quad \text{on} \quad \mathbf{\Gamma}_{inp} \cup \mathbf{\Gamma}_{out},$$

(2.10) $$v = 0 \quad \text{on} \quad \mathbf{\Gamma}_{upp+low}.$$

Conditions for density:

we suppose that

(2.13) $$\varrho = \varrho_0 \quad \text{on} \quad \mathbf{\Gamma}_{inp},$$

(2.14) $$\varrho = \varrho_0 \quad \text{in } \mathbf{\Omega} \text{ for } \quad t = 0.$$

Let us suppose that we are already given a solution ϱ, v $(\varrho \geqq 0)$ which is sufficienty smooth. Assume that v_0 is such a function that there exists its extension onto the whole cylinder $\mathbf{Q}_{t_0}$ so that this extension is an element of $\mathbf{L}^2((0, T), \mathbf{W}^{2,2}(\mathbf{\Omega}))$. We shall denote the extension by v_0 again.

Then we can write

(2.15) $$v = v^0 + w,$$

where

(2.16) $$w = 0 \quad \text{on} \quad (0, t_0) \times \partial\mathbf{\Omega}.$$

We assume another boundary condition:

(2.17) $$\tau^d_{ijk}\nu_j\nu_k = 0 \quad \text{on} \quad (0, t_0) \times \partial\mathbf{\Omega}.$$

Now we shall need apriori estimates.

Theorem 2.1. *Let ϱ, v, v^0 be smooth enough. Then*

(2.18) $$\int_{\mathbf{\Omega}_t} \varrho \, \mathrm{d}x \leqq \int_{\mathbf{\Omega}_0} \varrho \, \mathrm{d}x + \int_0^t \int_{\mathbf{\Gamma}_{inp}} \varrho_0 v_i^0 \nu_i \, \mathrm{d}s \, \mathrm{d}t,$$

(2.19) $$\frac{1}{2}\int_{\mathbf{\Omega}_t} \varrho |w|^2 \, \mathrm{d}x - \frac{1}{2}\int_{\mathbf{\Omega}_0} \varrho |w|^2 \, \mathrm{d}x +$$
$$+ \int_{\mathbf{Q}_t} \left(\frac{\partial v_i^0}{\partial t} \varrho w_i + \varrho \left(v_j^0 + w_j\right) w_i \frac{\partial v_i^0}{\partial x_j} \right) \mathrm{d}x \, \mathrm{d}t +$$
$$+ \lambda \int_{\mathbf{\Omega}_t} (\varrho \ln \varrho - \varrho) \, \mathrm{d}x - \lambda \int_{\mathbf{\Omega}_0} (\varrho \ln \varrho - \varrho) \, \mathrm{d}x +$$

$$+ \lambda \int_0^t \int_{\Gamma_{inp}} \varrho \ln \varrho v_i^0 v_i \, ds \, dt + \lambda \int_0^t \int_{\Gamma_{out}} \varrho \ln \varrho v_i^0 v_i \, ds \, dt +$$

$$+ \lambda \int_{\Omega_t} \varrho \, dx - \lambda \int_{\Omega_0} \varrho \, dx + \lambda \int_0^t \int_{\Omega} \varrho \frac{\partial v_i^0}{\partial x_i} \, dx \, dt +$$

$$+ \int_{Q_t} \Big\{ \gamma \, e^2(w_{ll}) + 2\mu \, e_{ij}(w) \, e_{ij}(w) + \gamma_1 \frac{\partial e_{ll}(w)}{\partial x_k} \frac{\partial e_{pp}(w)}{\partial x_k} +$$

$$+ 2\mu_1 \frac{\partial e_{ij}(w)}{\partial x_k} \frac{\partial e_{ij}(w)}{\partial x_k} + \gamma e_{ll}(v^0) \, e_{kk}(w) +$$

$$+ 2\mu \, e_{ij}(v^0) \, e_{ij}(w) + \gamma_1 \frac{\partial e_{ll}(v^0)}{\partial x_k} \frac{\partial e_{pp}(w)}{\partial x_k} +$$

$$+ 2\mu_1 \frac{\partial e_{ij}(v^0)}{\partial x_k} \frac{\partial e_{ij}(w)}{\partial x_k} \Big\} = 0 ,$$

where $\mathbf{\Omega}_t = \{(x, t), x \in \mathbf{\Omega}\}$, $\mathbf{\Omega}_0 = \{(x, 0), x \in \mathbf{\Omega}\}$.

Proof. Let us prove (2.18) (we denote $\boldsymbol{Q}_{t_0}$ by $\boldsymbol{Q}_t$). The proof of (2.18) is based on the integration of equations (2.3) over $\boldsymbol{Q}_t$, the use of Green's theorem and the boundary condition

$$\int_{Q_t} \left[\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho v_i)}{\partial x_i} \right] dx \, dt = 0 ,$$

hence

$$(2.19) \quad \begin{cases} \displaystyle\int_{\Omega t} \left[\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho v_i)}{\partial x_i} \right] dx \, dt = \int_0^t \left\{ \int_{\Omega} \left[\frac{\partial \varrho}{\partial t} \right] dx \right\} dt + \\ \displaystyle + \int_0^t \int_{\partial\Omega} \varrho v_i \nu_i \, ds \, dt = \int_{\Omega_t} \varrho \, dx - \int_{\Omega_0} \varrho \, dx + \int_0^t \int_{\partial\Omega} \varrho v_i \nu_i \, ds \, dt . \end{cases}$$

Now we write the last integral

$$(2.20) \quad \int_0^t \int_{\partial\Omega} \varrho v_i \nu_i \, ds \, dt = \int_0^t \int_{\Gamma_{inp}} \varrho_0 v_i^0 \nu_i \, ds \, dt +$$
$$+ \int_0^t \int_{\Gamma_{out}} \varrho v_i^0 \nu_i \, ds \, dt + \int_0^t \int_{\Gamma_{upp+low}} \varrho v_i \nu_i \, ds \, dt .$$

We know that the last integral in (2.20) is equal to zero.

Because

$$(2.21) \quad \int_0^t \int_{\Gamma_{inp}} \varrho_0 v_i^0 \nu_i \, ds \, dt \leqq 0 ,$$

$$(2.22) \quad \int_0^t \int_{\Gamma_{out}} \varrho_0 v_i^0 \nu_i \, ds \, dt \geqq 0 ,$$

it follows that

$$\int_{\Omega_t} \varrho \, dx \leqq \int_{\Omega_0} \varrho \, dx - \int_0^t \int_{\Gamma_{inp}} \varrho v_i^0 v_i \, ds \, dt \,. \tag{2.23}$$

Now we prove (2.19).

Let us multiply equations (2.8) by w, where $v = v^0 + w$, and integrate over Q_t. We have

$$0 = \int_{Q_t} \left[\frac{\partial}{\partial t} (\varrho v_i) \, w_i + \frac{\partial}{\partial x_j} (\varrho v_i v_j + \delta_{ij} p - \tau_{ij}^d) \, w_i \right] dx \, dt \,.$$

Let us divide the right hand side into three parts.

The first part:

$$\int_{Q_t} \left\{ \frac{\partial}{\partial t} (\varrho v_i) w_i + \frac{\partial}{\partial x_j} (\varrho v_i v_j) \, w_i \right\} dx \, dt =$$

$$= \int_{Q_t} \left\{ \frac{\partial v_i}{\partial t} \varrho w_i + \frac{\partial \varrho}{\partial t} v_i w_i + \frac{\partial}{\partial x_j} (\varrho v_j) \, v_i w_i + \frac{\partial v_i}{\partial x_j} \varrho v_j w_i \right\} dx \, dt \,.$$

We use the continuity equation and obtain

$$\int_{Q_t} \left(\frac{\partial v_i}{\partial x_j} \varrho v_j w_i + \frac{\partial v_i}{\partial t} \varrho w_i \right) dx \, dt = \int_{Q_t} \left(\frac{\partial (v_i^0 + w_i)}{\partial t} \varrho w_i + \right.$$

$$\left. + \frac{\partial v_i}{\partial x_j} \varrho v_j w_i \right) dx \, dt = \int_{Q_t} \frac{\partial w_i}{\partial t} \varrho w_i \, dx \, dt + \int_{Q_t} \frac{\partial v_i^0}{\partial t} \varrho w_i \, dx \, dt +$$

$$+ \int_{Q_t} \frac{\partial (v_i^0 + w_i)}{\partial x_j} \varrho (v_j^0 + w_j) \, w_i \, dx \, dt = \frac{1}{2} \int_{Q_t} \frac{\partial}{\partial t} (\varrho |w|^2) \, dx \, dt -$$

$$- \frac{1}{2} \int_{Q_t} \frac{\partial \varrho}{\partial t} |w|^2 \, dx \, dt + \int_{Q_t} \frac{\partial v_i^0}{\partial t} \varrho w_i \, dx \, dt +$$

$$+ \int_{Q_t} \left(\varrho w_j \frac{\partial v_i^0}{\partial x_j} w_i + \varrho w_j \frac{\partial w_i}{\partial x_i} w_i \right) dx \, dt +$$

$$+ \int_{Q_t} \left(\varrho \frac{\partial w_i}{\partial x_j} v_j^0 w_i + \varrho v_j^0 \frac{\partial v_i^0}{\partial x_j} w_i \right) dx \, dt =$$

$$= \frac{1}{2} \int_{\Omega_t} \varrho |w|^2 \, dx - \frac{1}{2} \int_{\Omega_0} \varrho |w|^2 \, dx - \frac{1}{2} \int_{Q_t} \frac{\partial \varrho}{\partial t} |w|^2 \, dx \, dt +$$

$$+ \int_{Q_t} \frac{\partial v_i^0}{\partial t} \varrho w_i \, dx \, dt + \frac{1}{2} \int_{Q_t} \varrho v_j^0 \frac{\partial |w|^2}{\partial x_j} \, dx \, dt +$$

$$+ \frac{1}{2} \int_{Q_t} \varrho w_j \frac{\partial |w|^2}{\partial x_j} \, dx \, dt + \int_{Q_t} \varrho v_j^0 \, w_i \frac{\partial v_i^0}{\partial x_j} + \varrho w_j w_i \frac{\partial v_i^0}{\partial x_j} \, dx \, dt =$$

$$= \frac{1}{2}\int_{\Omega_t} \varrho|w|^2\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_0} \varrho|w|^2\,\mathrm{d}x + \int_{Q_t} \frac{\partial v_i^0}{\partial t}\,\varrho w_i\,\mathrm{d}x\,\mathrm{d}t -$$

$$- \frac{1}{2}\int_{Q_t} \left(\frac{\partial \varrho}{\partial t}|w|^2 + \frac{\partial(\varrho v_j)}{\partial x_j}|w|^2\right)\mathrm{d}x\,\mathrm{d}t + \int_{Q_t} \varrho v_j w_i \frac{\partial v_i^0}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t =_{(2.3)}$$

$$=_{(2.3)} \frac{1}{2}\int_{\Omega_t} \varrho|w|^2\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_0} \varrho|w|^2\,\mathrm{d}x + \int_{Q_t} \frac{\partial v_i^0}{\partial t}\,\varrho w_i\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \int_{Q_t} \varrho v_j w_i \frac{\partial v_i^0}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t\,.$$

