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FUZZY EQUALITY AND CONVERGENCES FOR F-OBSERVABLES IN F-QUANTUM SPACES

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Summary. We introduce a fuzzy equality for F-observables on an F-quantum space which enables us to characterize different kinds of convergences, and to represent them by pointwise functions on an appropriate measurable space.

Keywords: F-quantum space, F-state, F-observable, representation theorem of F-observables, convergence of F-observables.

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Let $(\Omega, \mathscr{S}, E)$ be a probability space and $f\colon \Omega \to R^1$ a real valued, $\mathscr{S}$-measurable random variable, i.e. $f^{-1}(E) \in \mathscr{S}$ for any set $E \in B(R^1)$, where $B(R^1)$ is the Borel σ-algebra of the real line R^1. The mapping $x\colon B(R^1) \to \mathscr{S}$ defined as $x(E) = f^{-1}(E)$, $E \in B(R^1)$, is a σ-homomorphism, called an observable of $\mathscr{S}$.

Gudder and Mullikin [1] introduced many types of convergences for the observables in a quantum logic. Motivated by their definitions, we present some convergences of F-observables in F-quantum spaces.

1. F-QUANTUM SPACE

We recall that according to [2], an F-quantum space is a couple (Ω, M), where Ω is a nonvoid set and $M \subset [0, 1]^\Omega$ is a system of fuzzy subsets of Ω such that

(i) if $1(\omega) = 1$ for any $\omega \in \Omega$, then $1 \in M$,
(ii) $a \in M$ implies $a^\perp := 1 - a \in M$,
(iii) if $1/2(\omega) = 1/2$ for any $\omega \in \Omega$, then $1/2 \notin M$,
(iv) if $\{a_n\}_{n\in N} \subset M$, then $\bigcup_{n\in N} a_n := \sup_{n\in N} a_n \in M$.

The $\bigcap$ is a fuzzy union, and the fuzzy intersection, $\bigcap$, is defined via $\bigcap_{n\in N} a_n := \inf_{n\in N} a_n$. The set M is also called a soft fuzzy σ-algebra [4].

The soft fuzzy σ-algebra M can be regarded as a partially ordered set in which we define $a \leqq b$ iff $a(\omega) \leqq b(\omega)$ for any $\omega \in \Omega$.

Using the complementation $\perp: a \mapsto a^{\perp} = 1 - a$, we see that it satisfies two conditions:

(i) $(a^{\perp})^{\perp} = a$ for any $a \in M$,
(ii) $a \leqq b$ implies $b^{\perp} \leqq a^{\perp}$.

Two fuzzy sets a and b are called orthogonal or W-separated and we write $a \perp b$, iff $a \leqq b^{\perp}$.

It is clear that $a \perp a^{\perp}$ for any $a \in M$.

We say that a fuzzy set $a \in M$ is a W-empty set (W-universum), iff $a \leqq a^{\perp}(a \geqq a^{\perp})$. It is evident that the following assertions are equivalent:

(i) a is a W-empty set (W-universum),
(ii) $a \leqq 1/2 \, (a \geqq 1/2)$,
(iii) $a \cap a^{\perp} = a(a \cup a^{\perp} = a)$,
(iv) $a^{\perp}$ is a W-universum (W-empty set).

We denote by $W_0(M)$ and $W_1(M)$ the sets of all W-empty sets and W-universes, respectively, from M.

If $a, b \in M$, $a \leqq b$, $b \in W_0(M)$ then $a \in W_0(M)$ and if $a \leqq b$, $a \in W_1(M)$ then $b \in W_1(M)$.

Let $B(R^1)$ be the Borel σ-algebra of the real line R^1. By an F-observable of (Ω, M) we mean a mapping $x: B(R^1) \to M$ such that

(i) $x(E^c) = x(E)^{\perp}$, $E \in B(R^1)$, $E^c = R^1 - E$,
(ii) $x(E) \perp x(F)$ if $E \cap F = \emptyset$, $E, F \in B(R^1)$,
(iii) if $\{E_n\}_{n \in N} \subset B(R^1)$, $E_i \cap E_j = \emptyset$ for $i \neq j$, then $x(\bigcup_{n \in N} E_n) = \bigcup_{n \in N} x(E_n)$.

If a is a fuzzy set from M, then the mapping x_a defined via

$$x_a(E) = \begin{cases} a \cap a^{\perp} & \text{if } \; 0, 1 \notin E, \\ a^{\perp} & \text{if } \; 0 \in E, \; 1 \notin E, \\ a & \text{if } \; 0 \notin E, \; 1 \in E, \\ a \cup a^{\perp} & \text{if } \; 0, 1 \in E \end{cases}$$

for any $E \in B(R^1)$ is an F-observable of (Ω, M) and plays the role of the indicator of the fuzzy event $a \in M$.

If x is an F-observable and $f: R^1 \to R^1$ is a Borel function, then $f \circ x: E \mapsto x(f^{-1}(E))$, $E \in B(R^1)$, is an F-observable of (Ω, M), too. In particular, if $f(t) = |t|$, $t \in R^1$, we put $|x| = f \circ x$, etc. Similarly $-x$ is an F-observable defined via

$$-x(E) = x(\{t: -t \in E\}) \quad \text{for any} \quad E \in B(R^1)\,. \tag{1.1}$$

Let x and y be two F-observables. By the sum of x and y (see [5]) we mean an F-observable z such that

$$z((-\infty, t)) = \bigcup_{r \in Q} x((-\infty, r)) \cap y((-\infty, t - r)) \tag{1.2}$$

for any $t \in R^1$, where Q is the set of all rationals in the real line R^1 and we write $z = x + y$. In the paper [5] it has been proved that the sum of any pair of F-observables exists and is unique. We shall denote by $O(M)$ the set of all F-observables of (Ω, M).

An F-state on (Ω, M) is a mapping $m: M \to [0, 1]$ such that

(i) $m(a \cup a^{\perp}) = 1$ for any $a \in M$,

(ii) if $\{a_n\}_{n\in N} \subset M$, $a_i \leqq a_j^{\perp}$ for $i \neq j$, then $m(\bigcup_{n\in N} a_n) = \sum_{n\in N} m(a_n)$.

According to [6], we define $K(M)$ as the set of all subsets $A \subset \Omega$ such that there is a fuzzy set $a \in M$ satisfying

$$(1.3) \qquad \{a > 1/2\} \subset A \subset \{a \geqq 1/2\},$$

where $\{a > 1/2\} = \{\omega \in \Omega: a(\omega) > 1/2\}$, similarly for $\{a \geqq 1/2\}$.

The following result holds (see [6], [7]).

Theorem 1.1. *Let (Ω, M) be an F-quantum space. Then $K(M)$ is a σ-algebra of subsets of the set Ω. If m is an F-state, the function $P = P_m: K(M) \to [0, 1]$ defined via*

$$(1.4) \qquad P(A) = m(a), \quad A \in K(M),$$

where A and a satisfy (1.3), is a probability measure on $K(M)$ with

$$(1.5) \qquad P(\{a = 1/2\}) = 0 \quad \text{for any} \quad a \in M.$$

Moreover, if m, n are F-states such that $m \neq n$, then $P_m \neq P_n$.

Conversely, let P be any probability measure on $K(M)$ with (1.5), then the mapping $m = m_P: M \to [0, 1]$ defined via

$$(1.6) \qquad m(a) = P(A), \quad a \in M,$$

where a and A fulfil (1.3), is an F-state. Moreover, if $P \neq Q$, then $m_P \neq m_Q$. In addition, $m = m_{P_m}$ and $P = P_{m_P}$.