The second part:

$$\int_{Q_t} \frac{\partial}{\partial x_j}(\delta_{ij}p)\,w_i\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_t} \frac{\partial p}{\partial x_i}\,w_i\,\mathrm{d}x\,\mathrm{d}t =_{(2.1)} \lambda\int_{Q_t} \frac{\partial \varrho}{\partial x_i}\,w_i\,\mathrm{d}x\,\mathrm{d}t =$$

$$= \lambda\int_{Q_t} \frac{\partial \varrho}{\partial x_i}\frac{1}{\varrho}\,\varrho w_i\,\mathrm{d}x\,\mathrm{d}t = \lambda\int_{Q_t} \frac{\partial}{\partial x_i}(\ln\varrho)\,\varrho w_i\,\mathrm{d}x\,\mathrm{d}t =$$

$$= -\lambda\int_{Q_t} \ln\varrho\,\frac{\partial(\varrho w_i)}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t = -\lambda\int_{Q_t} \ln\varrho\,\frac{\partial(\varrho(v_i - v_i^0))}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t =$$

$$= -\lambda\int_{Q_t} \left\{\ln\varrho\,\frac{\partial}{\partial x_i}(\varrho v_i) - \ln\varrho\,\frac{\partial}{\partial x_i}(\varrho v_i^0)\right\}\mathrm{d}x\,\mathrm{d}t =$$

$$= \lambda\int_{Q_t} \ln\varrho\,\frac{\partial \varrho}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \lambda\int_{Q_t} \ln\varrho\,\frac{\partial}{\partial x_i}(\varrho v_i^0)\,\mathrm{d}x\,\mathrm{d}t =$$

$$= \lambda\int_{Q_t} \frac{\partial}{\partial t}(\varrho\ln\varrho - \varrho)\,\mathrm{d}x\,\mathrm{d}t + \lambda\int_{Q_t} \ln\varrho\,\frac{\partial}{\partial x_i}(\varrho v_i^0)\,\mathrm{d}x\,\mathrm{d}t =$$

$$= \lambda\int_{\Omega_t} (\varrho\ln\varrho - \varrho)\,\mathrm{d}x - \lambda\int_{\Omega_0} (\varrho\ln\varrho - \varrho)\,\mathrm{d}x +$$

$$+ \lambda\int_{Q_t} \ln\varrho\,\frac{\partial}{\partial x_i}(\varrho v_i^0)\,\mathrm{d}x\,\mathrm{d}t = \lambda\int_{\Omega_t} (\varrho\ln\varrho - \varrho)\,\mathrm{d}x -$$

$$- \lambda\int_{\Omega_0} (\varrho\ln\varrho - \varrho)\,\mathrm{d}x + \lambda\int_0^t\int_{\partial\Omega} (\varrho v_i^0)\ln\varrho v_i\,\mathrm{d}S\,\mathrm{d}t -$$

$$- \lambda\int_0^t\int_{\Omega} \frac{\partial \varrho}{\partial x_i}\,v_i^0\,\mathrm{d}x\,\mathrm{d}t = \lambda\int_{\Omega_t} (\varrho\ln\varrho - \varrho)\,\mathrm{d}x -$$

$$- \lambda\int_{\Omega_0} (\varrho\ln\varrho - \varrho)\,\mathrm{d}x + \lambda\int_0^t\int_{\Gamma_{inp}} \varrho v_i^0 v_i\ln\varrho\,\mathrm{d}S\,\mathrm{d}t +$$

$$+ \lambda\int_0^t\int_{\Gamma_{out}} \varrho v_i^0 v_i\ln\varrho\,\mathrm{d}S\,\mathrm{d}t - \lambda\int_0^t\int_{\Gamma_{inp}} \varrho v_i^0 v_i\,\mathrm{d}S\,\mathrm{d}t -$$

$$- \lambda \int_0^t \int_{\Gamma_{out}} \varrho v_i^0 v_i \,\mathrm{d}S\,\mathrm{d}t + \lambda \int_{Q_t} \varrho \frac{\partial v_i^0}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t =$$

$$=_{(2.19)(2.20)} \lambda \int_{\Omega_t} (\varrho \ln \varrho - \varrho)\,\mathrm{d}x - \lambda \int_{\Omega_0} (\varrho \ln \varrho - \varrho)\,\mathrm{d}x +$$

$$+ \lambda \int_0^t \int_{\Gamma_{inp}} \varrho v_i^0 v_i \ln \varrho \,\mathrm{d}S\,\mathrm{d}t + \lambda \int_0^t \int_{\Gamma_{out}} \varrho v_i^0 v_i \ln \varrho \,\mathrm{d}S\,\mathrm{d}t -$$

$$- \lambda \int_0^t \int_{\Gamma_{inp}} \varrho v_i^0 v_i \,\mathrm{d}S\,\mathrm{d}t + \lambda \int_{\Omega_t} \varrho \,\mathrm{d}x - \lambda \int_{\Omega_0} \varrho \,\mathrm{d}x +$$

$$+ \lambda \int_0^t \int_{\Gamma_{inp}} \varrho v_i^0 v_i \,\mathrm{d}S\,\mathrm{d}t + \lambda \int_0^t \int_{\Omega} \varrho \frac{\partial v_i^0}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t =$$

$$= \lambda \int_{\Omega_t} (\varrho \ln \varrho - \varrho)\,\mathrm{d}x - \lambda \int_{\Omega_0} (\varrho \ln \varrho - \varrho)\,\mathrm{d}x +$$

$$+ \lambda \int_0^t \int_{\Gamma_{inp}} \varrho v_i^0 \ln \varrho v_i \,\mathrm{d}S\,\mathrm{d}t + \lambda \int_0^t \int_{\Gamma_{out}} \varrho v_i^0 \ln \varrho v_i \,\mathrm{d}S\,\mathrm{d}t +$$

$$+ \lambda \int_{\Omega_t} \varrho \,\mathrm{d}x - \lambda \int_{\Omega_0} \varrho \,\mathrm{d}x + \lambda \int_0^t \int_{\Omega} \varrho \frac{\partial v_i^0}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t .$$

The third part:

$$- \int_{Q_t} \frac{\partial}{\partial x_j} (\tau_{ij}^d)\, w_i \,\mathrm{d}x\,\mathrm{d}t = - \int_0^t \int_{\partial\Omega} \tau_{ij}^d w_i v_i \,\mathrm{d}S\,\mathrm{d}t +$$

$$+ \int_{Q_t} \tau_{ij}^d \frac{\partial w_i}{\partial x_j} \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_t} \tau_{ij}^d \frac{\partial w_i}{\partial x_j} \,\mathrm{d}x\,\mathrm{d}t = \int_0^t \int_{\Omega} \left\{ \gamma \frac{\partial v_l}{\partial x_l} \delta_{ij} \frac{\partial w_i}{\partial x_j} + \right.$$

$$\left. + 2\mu e_{ij}(v) \frac{\partial w_i}{\partial x_j} - \gamma_1 \Delta \left(\frac{\partial v_l}{\partial x_l} \right) \delta_{ij} \frac{\partial w_i}{\partial x_j} - 2\mu_1 \Delta e_{ij} \frac{\partial w_i}{\partial x_j} \right\} \mathrm{d}x\,\mathrm{d}t =$$

$$= \int_0^t \int_{\Omega} \left\{ \gamma \frac{\partial v_l}{\partial x_l} \frac{\partial w_i}{\partial x_i} + 2\mu_1\, e_{ij}(v) \left(\frac{1}{2} \frac{\partial w_i}{\partial x_j} + \frac{1}{2} \frac{\partial w_j}{\partial x_i} \right) - \right.$$

$$\left. - \gamma_1 \Delta \left(\frac{\partial v_l}{\partial x_l} \right) \frac{\partial w_i}{\partial x_i} - 2\mu_1 \Delta e_{ij} \left(\frac{1}{2} \frac{\partial w_i}{\partial x_j} + \frac{1}{2} \frac{\partial w_j}{\partial x_i} \right) \right\} \mathrm{d}x\,\mathrm{d}t =$$

$$= \int_0^t \int_{\Omega} \left\{ \gamma \left(\frac{\partial v_l^0}{\partial x_l} \frac{\partial w_i}{\partial x_i} + \frac{\partial w_l}{\partial x_l} \frac{\partial w_i}{\partial x_i} \right) + 2\mu e_{ij}(v^0 + w)\, e_{ij}(w) \right\} \mathrm{d}x\,\mathrm{d}t -$$

$$- \int_0^t \int_{\partial\Omega} \left\{ \gamma_1 \frac{\partial}{\partial x_k} \left(\frac{\partial v_l}{\partial x_l} \right) \frac{\partial w_i}{\partial x_i} v_k + 2\mu_1 \frac{\partial}{\partial x_k} (e_{ij}(v)) \frac{\partial w_i}{\partial x_j} v_k \right\} \mathrm{d}S\,\mathrm{d}t +$$

$$+ \int_{Q_t} \gamma_1 \frac{\partial}{\partial x_k} \left(\frac{\partial v_l}{\partial x_l} \right) \frac{\partial}{\partial x_k} \left(\frac{\partial w_i}{\partial x_i} \right) +$$

$$+ 2\mu_1 \frac{\partial}{\partial x_l} (e_{ij}(v)) \frac{\partial}{\partial x_k} \left(\frac{\partial w_i}{\partial x_j}\right) \mathrm{d}x\, \mathrm{d}t =$$

$$= \int_0^t \int_\Omega \Big\{ \gamma e_{ll}^2(w) + 2\mu e_{ij}(w)\, e_{ij}(w) + \gamma_1 \frac{\partial}{\partial x_k} e_{ll}(w) \frac{\partial}{\partial x_k} e_{pp}(w) +$$

$$+ 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(w) \frac{\partial}{\partial x_k} e_{ij}(w) + \gamma\, e_{ll}(v^0)\, e_{kk}(w) +$$

$$+ 2\mu\, e_{ij}(v^0)\, e_{ij}(w) + \gamma_1 \frac{\partial}{\partial x_k} e_{ll}(v^0) \frac{\partial}{\partial x_k} e_{pp}(w) +$$

$$+ 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(v^0) \frac{\partial}{\partial x_k} e_{ij}(w)\, \mathrm{d}t\, \mathrm{d}x - \int_0^t \int_{\partial\Omega} \tau_{ijk}^d \frac{\partial w_i}{\partial x_j} v_k\, \mathrm{d}S\, \mathrm{d}t$$

Let us denote the last term by B.

$$B = \int_0^t \int_{\partial\Omega} \tau_{ijk}^d \frac{\partial w_i}{\partial x_j} v_k\, \mathrm{d}S\, \mathrm{d}t\,.$$

We know that $\frac{\partial w_i}{\partial x_j} = \frac{\partial w_i}{\partial v} v_j\,.$

This means that

$$B = \int_0^t \int_{\partial\Omega} \tau_{ijk}^d \frac{\partial w_i}{\partial v} v_j v_k\, \mathrm{d}S\, \mathrm{d}t$$

and we use condition (2.17). It implies that $B = 0$.