Lemma 1.2. *Let $\{a_n\}_{n\in N} \subset M$. Then*

(i) $\bigcup_{n\in N} \{a_n \geqq 1/2\} \subset \{\bigcup_{n\in N} a_n \geqq 1/2\}$,

(ii) $\bigcup_{n\in N} \{a_n > 1/2\} = \{\bigcup_{n\in N} a_n > 1/2\}$,

(iii) $\{\bigcap_{n\in N} a_n > 1/2\} \subset \bigcap_{n\in N} \{a_n > 1/2\}$,

(iv) $\{\bigcap_{n\in N} a_n \geqq 1/2\} = \bigcap_{n\in N} \{a \geqq 1/2\}$,

(v) *if $a_n \in W_1(M)$ for any $n \in N$, then*

$$\bigcup_{n\in N} \{a_n = 1/2\} \subset \{\bigcap_{n\in N} a_n = 1/2\}.$$

Proof. It is straightforward and, therefore, it is omitted.

A. Dvurečenskij proved the following representation theorem.

Theorem 1.3 [7]. *For any $x \in O(M)$ there is a $K(M)$-measurable, real-valued function f on Ω such that*

$$\{x(E) > 1/2\} \subset f^{-1}(E) \subset \{x(E) \geqq 1/2\} \tag{1.7}$$

for any $E \in B(R^1)$. If g is any $K(M)$-measurable, real-valued function on Ω satisfying (1.7), then

$$\{\omega \in \Omega: f(\omega) \neq g(\omega)\} \subset \{x(\emptyset) = 1/2\} .$$

Conversely, let $f: \Omega \to R^1$ be any $K(M)$-measurable function. Then there is an F-observable x satisfying (1.7). If y is any F-observable satisfying (1.7), then $x(E) \cap y(E^c) \in W_0(M)$ for any $E \in B(R^1)$.

We shall denote by $F(M)$ the set of all $K(M)$-measurable real-valued functions on Ω and write $x \sim f$ for $x \in O(M)$ and $f \in F(M)$ such that (1.7) holds.

Theorem 1.4 [7]. *Let $x \sim f$, $y \sim g$ and h be any Borel function. Then*

(i) $x + y \sim f + g$,

(ii) $h \circ x \sim h \circ f$,

(iii) $x \,.\, y \sim f \,.\, g$, *where* $x \,.\, y := 1/2((x + y)^2 - x^2 - y^2)$,

(iv) *if* $f \geqq 0$ *then* $x([0, \infty)) = x(R^1)$.

Lemma 1.5. *Let $a \in M$, $A \in K(M)$ be such that the condition (1.3) holds. Then $x_a \sim I_A$, where I_A is the indicator of the set A.*

Proof. By the assumptions of the lemma, $\{a > 1/2\} \subset A \subset \{a \geqq 1/2\}$. Let $E \in B(R^1)$ and $0, 1 \notin E$. Then $x_a(E) = a \cap a^\perp$ and $\{x_a(E) > 1/2\} = \{a \cap a^\perp > 1/2\} = \emptyset = I_A^{-1}(E) \subset \{x_a(E) \geqq 1/2\}$. If $E \in B(R^1)$ is such that $0 \in E$ and $1 \notin E$, then $x_a(E) = a^\perp$ and $\{x_a(E) > 1/2\} = \{a^\perp > 1/2\} \subset A^c = I_A^{-1}(E) \subset \{a^\perp \geqq 1/2\} = \{x_a(E) \geqq 1/2\}$. If $E \in B(R^1)$ is such that $0,1 \in E$, then $x_a(E) = a \cup a^\perp$, and due to Lemma 1.2, $\{x_a(E) > 1/2\} = \{a \cup a^\perp > 1/2\} \subset A \cup A^c = \Omega = I_A^{-1}(E) = \{a \cup a^\perp \geqq 1/2\} = \{x_a(E) \geqq 1/2\}$.