Thus

$$\frac{1}{2}\int_{\Omega_t} \varrho |w|^2\, \mathrm{d}x - \frac{1}{2}\int_{\Omega_0} \varrho |w|^2\, \mathrm{d}x + \int_{Q_t} \frac{\partial v_i^0}{\partial t} \varrho w_i\, \mathrm{d}x\, \mathrm{d}t +$$

$$+ \int_{Q_t} \varrho(v_j^0 + w_j)\, w_i \frac{\partial v_i^0}{\partial x_j} \mathrm{d}x\, \mathrm{d}t + \lambda \int_{\Omega_t} (\varrho \ln \varrho - \varrho)\, \mathrm{d}x -$$

$$- \lambda \int_{\Omega_0} (\varrho \ln \varrho - \varrho)\, \mathrm{d}x + \lambda \int_0^t \int_{\Gamma_{inp}} \varrho v_i^0 \ln \varrho v_i\, \mathrm{d}S\, \mathrm{d}t +$$

$$+ \lambda \int_0^t \int_{\Gamma_{out}} \varrho v_i^0 \ln \varrho v_i\, \mathrm{d}S\, \mathrm{d}t + \lambda \int_{\Omega_t} \varrho\, \mathrm{d}x - \lambda \int_{\Omega_0} \varrho\, \mathrm{d}x +$$

$$+ \lambda \int_{Q_t} \varrho \frac{\partial v_i^0}{\partial x_i} \mathrm{d}x\, \mathrm{d}t + \int_{Q_t} \Big\{ \gamma\, e_{ll}^2(w) + 2\mu\, e_{ij}(w)\, e_{ij}(w) +$$

$$+ \gamma_1 \frac{\partial}{\partial x_k} e_{ll}(w) \frac{\partial}{\partial x_k} e_{pp}(w) + 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(w) \frac{\partial}{\partial x_k} e_{ij}(w) +$$

$$+ 2\mu\, e_{ij}(v^0)\, e_{ij}(w) + \gamma\, e_{ll}(v^0)\, e_{kk}(w) +$$

$$+ 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(v^0) \frac{\partial}{\partial x_k} e_{ij}(w) + \gamma_1 \frac{\partial}{\partial x_k} e_{ll}(v^0) \frac{\partial}{\partial x_k} e_{pp}(w) \Big\}\, \mathrm{d}x\, \mathrm{d}t = 0\,.$$

The last term in denoted by $A_1 + B_1$, where $A_1 = ((w, w))$, $B_1 = ((v^0, w))$ are scalar products in $W^{2,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)$, see below and suppose that the following condition (C) is satisfied:

(C): $\varrho_0 \in C^1(\bar{Q})$ ($\varrho = \varrho_0$ on input and $\varrho = \varrho_0$ for $t = 0$); $\varrho_0 > 0$,
$v_0 \in C^1(Q)$,
$\varrho \in L^\infty(I. L^1(\Omega))$,
$w \in L^2(I, W^{2,2}(\Omega, R^N))$,
$v^0 \in L^2(I, W^{2,2}(\Omega, R^N))$.

Theorem 2.2. *Let us suppose* (C), *then*

$$\frac{1}{2}\int_{\Omega_t} \varrho|w|^2 \,dx + \int_{\Omega_t} \varrho \ln \varrho \,dx + \frac{1}{4}\int_0^t \|w\|^2 \,dt \leqq \tag{2.24}$$

$$\leqq h \int_{Q_t} \varrho |w|^2 \,dx\,dt + k \leqq l,$$

where $h, k. l \geqq 0$, h, k, l *are constants.*

Proof. From (2.19) we know that (we denote Q_t^n by Q_t)

$$\frac{1}{2}\int_{\Omega_t} \varrho|w|^2 \,dx - \frac{1}{2}\int_{\Omega_0} \varrho|w|^2 \,dx + \int_{Q_t} \varrho w_i \frac{\partial v_i^0}{\partial t} \,dx\,dt +$$
$$+ \int_{Q_t} \varrho v_j w_i \frac{\partial v_i^0}{\partial x_j} \,dx\,dt + \lambda \int_{\Omega_t} (\varrho \ln \varrho - \varrho) \,dx -$$
$$- \lambda \int_{\Omega_0} (\varrho \ln \varrho - \varrho) \,dx + \lambda \int_0^t \int_{\Gamma_{inp}} \varrho_0 v_i^0 \ln \varrho_0 v_i \,dS\,dt +$$
$$+ \lambda \int_0^t \int_{\Gamma_{out}} \varrho v_i^0 \ln \varrho v_i \,dS\,dt + \lambda \int_{\Omega_t} \varrho \,dx -$$
$$- \lambda \int_{\Omega_0} \varrho \,dx + \lambda \int_0^t \int_\Omega \varrho \frac{\partial v_i^0}{\partial x_i} \,dx\,dt + \int_0^t [((w, w)) + ((v^0, w))] \,dt = 0.$$

First we move the known terms to the right hand side

$$\frac{1}{2}\int_{\Omega_t} \varrho|w|^2 \,dx + \int_{Q_t} \varrho w_i \frac{\partial v_i^0}{\partial t} \,dx\,dt + \int_{Q_t} \varrho v_j w_i \frac{\partial v_i^0}{\partial x_j} \,dx\,dt +$$
$$+ \lambda \int_{\Omega_t} (\varrho \ln \varrho - \varrho) \,dx + \lambda \int_0^t \int_\Omega \varrho \frac{\partial v_i^0}{\partial x_i} \,dx\,dt +$$
$$+ \int_0^t [((w, w)) + ((v^0, w))] \,dt + \int_{\Omega_t} \varrho \,dx =$$

$$= \lambda \int_{\Omega_0} (\varrho \ln \varrho - \varrho)\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_0} \varrho |w|^2\,\mathrm{d}x + \lambda \int_{\Omega_0} \varrho\,\mathrm{d}x -$$

$$- \lambda \int_0^t \int_{\Gamma_{inp}} \varrho_0 v_i^0 \ln \varrho_0 v_i\,\mathrm{d}S\,\mathrm{d}t - \lambda \int_0^t \int_{\Gamma_{out}} \varrho v_i^0 \ln \varrho v_i\,\mathrm{d}S\,\mathrm{d}t\,.$$

Now let us estimate. From (2.18) we know that

(2.25) $$\|\varrho\|_{L^\infty(I,L^1(\Omega))} \leqq k\,;$$

(2.26) $$\left|\int_{Q_t} \varrho w_i \frac{\partial v_i^0}{\partial t}\right| \leqq c_1 \int_0^t \|w\|_{W^{2,2}(\Omega)} \int_\Omega |\varrho| \leqq$$

$$\leqq_{(2.18)} \hat{c}_1 \int_0^t \|w\|_{W^{2,2}(\Omega)} \leqq_{\text{Holder. ineq.}} \hat{c}_1 \sqrt{(t)}\, \|w\|_{L^2(I,W^{2,2}(\Omega))}\,.$$

(2.27) $$\left|\int_{Q_t} \varrho v_j^0 w_i \frac{\partial v_i^0}{\partial x_j}\right| \leqq c_2 \int_0^t \|w\|_{W^{2,2}(\Omega)} \int_\Omega |\varrho| \leqq$$

$$\leqq \hat{c}_2 \int_0^t \|w\|_{W^{2,2}(\Omega)} \leqq \hat{c}_2 \sqrt{(t)}\, \|w\|_{L^2(I,W^{2,2}(\Omega))}\,,$$

$$\int_0^t [((w, w)) + ((v^0, w))]\,\mathrm{d}t = \int_0^t \|w\|^2 + ((v^0, w))\,\mathrm{d}t \leqq$$

$$\leqq \int_0^t (\|w\|^2_{W^{2,2}(\Omega)} + \|v^0\|_{W^{2,2}(\Omega)} \|w\|_{W^{2,2}(\Omega)})\,\mathrm{d}t \leqq$$

$$\leqq \int_0^t \left(\|w\|^2 + \frac{1}{\varepsilon}\|v^0\|^2 + \varepsilon \|w\|^2\right)\mathrm{d}t\,,$$

(2.29) $$\left|\lambda \int_0^t \int_\Omega \varrho \frac{\partial v_i^0}{\partial x_i}\right| \leqq \hat{c}_4\,.$$

Thus

(2.30) $$\frac{1}{2}\int_{\Omega_t} \varrho |w|^2\,\mathrm{d}x + \lambda \int_{\Omega_t} \varrho \ln \varrho\,\mathrm{d}x + \int_0^t \|w\|^2\,\mathrm{d}t \leqq$$

$$\leqq \frac{1}{2}\int_{\Omega_0} \varrho |w|^2\,\mathrm{d}x + \lambda \int_{\Omega_0} \varrho \ln \varrho\,\mathrm{d}x - \lambda \int_0^t \int_{\Gamma_{inp}} \varrho_0 v_i^0 \ln \varrho_0 v_i\,\mathrm{d}S\,\mathrm{d}t -$$

$$- \lambda \int_0^t \int_{\Gamma_{out}} \varrho v_i^0 \ln \varrho v_i\,\mathrm{d}S\,\mathrm{d}t + \hat{c}_1 \sqrt{(t)}\, \|w\|_{L^2(I,W^{2,2})} +$$

$$+ \hat{c}_2 \sqrt{(t)}\, \|w\|_{L^2(I,W^{2,2})} + c_3 \int_0^t \int_\Omega \varrho w_i w_i\,\mathrm{d}x\,\mathrm{d}t +$$

$$+ \int_0^t \left(\frac{1}{\varepsilon}\|v^0\|^2 + \varepsilon \|w\|^2\right)\mathrm{d}t + c\,.$$

Now we use the Gronwall lemma:

$$f'(t) \leqq K_1 f(t) + K_2 ,$$

where

$$f'(t) = \frac{1}{2}\int_{\Omega_t} \varrho |w|^2 \, dt ,$$

$$f(t) = \int_{Q_t} \varrho |w|^2 \, dx \, dt ,$$

$$f'(t) \leqq f'(t) + \lambda \int_{\Omega_t} \varrho \ln \varrho \, dx + \int_0^t \|w\|^2 \, dt \leqq_{\varepsilon = 1/2}$$

$$\leqq_{\varepsilon=1/2} c_3 \int_{Q_t} \varrho |w|^2 \, dx \, dt + K + c_5 \|w\|_{L^2(I, W^{2,2}(\Omega))} + \int_0^t \frac{1}{2} \|w\|^2 \, dt ,$$

where

$$K = \frac{1}{2}\int_{\Omega_0} \varrho |w|^2 \, dx + \lambda \int_{\Omega_0} \varrho \ln \varrho \, dx - \lambda \int_0^t \int_{\Gamma_{inp}} \varrho_0 v_i^0 \ln \varrho_0 v_i \, dS \, dt -$$

$$- \lambda \int_0^t \int_{\Gamma_{out}} \varrho v_i^0 \ln \varrho v_i \, dS \, dt + \int_0^t 2 \|v^0\|^2 \, dt + c .$$

For $\varrho \leqq 1$ we have $|\lambda \int_{\Omega_t} \varrho \ln \varrho \, dx| \leqq c_6$. Thus

$$f'(t) + \lambda \int_{\Omega_t} \underset{\varrho \geqq 1}{\varrho \ln \varrho} \, dx + \frac{1}{2}\int_0^t \|w\|^2 \, dt \leqq c_3 \int_{Q_t} \varrho |w|^2 \, dx \, dt +$$

$$+ K_1 + c_5 \|w\|_{L^2(I, W^{2,2}(\Omega))} ,$$

$$K_1 = K + c_6 ,$$

$$c_5 \|w\|_{L^2(I, W^{2,2}(\Omega))} \leqq \frac{1}{4}\int_0^t \|w\|^2 \, dt + c_6 .$$

Thus

$$(2.31) \qquad \frac{1}{2}\int_{\Omega_t} \varrho |w|^2 \, dx \leqq \frac{1}{2}\int_{\Omega_t} \varrho |w|^2 \, dx + \frac{1}{4}\int_0^t \|w\|^2 \, dt +$$

$$+ \lambda \int_{\Omega_t} \underset{\varrho \geqq 1}{\varrho \ln \varrho} \, dx \leqq c_3 \int_0^t \int_{\Omega} \varrho |w|^2 \, dx \, dt + c_7 .$$

This implies (use Gronwall lemma) that

$$\frac{1}{2}\int_{\Omega_t} \varrho |w|^2 \, dx - 2c_3 \frac{1}{2}\int_0^t \int_{\Omega} \varrho |w|^2 \, dx \, dt \leqq c_7 .$$

Multiply this inequality by e^{-2c_3t}:

$$\left\{\frac{1}{2}\int_0^t\int_\Omega \varrho|w|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{e}^{-2c_3t}\right\}' \leqq c_7e^{-2c_3t}\,,$$

$$\int_0^t \frac{1}{2}\int_\Omega \varrho|w|^2\,\mathrm{d}x\,\mathrm{d}t\,e^{-2c_3t}\Big|_0^t \leqq \frac{c_7e^{-2c_3t}}{-2c_3}\Big|_0^t = \frac{c_7}{2c_3} - \frac{c_7e^{-2c_3t}}{2c_3}\,.$$

Now we multiply by e^{2c_3t}:

$$\text{(2.32)}\qquad \frac{1}{2}\int_0^t\int_\Omega \varrho|w|^2\,\mathrm{d}x\,\mathrm{d}t \leqq \frac{c_7}{2c_3}\,e^{2c_3t} - \frac{c_7}{2c_3}\,,$$

$$\text{(2.33)}\qquad f'(t) = \frac{1}{2}\int_{\Omega_t} \varrho|w|^2\,\mathrm{d}x \leqq 2c_3\,\frac{1}{2}\int_{Q_t} \varrho|w|^2\,\mathrm{d}x\,\mathrm{d}t + c_7 \leqq$$

$$\leqq 2c_3\left[\frac{c_7}{2c_3}\,e^{2c_3t} - \frac{c_7}{2c_3}\right] + c_7 = c_7e^{2c_3t}\,,$$

$$f'(t) + \frac{1}{4}\int_{\Omega_t} \|w\|^2\,\mathrm{d}t + \lambda\int_{\Omega_t\ \varrho\geqq 1} \varrho\ln\varrho\,\mathrm{d}x \leqq 2c_3f(t) + c_7\,,$$

$$\text{(2.34)}\qquad \frac{1}{4}\int_{\Omega_t} \|w\|^2\,\mathrm{d}t + \lambda\int_{\Omega_t\ \varrho\geqq 1} \varrho\ln\varrho\,\mathrm{d}x \leqq 2c_3c_7\,e^{2c_3t} + c_7\,.$$ ■

Let $\varphi(t)$ and $\psi(t)$ be two right continuous $(s, t > 0)$ nondecreasing functions such that

$$\text{(2.35)}\qquad \varphi(t) = \sup_{\psi(s)\leqq t} s\,,\quad \psi(s) = \sup_{\varphi(t)\leqq s} t$$

which satisfy the conditions

$$\text{(2.36)}\qquad \varphi(0) = \psi(0) = 0\,,$$

$$\varphi(\infty) = \psi(\infty) = \infty\,.$$

The convex functions $\Phi(u)$ and $\Psi(v)$ defined by the relations

$$\text{(2.37)}\qquad \Phi(t) = \int_0^{|u|} \varphi(t)\,\mathrm{d}t$$

$$\Psi(t) = \int_0^{|v|} \psi(s)\,\mathrm{d}s$$

are called mutually complementary Young (or Ψ –) functions [2].

A convex function $\Phi_1(t)$ will be called the principal part of the Ψ-function $\Phi_2(t)$ if

$$\Phi_1(t) = \Phi_2(t)\quad \text{for sufficiently large } t\,.$$

By $\tilde{L}_\Phi(\Omega)$ we denote the Orlicz class corresponding to Φ, i.e. the set of all Lebesgue measurable functions u in $\mathbf{\Omega}$ such that

$$(2.38)\qquad \int_{\Omega} \Phi(u)\,\mathrm{d}x < \infty\,.$$

The Orlicz space $L_\Phi(\mathbf{\Omega})$ is defined to be the linear hull of the Orlicz class $\tilde{L}_\Phi(\mathbf{\Omega})$ with the Luxemburg norm

$$(2.39)\qquad |||u|||_\Phi = \inf\left\{h > 0: \int_{\Omega} \Phi(u(x))/h\,\mathrm{d}x \leqq 1\right\},$$

$$(2.40)\qquad |||u|||_\Psi = \inf\left\{h > 0: \int_{\Omega} \Psi(u(x))/h\,\mathrm{d}x \leqq 1\right\}.$$

For $|||u|||_\Phi > 1$ we have

$$(2.41)\qquad \int_{\Omega} \Phi(u(x))/|||u|||_\Phi\,\mathrm{d}x \leqq 1\,.$$

Let $C(\mathbf{\Omega})$ be the set of all functions u continuous on $\mathbf{\Omega}$ up to the boundary. The space C_Φ and C_Ψ are defined as the closures of the set $C(\mathbf{\Omega})$ with respect to the Luxemburg norms $|||\cdot|||_\Phi$ and $|||\cdot|||_\Psi$, respectively.

The following inclusions hold:

$$(2.42)\qquad C_\Phi(\mathbf{\Omega}) \subset \tilde{L}_\Phi(\mathbf{\Omega}) \subset L_\Phi(\mathbf{\Omega})\,.$$

Definition. *We say that Φ satisfies the Δ_2-condition if for large values of t we have*

$$(2.43)\qquad \exists a > 0:\ \Phi(2t) \leqq a\,\Phi(t)\,.$$ ■

We use

$$(2.44)\qquad \Psi(t) = (1 + t)\ln(1 + t) - t\,,$$
$$\Phi(t) = e^t - t - 1\,.$$

Remark.

(2.45) L_Φ is a Banach space [2].

(2.46) $L_\Psi = (C_\Phi)^*$ [2].

(2.47) If Φ satisfies the Δ_2-condition, then L_Φ is separable [2].

Theorem 2.3. *If Ψ satisfies the Δ_2-condition, then*

$$(2.48)\qquad C_\Psi(\mathbf{\Omega}) = \tilde{L}_\Psi(\mathbf{\Omega}) = L_\Psi(\mathbf{\Omega})\,.$$

Proof. See [2]. ■

Theorem 2.4. *If Ψ satisfies the Δ_2-condition, then*

$$L_\Phi(\Omega) = (L_\Psi(\Omega))^* . \tag{2.49}$$

Proof. See [2]. ∎

Of course $C_\Phi(\Omega)$, $C_\Psi(\Omega)$ are separable Banach spaces. We realize that Ψ satisfies the Δ_2-condition, hence

$$C_\Psi(\Omega) = L_\Psi(\Omega) . \tag{2.50}$$

Definition. *If X is any Banach space, we set $\boldsymbol{X} = [X]^d$ while X^* will denote the dual space. Moreover, the symbols $(.,.)$ and $|\cdot|_2$ will denote, as customary, the scalar product and the norm in $\boldsymbol{L}_2(\Omega)$, respectively. For $1 \leqq p < \infty$ and X a Banach space will the norm $|\cdot|_X$, we denote by $L_p(I, X)$ the set of all mappings f: $I = (0, t) \to \to X$ which are strongly measurable and such that*

$$\int_0^t |f|_X^p \, dt < \infty . \tag{2.51}$$ ∎

Theorem 2.5. *If X is a separable space and $1 < p < \infty$, then*

$$(L^p(I, X))^* = L^q(I, X^*) , \quad \frac{1}{p} + \frac{1}{q} = 1 .$$

Proof. See [10]. ∎

Remark. This theorem implies that

$$L^2(I, C_\Phi(\Omega))^* = L^2(I, L_\Psi(\Omega)) . \tag{2.52}$$

Definition. *Let $1 \leqq p \leqq \infty$. The space $W^{k,p}(\Omega)$ is the subspace of $L^p(\Omega)$ of functions u for which there exists $\omega_\alpha \in L^p(\Omega)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$, $\alpha_i \geqq 0$, $|\alpha| = = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n$, $1 \leqq |\alpha| \leqq k$, such that $\forall \phi \in \mathcal{D}(\Omega)$,*

$$\int_\Omega D^\alpha \phi u \, dx = (-1)^{|\alpha|} \int_\Omega \phi \omega_\alpha \, dx , \tag{2.53}$$

where $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$.