Finally, if $E \in B(R^1)$ is such that $0 \notin E$ and $1 \in E$, then $x_a(E) = a$ and $I_A^{-1}(E) = A$.

We see that $\{x_a(E) > 1/2\} \subset I_A^{-1}(E) \subset \{x_a(E) \geqq 1/2\}$ for any $E \in B(R^1)$, which implies $x_a \sim I_A$.

2. FUZZY EQUALITIES AND FUZZY INEQUALITIES

Let (Ω, M) be an F-quantum space. According to [8], a non-void subset I of M is said to be an F-ideal (F-σ-ideal) if:

(i) $a \cap a^{\perp} \in I$ for any $a \in M$,
(ii) if $a \in M$ and $a \leqq b$, $b \in I$, then $a \in I$,
(iii) if $a \cap b \in I$ for some $b \in W_1(M)$, then $a \in I$,
(iv) $a, b \in I$ implies $a \cup b \in I$ ($\bigcup_{n \in N} a_n \in I$ whenever $\{a_n\}_{n \in N} \subset I$).

Suppose that I is an F-σ-ideal and put $a \sim_I b$ iff $a \cap b^{\perp}$ and $a^{\perp} \cap b$ are from I. Then $\sim_I$ is a congruence (σ-)relation on M (see [8]), i.e.,

(i) $\sim_I$ is an equivalence relation on M,
(ii) $a \cap a^{\perp} \sim_I 0$ for any $a \in M$,
(iii) $a \sim_I b$ implies $a^{\perp} \sim_I b^{\perp}$,
(iv) $a_1 \sim_I b_1$ and $a_2 \sim_I b_2$ imply $a_1 \cup a_2 \sim_I b_1 \cup b_2$
($a_n \sim_I b_n$, $n \in N$, imply $\bigcup_{n \in N} a_n \sim_I \bigcup_{n \in N} b_n$).

Denote by

$$I_0 = \{a \in M\colon \text{there is a } c \in W_1(M) \text{ such that } a \cap c \in W_0(M)\}.$$

Then I_0 is an F-σ-ideal, $I_0 \subset I$ for any F-σ-ideal and $1 \notin I_0$. In particular, if M consists exclusively from crisp subsets of Ω, then $I_0 = \{\emptyset\}$.

Definition 2.1. *We say that two fuzzy sets $a, b \in M$ are fuzzy equal and we write $a =_F b$, iff $a \cap b^{\perp}$, $a^{\perp} \cap b \in I_0$.*

Let $x, y \in O(M)$. We say that x and y are fuzzy equal and we write $x =_F y$, iff $x(E) \cap y(E^c) \in I_0$ for every $E \in B(R^1)$.

Let $A, B \in K(M)$. We say that A and B are fuzzy equal and we write $A =_F B$, iff there is a $c \in W_1(M)$ such that $A \triangle B \subset \{\omega \in \Omega\colon c(\omega) = 1/2\}$, where $A \triangle B = A \cap B^c \cup A^c \cap B$.

Let $f, g\colon \Omega \to R^1$ be $K(M)$-measurable functions. We say that f and g are fuzzy equal and write $f =_F g$, iff there is a $c \in W_1(M)$ such that $\{\omega \in \Omega\colon f(\omega) \neq g(\omega)\} \subset \{\omega \in \Omega\colon c(\omega) = 1/2\}$.

The relation $=_F$ is an equivalence relation on M, $O(M)$, $K(M)$ and $F(M)$. It is simple to verify that the following assertions hold:

(i) $a \cap a^{\perp} =_F 0$ and $a \cup a^{\perp} =_F 1$ for any $a \in M$,
(ii) $x(\emptyset) =_F 0$ and $x(R^1) =_F 1$ for any $x \in O(M)$.