For $1 \leqq p \leqq \infty$ *we put*

$$\|u\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \left[\sum_{1 \leqq |\alpha| \leqq k} |\omega_\alpha|^p \, dx + |u|^p \right] dx \right)^{1/p} \tag{2.54}$$

and for $p = \infty$

$$\|u\|_{k,\infty} = \sup_{x \in \Omega} \text{ess} \, |u(x)| + \sum_{1 \leqq |\alpha| \leqq k} \sup \text{ess} \, |\omega_\alpha(x)| . \tag{2.55}$$ ∎

Definition. *Let* $\Omega \subset R^N$ *be a bounded domain. Let* $1 \leqq p \leqq \infty$. *We introduce*

$$W^{k,p}(\Omega) = \overline{C^k(\Omega)}^{\| \ \|_{k,p}} .$$

∎

We shall work with

$$w \in L^2(I, W^{2,2}(\Omega, R^N)) , \tag{2.56}$$

$$v^0 \in L^2(I, W^{2,2}(\Omega, R^N)) , \tag{2.57}$$

$$v^0 \in C^1(\overline{Q_t}) , \tag{2.58}$$

$$\varrho_0 \in C(\overline{Q_t}) , \tag{2.59}$$

$$\varrho \in L^\infty(I, L_\Psi(\Omega)) , \ \varrho \geqq 0 . \tag{2.60}$$

The weak formulation of the equation (2.8), (2.4) will be the following:

$$- \int_{Q_{t_0}} \varrho v_i \frac{\partial z_i}{\partial t} \, dx \, dt - \int_\Omega \varrho_0 \tilde{v}_i^0 \, z_i(0) \, dx + \int_0^t ((v, z)) \, dt - \tag{2.61}$$

$$- \int_{Q_{t_0}} (\varrho v_i v_j + p \delta_{ij}) \frac{\partial z_i}{\partial x_j} \, dx \, dt = 0 ,$$

$$- \int_{\Omega_0} \varrho_0 \, z_i(0) \, dx - \int_{Q_{t_0}} \varrho \frac{\partial z_i}{\partial t} - \int_{Q_{t_0}} \varrho v_j \frac{\partial z_i}{\partial x_j} = 0$$

for every $z \in C^\infty(\overline{Q_t}, R^N)$, $z(t) = 0$, $v, z \in W^{2,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)$, $\tilde{v}^0 = v(0)$, $z = 0$ on $\partial\Omega \times (0, t_0)$, $((v, z))$ is defined by (3.1) in the next section.

3. A MODIFIED GALERKIN METHOD

First we construct a sequence of suitable approximations. Let us denote $V = W^{2,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)$ and for $w, z \in V$ let

$$((w, z)) = \int_\Omega \Big[\gamma \, e_{ll}(w) \, e_{kk}(z) + 2\mu \, e_{ij}(w) \, e_{ij}(z) + \tag{3.1}$$

$$+ \gamma_1 \frac{\partial}{\partial x_k} e_{ll}(w) \frac{\partial}{\partial x_k} e_{pp}(z) + 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(w) \frac{\partial}{\partial x_k} e_{ij}(z) \Big] \, dx .$$

(3.1) is a scalar product in V.

It may be shown that the bilinear form (3.1) is coercive in V.

$$\int_\Omega [\gamma \, e_{ll}^2(w) + 2\mu \, e_{ij}(w) \, e_{ij}(w)] \, dx \geqq_{\text{we use Korn's ineq., see [4]}} \tag{3.2}$$

$$\geqq \int_\Omega \Big[\gamma \, e_{ll}^2(w) + \mu \left(\frac{\partial w_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \right) + \frac{1}{2} \left(\frac{\partial w_i}{\partial x_i} \right)^2 dx \geqq_{\mu > 0, \gamma \geqq -2/3\mu}$$

$$\geqq \mu \int_{\Omega} \left(|\nabla w|^2 + \frac{1}{2} \left(\frac{\partial w_i}{\partial x_i} \right)^2 - \frac{2}{3} e_{ll}^2(w) \right) dx \geqq$$

$$\geqq \mu \int_{\Omega} (|\nabla w|^2) \, dx \geqq k \int_{\Omega} (\nabla |w|^2 + |w|^2) \, dx \, .$$

(3.3)
$$\int_{\Omega} \left(2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(w) \frac{\partial}{\partial x_k} e_{ij}(w) + \gamma_1 \frac{\partial}{\partial x_k} e_{ll}(w) \frac{\partial}{\partial x_k} e_{ll}(w) \right) dx =$$

$$= \int_{\Omega} \left(2\mu_1 e_{ij} \left(\frac{\partial w}{\partial x_k} \right) e_{ij} \left(\frac{\partial w}{\partial x_k} \right) + \gamma_1 e_{ll} \left(\frac{\partial w}{x_k} \right) e_{ll} \left(\frac{\partial w}{x_k} \right) \right) dx \geqq$$

$$\geqq 2\mu_1 \int_{\Omega} \left(e_{ij} \left(\frac{\partial w}{\partial x_k} \right) e_{ij} \left(\frac{\partial w}{\partial x_k} \right) + \right.$$

$$\left. + \frac{\gamma_1}{2\mu_1} e_{ll} \left(\frac{\partial w}{\partial x_k} \right) e_{ll} \left(\frac{\partial w}{\partial x_k} \right) \right) dx \geqq_{(\text{we use } (\gamma_1/2\mu_1) > -1/3)}$$

$$\geqq 2\mu_1 \int_{\Omega} \left(e_{ij} \left(\frac{\partial w}{\partial x_k} \right) e_{ij} \left(\frac{\partial w}{\partial x_k} \right) + \left(-\frac{1}{3} + \varepsilon \right) e_{ll} \left(\frac{\partial w}{\partial x_k} \right) e_{ll} \left(\frac{\partial w}{\partial x_k} \right) \right) dx \geqq$$

$$\geqq 2\mu_1 \int_{\Omega} \left(e_{ij} \left(\frac{\partial w}{\partial x_k} \right) e_{ij} \left(\frac{\partial w}{\partial x_k} \right) . \, e_{ij} \left(\frac{\partial w}{\partial x_k} \right) - e_{ij} \left(\frac{\partial w}{\partial x_k} \right) (-3\varepsilon + 1) \right) dx \geqq$$

$$\geqq 6\mu_1 \varepsilon \int_{\Omega} \left(e_{ij} \left(\frac{\partial w}{\partial x_k} \right) e_{ij} \left(\frac{\partial w}{\partial x_k} \right) \right) dx \, .$$

Now

(3.4)
$$6\mu_1 \varepsilon \int_{\Omega} \left(e_{ij} \left(\frac{\partial w}{\partial x_k} \right) e_{ij} \left(\frac{\partial w}{\partial x_k} \right) \right) dx + \frac{\mu_0}{2} \int_{\Omega} |\nabla w|^2 \, dx \geqq$$

$$\geqq 6\mu_1 \varepsilon \int_{\Omega} \left(\sum_{k=1}^{3} e_{ij} \left(\frac{\partial w}{\partial x_k} \right) e_{ij} \left(\frac{\partial w}{\partial x_k} \right) + \frac{\mu_0}{2 \, . \, 6 \, . \, \varepsilon \, . \, \mu_1} \sum_{k=1}^{3} |\nabla w_k|^2 \, dx \geqq \right.$$

$$\geqq 6\varepsilon\mu_1 \sum_{k=1}^{3} \int_{\Omega} \left\{ e_{ij} \left(\frac{\partial w}{\partial x_k} \right) e_{ij} \left(\frac{\partial w}{\partial x_k} \right) + |\nabla w_k|^2 \right\} dx \geqq c \, 6\varepsilon\mu_1 \sum_{k=1}^{3} \left\| \frac{\partial w}{\partial x_k} \right\|_{W^{1,2}}^2$$

(we use the coerciveness of deformations, see [4]). (3.2), (3.4) imply that

$$((w, w)) \geqq c_1 \int_{\Omega} (|w|^2 + |\nabla w|^2 + |\nabla_2 w|^2) \, dx \ \ \forall w \in \boldsymbol{W}^{1,2}(\boldsymbol{\Omega}, \boldsymbol{R}^N)$$

(this implies that the bilinear form is coercive in $\boldsymbol{V}$).

Let $(z^k)_{k=1}^{\infty}$ be a complete orthonormal system of eigenfunctions in $\boldsymbol{V}$ with the scalar product $((.,.))$, we seek the solution to the eigenproblem in $\boldsymbol{V}$

(3.5) $\quad ((w, z)) = \lambda(w, z) \quad \forall w \in \boldsymbol{V}, \quad \forall z \in \boldsymbol{V},$

where
$$(w, z) = \int_{\Omega} w_i z_i \, dx \, .$$

We have

$$(3.6)\qquad ((w, z^k)) = \lambda_k(w, z^k),\ (z^k, z^l) = \delta_{kl}\,.$$

For $z \in L^2(\Omega, R^N)$ let us put

$$(3.7)\qquad P_m z = \sum_{k=1}^{m} \lambda_k(z^k, z)\, z^k\,.$$

If

$$L_m^2 = \text{span}\,\{z^1, \ldots, z^m\} \quad \text{in} \quad L^2\,,$$
$$V_m = \text{span}\,\{z^1, \ldots, z^m\} \quad \text{in} \quad V\,,$$

then P_m is a projector of L^2 onto L_m^2 and of V onto V_m. From the regularity of solutions to the linear elliptic problem we get

$$(3.8)\qquad z^k \in C^\infty(\overline{\Omega}; R^n)$$

from (3.6), see [7].

Remark.

$$\sqrt{(\lambda_n)}\, z^k \text{ is an orthonormal system in } L^2\,.$$

Remark. For the construction of the base for the Galerkin method we have used the following regularity property of the weak solution to the elliptic problem

$$(3.9)\qquad u \in V\,,\quad f \in L^2(\Omega, R^N)$$
$$((v, u)) = \int_\Omega v_i f_i \, \mathrm{d}x \quad \text{for every} \quad v \in V\,.$$

Theorem 3.1. *Let $u \in V$ be a solution to (3.9). Then $u \in W^{4,2}(\Omega, R^N)$, $c > 0$ and*

$$(3.10)\qquad \|u\|_{W^{4,2}(\Omega, R^N)} \leqq c\|f\|_{L^2(\Omega, R^N)}\,.$$

Proof. See [7].

By (3.9) one defines the operator $\mathscr{A}$:

$$(3.11)\qquad ((w, z)) = (\mathscr{A}w, z) \quad \text{for every} \quad z \in V\,.$$

Its domain of definition is denoted by $D(\mathscr{A})$, of course $W_0^{4,2}(\Omega, R^N) \subset D(\mathscr{A})$. It is a consequence of Theorem 3.1 that

$$(3.11)\qquad \|w\|_{W^{4,2}(\Omega, R^N)} \leqq k_1\|\mathscr{A}w\|_{L^2(\Omega, R^N)}\,,$$
$$k_1 > 0\,,\quad \forall w \in D(\mathscr{A}) \quad \text{hence}$$

$$(3.12)\qquad \|P_m w\|_{W^{4,2}(\Omega, R^N)} \leqq k_1\|\mathscr{A}P_m w\|_{L^2(\Omega, R^N)} \leqq_{\mathscr{A}P_m = P_m\mathscr{A}}$$
$$\leqq k_1\|P_m\mathscr{A}w\|_{L^2(\Omega, R^N)} \leqq k_2\|\mathscr{A}w\|_{L^2(\Omega, R^N)} \leqq k_3\|w\|_{W^{4,2}(\Omega, R^n)}$$
$$\text{for every} \quad w \in W_0^{4,2}(\Omega, R^N)\,,\quad k_2, k_3 > 0\,.$$

Remark. Proof of the property $\mathscr{A}P_m = P_m\mathscr{A}$.

Proof.

$$\mathscr{A} P_m w = \mathscr{A} \sum_{i=1}^{m} a_i w_i\,, \quad \text{where} \quad a_i = \sqrt{(\lambda_i)}\,(w^i, w)\,.$$

This implies

$$\mathscr{A} P_m w = \mathscr{A} \sum_{i=1}^{m} a_i w_i = \mathscr{A} \sum_{i=1}^{m} \lambda_i (w^i, w)\, w^i = \sum_{i=1}^{m} \lambda_i (w^i, w)\, \mathscr{A} w^i =$$

$$= \sum_{i=1}^{m} \lambda_i^2 (w, w^i)\, w^i\,,$$

$$\mathscr{A} w^i = \lambda_i w^i\,,$$

$$\mathscr{A} w = \sum_{i=1}^{\infty} (\mathscr{A} w, w^i)\, \lambda_i w^i\,,$$

$$P_m \mathscr{A} w = \sum_{i=1}^{m} (\mathscr{A} w, w^i)\, \lambda_i w^i = \sum_{i=1}^{m} \lambda_i^2 (w, w^i)\, w^i\,.$$

Thus

$$P_m \mathscr{A} w = \mathscr{A} P_m w\,. \tag{3.13}$$

Due to the interpolation theorem, see [8], we thus have for every $v \in W_0^{3,2}(\Omega)$

$$\|P_m w\|_{W_0^{3,2}(\Omega, R^N)} \leqq k_4 \|w\|_{W_0^{3,2}(\Omega, R^N)}\,, \quad k_4 > 0\,. \tag{3.14}$$

Let $c_i \in C^1(I)$, $I = (0, t_0)$ and let us put

$$w^m(t, x) = \sum_{i=1}^{m} c_i(t)\, z^i(x)$$

and

$$v^m(t, x) = v^0 + w^m(t, x)\,.$$

We suppose that we know the velocity v^m and want to obtain ϱ_m. Let us first look for $\varrho_m \in C^1(\overline{Q}_t)$ such that

$$\frac{\partial \varrho_m}{\partial t} + \frac{\partial}{\partial x_i} (\varrho_m v_i^m) = 0\,. \tag{3.15}$$

We suppose that

$$\varrho_m(0, t) = \varrho_0(x) \in C^1(\overline{\Omega})\,, \quad \varrho_0(x) > 0 \text{ in } \overline{\Omega}\,. \tag{3.16}$$

Let

$$x'^m(\tau) = -v^m(t - \tau, x^m(\tau))\,, \quad x^m(0) = x\,, \quad x \in \Omega\,. \tag{3.17}$$

ϱ_m may be obtained by integration along chatacteristics. These characteristics pass through Q_{t_0} and start either in Ω_0 or in Γ_{inp}. Thus it is possible to use the fact that we know ϱ_m on the sets Ω_0, Γ_{inp}.

For $\tau \in I_{\tilde{t}}$ where $I_{\tilde{t}} \subset I$ and $I_{\tilde{t}} = (0, \tilde{t})$, $\tilde{t} > 0$, $x \to x(\tau)$ is a local diffeomorphism of $\overline{\Omega}$ onto $\overline{\Omega}$ and for $\sigma_m = \ln \varrho_m$ we have

$$\frac{\partial \varrho_m}{\partial t} + \frac{\partial \varrho_m}{\partial x_i} v_i^m + \varrho_m \frac{\partial v_i^m}{\partial x_i} = 0\,, \tag{3.18}$$

$$- \frac{\partial \sigma_m}{\partial t} + \frac{\partial \sigma_m}{\partial x_i} v_i^m = - \frac{\partial \sigma_m}{\partial t} + \frac{\partial \sigma_m}{\partial x_i} \frac{\partial x_i^m}{\partial \tau} = \frac{\partial v_i^m}{\partial x_i} (t - \tau,\ x^m(\tau)) = \tag{3.19}$$

$$= \frac{\mathrm{d}\sigma_m}{\mathrm{d}\tau} (t - \tau,\ x^m(\tau))\,.$$

Hence

$$\int_0^{\tilde{t}} \frac{\mathrm{d}}{\mathrm{d}\tau} \sigma_m(t - \tau,\ x^m(\tau))\,\mathrm{d}\tau = \int_0^{\tilde{t}} \frac{\partial v_i^m}{\partial x_i} (t - \tau,\ x^m(\tau))\,\mathrm{d}\tau\,.$$

Thus

$$\sigma_m(t, x^m(0)) = \sigma_m(t - \tau,\ x^m(\tilde{t})) - \int_0^{\tilde{t}} \frac{\partial v_i^m}{\partial x_i} (t - \tau,\ x^m(\tau))\,\mathrm{d}\tau\,.$$

There is a unique characteristic passing through the point $[x, t]$. Let us denote by t the time it takes for a particle of liquid to reach the point $[x, t]$ along this characteristic from the initial point of the characteristic (i.e. from $\boldsymbol{\Omega}_0$ or from $\boldsymbol{\Gamma}_{inp}$).

$$\varrho_m(t, x) = \varrho_0(t - \tilde{t},\ x^m(\tilde{t})) \exp\left(- \int_0^{\tilde{t}} \frac{\partial v_i^m}{\partial x_i} (t - \tau, x^m(\tau) \right) \mathrm{d}\tau\,, \tag{3.20}$$

where $x = x^m(0)$.

Theorem 3.2. $\varrho_m \in \boldsymbol{C}(\overline{\boldsymbol{Q}_t})$, $\varrho_m \in \boldsymbol{W}^{1,\infty}(\boldsymbol{Q}_t)$.

Proof. For the sake of simplicity we assume that $\boldsymbol{\Gamma}_{inp}$ is an interval and $N = 2$ For the proof that $\varrho_m \in \boldsymbol{C}(\boldsymbol{Q}_t)$ see [11].

First we prove that $\varrho_m \in \boldsymbol{W}^{1,\infty}(\boldsymbol{Q}_1)$ and $\varrho_m \in \boldsymbol{W}^{1,\infty}(\boldsymbol{Q}_2)$, where $\boldsymbol{Q}_1 \cup \boldsymbol{Q}_2 \cup \boldsymbol{S} = \boldsymbol{Q}_t$ and $\boldsymbol{S}$ is the surface described by the trajectories of solutions of the equations

$$x'^m(t) = v^m(t, x^m(t))\,, \quad x \in \boldsymbol{\Gamma}_{inp} \quad \text{closed}\,.$$

For t fixed, $t = \tilde{t}$,

$$\frac{\partial \varrho_m}{\partial x_i}(t, x) = \frac{\partial \varrho_0}{\partial X_k} \frac{\partial X_k}{\partial x_i} \exp\left(- \int_0^t \frac{\partial v_i^m}{\partial x_i} (t - \tau, x^m(\tau) \right) \mathrm{d}\tau + \tag{3.21}$$

$$+ \varrho_0(0, x^m(t)) \exp\left(- \int_0^t \frac{\partial v_i^m}{\partial x_i} (t - \tau, x^m(\tau))\,\mathrm{d}\tau \right) \times$$

$$\times \left(- \int_0^t \frac{\partial^2 v_j^m}{\partial x_j\, \partial x_k} (t - \tau,\ x^m(\tau) \right) \frac{\partial x_k^m}{\partial x_i} (\tau, x)\,\mathrm{d}\tau =$$

$$= \frac{\partial \varrho_0}{\partial X_k} \frac{\partial X_k}{\partial x_i} \exp\left(- \int_0^t \frac{\partial v_i^m}{\partial x_i} (t - \tau, x^m(\tau))\, \mathrm{d}\tau \right) +$$

$$+ \varrho_0(0, x^m(t)) \exp\left(- \int_0^t \frac{\partial v_i^m}{\partial x_i} (t - \tau, x^m(\tau))\, \mathrm{d}\tau \right) \times$$

$$\times \left(- \int_0^t \frac{\partial^2 v_j}{\partial x_j \partial x_k} (t - \tau, x^m(\tau)) \frac{\partial x_k^m}{\partial X_l} \frac{\partial X_l}{\partial x_i}\, \mathrm{d}\tau \right).$$

We assume that $\varrho_0 \in \boldsymbol{C}^1(\overline{\boldsymbol{Q}_t})$, $\partial X_k/\partial x_i$ is the inverse matrix to $\partial x_i/\partial X_k = \mathscr{B}$, where $x'^m(\tau) = v(\tau, x^m(\tau))$.

$$\text{(3.22)} \qquad \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial^m x_j}{\partial X_l} \right) = \frac{\partial v_j}{\partial x_m} \frac{\partial x_m}{\partial X_k}.$$

It implies that $\partial x_k/\partial X_i$ are bounded and continuous functions depending on parameters, see [11]. Further $\mathscr{B}^{-1}(\partial X_i/\partial x_n)$ is bounded which follows from conversation of the mass $\varrho_0(X)\, \mathrm{d}X = \varrho(t, x)\, \mathrm{d}x$ and $\varrho_0(X)/\varrho(t, x) = \mathrm{d}x/\mathrm{d}X \neq 0$ because $\varrho_0 > 0$ and $\varrho \neq 0$ (from (3.20)). Thus ϱ_m has a continuous first derivative with respect to x_i and this derivative is bounded on $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ (analogously for $\partial \varrho_m/\partial t$; $\partial \varrho_m/\partial t$, $\partial \varrho_m/\partial x_i$ on input).