Lemma 2.2. *Let $a, b \in M$ and let $A, B \in K(M)$ be such that (1.3) holds. Then*

(i) $a =_F b$ if and only if $A =_F B$,
(ii) $a =_F b$ implies $m(a) = m(b)$, where m is an F-state on M.

Proof. (i) If $a =_F b$, then there are $c, d \in W_1(M)$ such that $a \cap b^{\perp} \cap c \in W_0(M)$ and $a^{\perp} \cap b \cap d \in W_0(M)$, which implies $\{a \cap b^{\perp} > 1/2\} \subset \{c = 1/2\}$ and $\{a^{\perp} \cap b > 1/2\} \subset \{d = 1/2\}$. From (1.3) we have $\{a > 1/2\} \subset A \subset \{a \geqq 1/2\}$ and $\{b^{\perp} > 1/2\} \subset B^c \subset \{b^{\perp} \geqq 1/2\}$, which gives, by Lemma 1.2, $\{a \cap b^{\perp} > 1/2\} \subset$

$\subset A \cap B^c \subset \{a \cap b^\perp \geqq 1/2\} = \{a \cap b^\perp > 1/2\} \cup \{a \cap b^\perp = 1/2\} \subset \{c = 1/2\} \cup \cup \{a \cap b^\perp \cup (a \cap b^\perp)^\perp = 1/2\} = \{c \cap ((a \cap b^\perp) \cup (a \cap b^\perp)^\perp) = 1/2\}$. Put $u = c \cap \cap ((a \cap b^\perp) \cup (a \cap b^\perp)^\perp)$. Then $u \in W_1(M)$ and $A \cap B^c \subset \{u = 1/2\}$. Similarly $A^c \cap B \subset \{v = 1/2\}$, where $v = d \cap ((a^\perp \cap b) \cup (a^\perp \cap b)^\perp) \in W_1(M)$. Therefore, $A \triangle B = A \cap B^c \cup A^c \cap B \subset \{u = 1/2\} \cup \{v = 1/2\} = \{u \cap v = 1/2\}$, where $u \cap \cap v \in W_1(M)$. which gives $A =_F B$.

Let now $A =_F B$. Then there is a $c \in W_1(M)$ such that $A \cap B^c \cup A^c \cap B \subset \subset \{c = 1/2\}$. We have $\{a \cap b^\perp > 1/2\} \subset A \cap B^c \subset \{c = 1/2\}$ and $\{a^\perp \cap b > 1/2\} \subset \subset A^c \cap B \subset \{c = 1/2\}$ and this is equivalent to $a \cap b^\perp \cap c \leqq 1/2$ and $a^\perp \cap b \cap \cap c \leqq 1/2$.

(ii) If $a =_F b$ then from (i) we have $A =_F B$, which implies that there is a $c \in W_1(M)$ such that $A \triangle B \subset \{c = 1/2\}$. Let P be a probability measure on $K(M)$ defined via (1.4). From (1.5) we have $P(A \cap B^c) = 0$ and $P(A^c \cap B) = 0$. Then $P(A) = = P(A \cap B^c \cup A \cap B) = P(A \cap B^c) + P(A \cap B) = P(A \cap B)$ and similarly $P(B) = = P(A \cap B)$, which gives the equality $P(A) = P(B)$ and by (1.6) $m(a) = m(b)$. □

Lemma 2.3. *Let x_a and x_b be two indicators of fuzzy sets a and b, respectively. Then $x_a =_F x_b$ if and only if $a =_F b$.*

Proof. It is evident. □

Lemma 2.4. *Let $x, y \in O(M)$. Then $x =_F y$ if and only if there is a $c \in W_1(M)$ such that $x(E) \cap y(E^c) \cap c \in W_0(M)$ for any $E \in B(R^1)$.*