Thus

$$\text{(3.23)} \qquad \varrho_m \in \boldsymbol{W}^{1,\infty}(\boldsymbol{Q}_1) \quad \text{and} \quad \varrho_m \in \boldsymbol{W}^{1,\infty}(\boldsymbol{Q}_2).$$

By second, we verify that the surface $\boldsymbol{S}$ is differentiable. We have to verify, see [13], that $|\mathrm{D}| \neq 0$ where X and t are parameters of the surface and the surface is described by the following equations:

$$x_1 = x_1(X, t),$$
$$x_2 = x_2(X, t),$$
$$T = t.$$

Then

$$D = \begin{pmatrix} \frac{\partial x_1}{\partial X}, & \frac{\partial x_2}{\partial X}, & \frac{\partial t}{\partial X} \\ \frac{\partial x_1}{\partial t}, & \frac{\partial x_2}{\partial t}, & \frac{\partial t}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial X}, & \frac{\partial x_2}{\partial X}, & 0 \\ \frac{\partial x_1}{\partial t}, & \frac{\partial x_2}{\partial t}, & 1 \end{pmatrix}$$

$|\mathrm{D}| \neq 0$ because $|\mathscr{B}^{-1}| \neq 0$.

Thirdly; we use the following theorem:

Let $\vartheta \in \boldsymbol{W}^{1,\infty}(\boldsymbol{Q}_1)$, $\vartheta \in \boldsymbol{W}^{1,\infty}(\boldsymbol{Q}_2)$, *then* $\vartheta \in \boldsymbol{W}^{1,\infty}(\boldsymbol{Q}_t)$.

Proof. See [12].

Only the outline of the proof is presented. We wish to prove that

$$\int_{\Omega} \frac{\partial f}{\partial x_1} \phi = - \int_{\Omega} f \frac{\partial \phi}{\partial x_1}, \quad \operatorname{supp} \phi \subset \boldsymbol{M}$$

and

$$\int_{M_1} \frac{\partial f}{\partial x_1} \phi = \int_{\Gamma} f \phi \tilde{v}_1 \, dS - \int_{M_1} f \frac{\partial \Phi}{\partial x_1},$$

$$\int_{M_2} \frac{\partial f}{\partial x_1} \phi = \int_{\Gamma} f \phi v_1 \, dS - \int_{M_2} f \frac{\partial \phi}{\partial x_1},$$

and $\tilde{v}_1 = -v_1$.

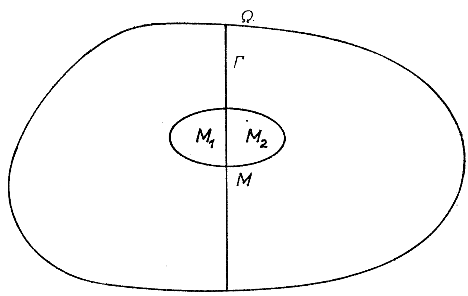

Fig. 1.

Then

$$\int_{\Omega} \frac{\partial f}{\partial x_1} \phi = \int_{M_1} \frac{\partial f}{\partial x_1} \phi + \int_{M_2} \frac{\partial f}{\partial x_1} = - \int_{M_1} f \frac{\partial \phi}{\partial x_1} - \int_{M_2} f \frac{\partial \phi}{\partial x_1} =$$
$$= - \int_{\Omega} f \frac{\partial \phi}{\partial x_1}.$$

Thus $\varrho_m \in \boldsymbol{W}^{1,\infty}(\boldsymbol{Q}_t)$. ∎

Now, let us look for $\bar{v}^m$ such that $\forall t \in \boldsymbol{I}$,

$$\int_{\Omega} \left(\varrho_m \frac{\partial \bar{v}_i^m}{\partial t} + \varrho_m v_j^m \frac{\partial \bar{v}_i^m}{\partial x_j} + \lambda \frac{\partial \varrho_m}{\partial x_i} \right) z_i^k \, dx = -((\bar{v}^m, z^k)), \quad k = 1, \ldots, m \tag{3.24}$$

(i.e. we suppose that we know ϱ_m and want to obtain $\bar{v}^m$). This equation is a system or ordinary differential equations where the unknown is $\bar{c}^i(t)$. Since $\varrho_m \in \boldsymbol{C}^{0,1}(\boldsymbol{Q}_t)$, then $\bar{c}^i(t) \in \boldsymbol{C}^1(\boldsymbol{I})$.

The initial conditions are

$$\int_{\Omega} \sum_{j=1}^{m} \bar{c}_l(0) \, z_j^k z_j^l \, dx = \int_{\Omega} v_k(0, x) \, z_k^l(x) \, dx. \tag{3.25}$$

In the sequel we shall assume that

$$(3.26) \qquad v(0, x) \in \boldsymbol{L}^2(\boldsymbol{\Omega}, \boldsymbol{R}^N)\,.$$

Because $\det \int_{\Omega} \varrho_m z_i^k z_i^l \, \mathrm{d}x \not\equiv 0$, we can solve (3.24), (3.25) uniquely in $\boldsymbol{I}$. We have $c_i \in \boldsymbol{C}^1(\boldsymbol{I})$. If we start with $c_i(t)$ in the ball

$$(3.27) \qquad \max_{[0,\alpha]} |c_i(t) - c_i(0)| \leqq 1\,, \quad i = 1, 2, \ldots, m$$

we get

$$(3.28) \qquad \max_{[0,\alpha]} |\bar{c}_i(t) - c_i(0)| \leqq 1\,, \quad i = 1, 2, \ldots, m\,,$$

$$(3.29) \qquad \max_{[0,\alpha]} |c_i'(t)| \leqq K(\alpha)\,,$$

provided α is sufficiently small. Thus applying Schauder's fixed point theorem we obtain $\bar{c}_i = c_i$ on $[0, \alpha]$. But for such solutions we obtain

$$(3.30) \qquad \int_{\Omega_t} \varrho_m \, \mathrm{d}x \leqq \int_{\Omega_0} \varrho_0 \, \mathrm{d}x + \int_0^t \int_{\Gamma_{inp}} \varrho_0 v_i^0 v_i \, \mathrm{d}S \, \mathrm{d}t\,,$$

$$\begin{aligned}(3.31) \qquad & \frac{1}{2} \int_{\Omega_t} \varrho_m |w^m|^2 \, \mathrm{d}x - \frac{1}{2} \int_{\Omega_0} \varrho_0 |w^m|^2 \, \mathrm{d}x + \int_0^t ((v^m, w^m)) \, \mathrm{d}t + \\ & + \int_{Q_t} \left(\frac{\partial v_i^0}{\partial t} \varrho_m w_i^m + \varrho_m (v_j^0 + w_j) \, w_i^m \frac{\partial v_i^0}{\partial x_j} \right) \mathrm{d}x \, \mathrm{d}t + \\ & + \lambda \int_{\Omega_t} (\varrho_m \ln \varrho_m - \varrho_m) \, \mathrm{d}x - \lambda \int_{\Omega_0} (\varrho_m \ln \varrho_m - \varrho_m) \, \mathrm{d}x + \\ & + \lambda \int_0^t \int_{\Gamma_{inp}} \varrho_m v_i^0 \ln \varrho_m v_i \, \mathrm{d}S \, \mathrm{d}t + \\ & + \lambda \int_0^t \int_{\Gamma_{out}} \varrho_m v_i^0 \ln \varrho_m v_i \, \mathrm{d}S \, \mathrm{d}t + \lambda \int_{\Omega_t} \varrho_m \, \mathrm{d}x - \\ & - \lambda \int_{\Omega_0} \varrho_m \, \mathrm{d}x + \lambda \int_0^t \int_{\Omega} \varrho_m \frac{\partial v_i^0}{\partial x_i} \, \mathrm{d}x \, \mathrm{d}t = 0\,.\end{aligned}$$

Now we estimate as in the previous section.

We obtain

$$\begin{aligned}(3.32) \qquad & \frac{1}{2} \int_{\Omega_t} \varrho_m |w^m|^2 \, \mathrm{d}x + \int_{\Omega_t} \varrho_m \ln \varrho_m \, \mathrm{d}x + c_1 \int_0^t \|w^m\|^2 \, \mathrm{d}t \leqq \\ & \leqq c_2 \int_{Q_t} \varrho_m |w^m|^2 \, \mathrm{d}x \, \mathrm{d}t + c_3 \leqq c_4\,, \quad c_1, c_2, c_3, c_4 > 0\,.\end{aligned}$$

Now we denote by $\boldsymbol{M}$ the set of the maximal α's. $\boldsymbol{M}$ is closed, which follows from (3.32) and $\boldsymbol{M}$ is open as follows from the theory of ordinary differential equations, see [11]. This implies that $\alpha = t_0$.

Lemma 4.1. *Let B be a Banach space, B_i $(i = 0, 1)$ reflexive Banach spaces. Let $B_0 \subset\subset B \subset B_1$ ($\subset\subset$ denotes a compact imbedding), $1 < p_i < \infty$. Let $W = \{v, v \in L^{p_0}(I, B_0), \partial v/\partial t \in L^{p_1}(I, B_1)\}$. Then $W \subset\subset L^{p_0}(I, B)$.*

Proof. See [6].

Main theorem. *Let (2.2), (2.5), (2.57), (2.59) be satisfied. Then there exists*

$$\varrho \in L^\infty(I, L_\Psi(\Omega)), \quad \varrho \geqq 0 \text{ a.e. in } Q_t, \tag{4.1}$$

$$v \in L^2(I, W^{2,2}(\Omega)) \cap W_0^{1,2}(\Omega, R^N)), \tag{4.2}$$

$$\frac{\partial \varrho}{\partial t} \in L^2(I, W^{-3,2}(\Omega)), \tag{4.3}$$

$$\frac{\partial}{\partial t}(\varrho v) \in L^2(I, W^{-3,2}(\Omega, R^N)) \tag{4.4}$$

satisfying (2.3), (2.8) in the sense of distributions and being such that (2.6) holds. Moreover we have

$$\|\varrho\|_{L^\infty(I, L_1(\Omega))} \leqq \int_{\Omega_0} \varrho_0 \, dx, \tag{4.5}$$

$$\frac{1}{2}\|\varrho|v|^2\|_{L^\infty(I, L^1(\Omega))} + \|w\|^2_{L^2(I, W^{2,2}(\Omega, R^N))} + \tag{4.6}$$

$$+ \lambda \operatorname*{supess}_{I} \int_{\Omega_t} \varrho \ln \varrho \, dx \leqq h_1 \quad (h_1 \geqq 0).$$

Proof. Let $0 \leqq k \leqq 2$ and let $W^{k,2}(\Omega)$, $W_0^{k,2}(\Omega)$ be the usual Sobolev spaces with fractional derivatives, see [2]; let $V^k = \bar{V}$, where the closure is taken in $W^{k,2}(\Omega, R^N)$; naturally the traces are zero for $k \geqq \frac{1}{2}$ only.

Since

$$W_0^{2,2}(\Omega) \subset\subset W_0^{k_1,2}(\Omega) \subset\subset W^{k_2,2}(\Omega) \subset\subset C(\overline{\Omega}) \subset C_\Phi(\Omega) \quad \text{for} \tag{4.7}$$
$$N/2 < k_2 < k_1 < 2$$

we have

$$L_\Psi(\Omega) \subset\subset W^{-k_2,2}(\Omega) \subset W^{-k_1,2}(\Omega) \subset\subset W^{-2,2}(\Omega); \tag{4.8}$$

obviously

$$W^{k,2}(\Omega) \subset C_\Psi(\Omega), \tag{4.9}$$

hence

$$L_\Psi(\Omega) \subset (W_0^{k,2}(\Omega))^* \quad N/2 < k. \tag{4.10}$$

It follows from the interpolation theorem (see [8]) that

$$(4.11)\qquad \sup_{v\in V^k} \|P_m v\|_{W^{k,2}(\Omega,R^N)} \leqq h_2\,, \quad h_2 > 0\ (0 \leqq k \leqq 2)\,,$$

where $\|v\|_{W^{k,2}(\Omega,R^N)} \leqq 1$

and

$$(4.12)\qquad \sup_{v\in (V^k)^*} \|P_m^* v\|_{(V^k)^*} \leqq h_3\,, \quad h_3 > 0\ (0 \leqq k \leqq 2)$$

$\|v\|_{(V^k)^*} \leqq 1$

(P_m^* is the dual operator to P_m).

Lemma 4.2. *Let* $0 \leqq k \leqq 2$. *Then for* $\varepsilon > 0$ *there exists* l_0 *such that for* $l \geqq l_0$ *we have* $\|u - P_l u\|_{W^{k,2}(\Omega,R^N)} < \varepsilon$ provided $\|u\|_{W_0^{2,2}(\Omega,R^N)} \leqq 1$.

Proof. Let us assume the contrary. Then there exists $\varepsilon_0 > 0$ and P_{l_j}, $l_j \to \infty$ and $\|u_{l_j}\|_{W_0^{2,2}(\Omega,R^N)} \leqq 1$ such that $\|u_{l_j} - P_{l_j} u_{l_j}\|_{W_0^{k,2}(\Omega,R^N)} \geqq \varepsilon_0$. Since $W_0^{2,2}(\Omega, R^N) \subset\subset W_0^{k,2}(\Omega, R^N)$, we can assume $u_{l_j} \to u$ strongly in $W_0^{k,2}(\Omega, R^N)$, hence $P_{l_j} u_{l_j} \to u$ strongly in $W_0^{k,2}(\Omega, R^N)$ which is a contradiction.

Let ϱ_m, v^m be an approximate solution from Section 3. Then there exist subsequences (denoted again by $(\varrho^m)_{m=1}^\infty$, $(v^m)_{m=1}^\infty$) such that

$$(4.13)\qquad \varrho_m \to \varrho * - \text{weakly in}\quad L^2(I, L_\Psi(\Omega)) = (L^2(I, C_\Phi(\Omega)))^*\,,$$

$$(4.14)\qquad v^m \to v \quad \text{weakly in} \quad L^2(I, W^{2,2}(\Omega))\,,$$

$$(4.15)\qquad v^m \to v\,, \quad \frac{\partial}{\partial x_i} v^m \to \frac{\partial}{\partial x_i} v\,, \quad \frac{\partial^2 v^m}{\partial x_i\, \partial x_j} \to \frac{\partial^2 v}{\partial x_i\, \partial x_j}\,, \quad i, j = 1, \ldots, M$$

weakly in $L^2(Q_t)$.

Due to (3.30) and (4.8) we obtain

$$(4.16)\qquad \|\varrho_m\|_{L^\infty(I,W^{-k,2}(\Omega))} \leqq c_1\,, \quad c_1 > 0\,, \quad N/2 < k \leqq 2$$

$(\varrho_m \in L^\infty(I, L_\Psi(\Omega)) \subset L^\infty(I, W^{-k,2}(\Omega))$.

For $N = 2,3$ we have

$$(4.17)\qquad \|v^m\|_{C^0(\bar{\Omega},R^N)} \leqq c_2 \|v^m\|_{W^{2,2}(\Omega,R^N)}\,, \quad c_2 > 0$$

(Sobolev imbedding),

hence

$$(4.18)\qquad \|\varrho_m v^m\|_{L^2(I,L_\Psi(\Omega,R^N))} \leqq c_3\,, \quad c_3 > 0$$

$$\left(\sup_{\|\Phi\|_{L_\Phi(\Omega,R^N)} \leqq 1} \left| \int_\Omega \varrho_m v^m \phi \right| \leqq \|v^m\|_{W^{2,2}(\Omega)} \left| \int_\Omega \varrho_m \phi \right| \leqq \right.$$

$$\left. \leqq c_3 \|v^m\|_{W^{2,2}(\Omega)}\, \|\varrho_m\| \;\&\; \int_0^t \|v^m\|_{W^{2,2}}^2 < \infty \right).$$

It follows from (3.15) that

$$(4.19)\qquad \left\|\frac{\partial \varrho_m}{\partial t}\right\|_{L^2(I,W^{-3,2}(\Omega,R^N))} \leqq c_4\,, \quad c_4 > 0$$

$$\left(\left(\frac{\partial \varrho}{\partial t}, w\right)\right) = -\int_\Omega \varrho_m v_i^m \frac{\partial w_i}{\partial x_i}\,\mathrm{d}x\,,$$

$$\sup_{\|w\|_{W^{3,2}}\leqq 1} \left|\left(\frac{\partial \varrho}{\partial t}, w\right)\right| \leqq c \max_{\bar\Omega} |v_i^m| \max_{\bar\Omega} \left|\frac{\partial w_i}{\partial x_j}\right| \leqq c\,\|v\|_{W^{2,2}}\,\|w\|_{W^{3,2}} \leqq c\|v^m\|_{W^{2,2}}\,,$$

and this implies $\partial\varrho_m/\partial t \in L^2(I, W^{-3,2}))$.

Due to Lemma 4.1, $\varrho_m \to \varrho$ strongly in $L^2(I, W^{-2,2}(\Omega))$. By (4.8), (4.18) we get

$$\varrho_m v^m \to \varrho v \quad \text{in} \quad L^2(I, W^{-2,2}(\Omega))\,.$$

Also

$$(4.20)\qquad \|\varrho_m v^m\|_{L^2(I,W^{-k,2}(\Omega,R^N))} \leqq c_5\,, \quad c_5 > 0\,, \quad N/2 < k/2\,,$$

hence by (4.12)

$$(4.21)\qquad \|P_m^*(\varrho_m v^m\|_{L^2(I,W^{-k,2}(\Omega,R^N))} \leqq c_6\,, \quad c_6 > 0\,.$$

According to (3.31), $\varrho_m|v^m|^2$ is bounded in $L^\infty(I, L^1(\Omega))$, therefore

$$(4.22)\qquad \|\varrho_m|v^m|^2\|_{L^2(I,W^{-2,2}(\Omega,R^N))} \leqq \|\varrho_m|v^m|^2\|_{L^\infty(I,L^1(\Omega))} \leqq c_7\,, \quad c_7 > 0\,.$$

By (3.24), (3.14), (3.32), (4.16), (4.22)

$$(4.23)\qquad \left\|\frac{\partial}{\partial t}(P_m^*\varrho_m v^m)\right\|_{L^2(I,W^{-3,2}(\Omega,R^N))} \leqq c_8\,, \quad c_8 > 0$$

holds.

Thus, by Lemma 4.1, $P_m^*(\varrho_m v^m) \to a$ strongly in $L^2(I, W^{-2,2}(\Omega, R^N))$. Let $z \in L^2(I, W_0^{2,2}(\Omega, R^N))$. Because of Lemma 4.2 for m sufficiently large, $k < 2$, we have

$$(4.24)\qquad \int_0^t \|P_m z - z\|^2_{W^{k,2}(\Omega,R^N)}\,\mathrm{d}\tau \leqq c^2 \int_0^t \|z\|^2_{W^{2,2}(\Omega,R^N)}\,\mathrm{d}\tau\,,$$

hence for $\int_0^t \|z\|^2_{W^{2,2}(\Omega,R^N)}\,\mathrm{d}\tau \leqq 1$, it follows that

$$(4.25)\qquad \lim_{m\to\infty} \int_0^t \int_\Omega (P_m^*(\varrho_m v_i^m) - \varrho_m v_i^m)\, z_i\,\mathrm{d}x\,\mathrm{d}\tau = 0$$

uniformly with respect to z.

Therefore, $\varrho_m v^m$ is a Cauchy sequence in $L^2(I, W^{-2,2}(\Omega, R^N))$ and $\varrho_m v^m \to a$ strongly in $L^2(I, W^{-2,2}(\Omega, R^N))$. But $\varrho_m v^m \to \varrho v$ in $D'(Q_t)$ in the sense of distributions, hence $a = \varrho v$. Therefore due to (4.22)

$$(4.26)\qquad \varrho_m v_i^m v_j^m \to \varrho v_i v_j \quad \text{weakly in} \quad L^2(I, W^{-2,2}(\Omega, R^N))\,.$$

It follows from (3.24) that for every $\phi \in C^\infty(Q_t, R^N)$ satisfying $\phi(t) \in V_m$ for every $t \in [0, T]$ and $\phi(T) = 0$, we have

$$\int_{Q_t} \varrho v_i \frac{\partial \phi_i}{\partial t} \,\mathrm{d}x \,\mathrm{d}t + \int_{Q_t} \varrho v_i v_j \frac{\partial \phi_i}{\partial x_j} \,\mathrm{d}x \,\mathrm{d}t + \lambda \int_{Q_t} \varrho \frac{\partial \phi_i}{\partial x_j} \,\mathrm{d}x \,\mathrm{d}t =$$

$$= \int_0^t ((v, \phi)) \,\mathrm{d}t - \int_\Omega \varrho_0 \tilde{v}_i^0 \phi_i \,\mathrm{d}x$$

and

$$- \int_{\Omega_0} \varrho_0 \, \phi_i(0) \,\mathrm{d}x - \int_{Q_t} \varrho \frac{\partial \phi_i}{\partial t} \,\mathrm{d}x \,\mathrm{d}t - \int_{Q_t} \varrho v_i \frac{\partial \phi_i}{\partial x_j} = 0 \,.$$

Due to the density argument (2.61) holds and (2.8) is satisfied in the sense of distributions. The continuity equation is obviously satisfied in the sense of distributions. ■

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Souhrn

GLOBÁLNÍ ŘEŠENÍ IZOTERMICKÉ STLAČITELNÉ BIPOLÁRNÍ TEKUTINY NA KONEČNÉM KANÁLU S NENULOVÝMI VSTUPY A VÝSTUPY

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V práci je dokázána globální existence slabého řešení vazké stlačitelné izotermické bipolární tekutiny smíšené počáteční okrajové úlohy na konečném kanálu.

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