Proof. Let $\{E_n\}_{n\in N}$ be a generator of $B(R^1)$. If $x =_F y$, then there are $u_n, v_n \geqq 1/2$ such that $x(E_n) \cap y(E_n^c) \cap u_n \leqq 1/2$ and $x(E_n^c) \cap y(E_n) \cap v_n \leqq 1/2$ for any $n \in N$. Denote $c_n = u_n \cap v_n$ and put $c = \bigcap_{n\in N} c_n$. Then $c \geqq 1/2$ and $x(E_n) \cap y(E_n^c) \cap c \leqq \leqq x(E_n) \cap y(E_n^c) \cap c_n \leqq x(E_n) \cap y(E_n^c) \cap u_n \leqq 1/2$ and similarly $x(E_n^c) \cap y(E_n) \cap c \leqq \leqq 1/2$. Denote

$$K = \{E \in B(R^1): x(E) \cap y(E^c) \cap c \leqq 1/2,\ x(E^c) \cap y(E) \cap c \leqq 1/2\}\,.$$

The system K is a non-empty set containing the generator $\{E_n\}_{n\in N}$. Moreover, $x(\emptyset) \cap \cap y(R^1) \cap c = x(\emptyset) \leqq 1/2$ and $x(R^1) \cap y(\emptyset) \cap c = y(\emptyset) \leqq 1/2$ imply that $\emptyset \in K$ and $R^1 \in K$. If $\{A_n\}_{n\in N} \subset K$, then $x(\bigcup_{n\in N} A_n) \cap y(\bigcap_{n\in N} A_n^c) \cap c = \bigcup_{n\in N} x(A_n) \cap \bigcap_{n\in N} y(A_n^c) \cap c \leqq \leqq \bigcup_{n\in N} x(A_n) \cap y(A_n^c) \cap c \leqq 1/2$, therefore, $\bigcup_{n\in N} A_n \in K$. We have proved that K is a σ-algebra, consequently, $K = B(R^1)$.

The converse assertion is obvious. □

Proposition 2.5. *Let $x, y \in O(M)$ and $f, g \in F(M)$ be such that $x \sim f$, $y \sim g$. The following statements are equivalent:*

(i) $x =_F y$.

(ii) $f =_F g$.

Proof. Suppose that (i) holds. From $x \sim f$, $y \sim g$ we have $\{x(E) > 1/2\} \subset \subset f^{-1}(E) \subset \{x(E) \geqq 1/2\}$, $\{y(E^c) > 1/2\} \subset g^{-1}(E^c) \subset \{y(E^c) \geqq 1/2\}$ and $\{x(E) \cap y(E^c) > 1/2\} \subset f^{-1}(E) \cap g^{-1}(E^c) \subset \{x(E) \cap y(E^c) \geqq 1/2\}$, too. By Lemma 2.4, there is a $c \geqq 1/2$ such that $x(E) \cap y(E^c) \cap c \leqq 1/2$ for any $E \in B(R^1)$, which gives $\{x(E) \cap y(E^c) > 1/2\} \subset \{c = 1/2\}$ for any $E \in B(R^1)$. Since $\{x(E) = 1/2\} = = \{x(R^1) = 1/2\}$ and $\{y(E^c) = 1/2\} = \{y(R^1) = 1/2\}$, we have $\{x(E) \cap y(E^c) \geqq \geqq 1/2\} = \{x(E) \cap y(E^c) > 1/2\} \cup \{x(E) \cap y(E^c) = 1/2\} \subset \{c = 1/2\} \cup \cup \{x(R^1) \cap y(R^1) = 1/2\} = \{c \cap x(R^1) \cap y(R^1) = 1/2\}$ for any $E \in B(R^1)$. Put $d = c \cap x(R^1) \cap y(R^1)$, then $d \in W_1(M)$ and we have $f^{-1}(E) \cap g^{-1}(E^c) \subset \subset x(E) \cap y(E^c) \geqq 1/2\} \subset \{d = 1/2\}$ for any $E \in B(R^1)$, which implies $f^{-1}(E) \cap \cap g^{-1}(E^c) \cup f^{-1}(E^c) \cap g^{-1}(E) \subset \{d = 1/2\}$, too.

Finally, $\{\omega \in \Omega: f(\omega) \neq g(\omega)\} = \bigcup_{r \in Q} (\{\omega: f(\omega) < r \leqq g(\omega)\} \cup \{\omega: g(\omega) < r \leqq \leqq f(\omega)\}) = \bigcup_{r \in Q} (f^{-1}((-\infty, r)) \cap g^{-1}((-\infty, r)^c) \cup f^{-1}((-\infty, r)^c) \cap g^{-1}((-\infty, r))) \subset \subset \{d = 1/2\}$, where Q is the set of all rationals in the real line, and this gives $f =_F g$.

Suppose now that (ii) holds. By definition, there is a $c \in W_1(M)$ such that $\{\omega: f(\omega) \neq g(\omega)\} \subset \{\omega: c(\omega) = 1/2\}$. For any $E \in B(R^1)$ we have $\{x(E) \cap y(E^c) > > 1/2\} \subset f^{-1}(E) \cap g^{-1}(E^c) \subset \{\omega: f(\omega) \neq g(\omega)\} \subset \{\omega: c(\omega) = 1/2\}$, which implies $x(E) \cap y(E^c) \cap c \leqq 1/2$. □

Corollary 2.6. *Let $x, y \in O(M)$ and $f, g \in F(M)$ be such that $x \sim f$, $y \sim g$. Let m be an F-state on M and let P be the probability measure on $K(M)$ defined via* (1.4). *If $x =_F y$ then $f = g$ almost everywhere with respect to the measure P, i.e. $P(\{\omega: f(\omega) \neq g(\omega)\}) = 0$.*

We define a mapping $o: B(R^1) \to M$ via

$$o(E) = \begin{cases} 1 & \text{if } \; 0 \in E, \\ 0 & \text{if } \; 0 \notin E \end{cases}$$

for any $E \in B(R^1)$. The mapping o is an F-observable of M. Moreover, if $f_0(\omega) = 0$ for any $\omega \in \Omega$, then f_0 is $K(M)$-measurable real-valued function from Ω into the real line R^1 and $o \sim f_0$.

Lemma 2.7. *Let $f, g \in F(M)$. Then $f =_F g$ if and only if $f - g =_F 0$.*

Proof. If $f =_F g$ then there is a $c \in W_1(M)$ such that $\{\omega: f(\omega) \neq g(\omega)\} \subset \subset \{c = 1/2\}$. But $\{\omega: (f - g)(\omega) \neq 0\} = \{\omega: f(\omega) \neq g(\omega)\} \subset \{c = 1/2\}$, which implies $f - g =_F 0$.

The converse assertion is evident. □

Proposition 2.8. *Let $x, y \in O(M)$. Then $x =_F y$ if and only if $x - y =_F o$.*

Proof. It follows from Lemma 2.7 and Proposition 2.5. □

Proposition 2.9. *Let $x, y \in O(M)$. The following statements are equivalent:*

(i) $x =_F y$,

(ii) $(x - y)(\{0\}) =_F 1$.

Proof. Let (i) hold and let $f, g \in F(M)$ be such that $x \sim f$, $y \sim g$. By Proposition 2.5, $f =_F g$ and by Lemma 2.7, $f - g =_F 0$, which is equivalent to $x - y =_F o$. Lemma 2.4 yields $(x - y)(\{0\}) =_F o(\{0\}) = 1$.

Suppose now that (ii) holds. Then there is a $c \in W_1(M)$ such that $(x - y)(\{0\}^c) \cap \cap 1 \cap c \leqq 1/2$, which implies $\{(x - y)(\{0\}^c) > 1/2\} \subset \{c = 1/2\}$. If $E \in B(R^1)$, $0 \notin E$, then $E \subset \{0\}^c$ and $(x - y)(E) \leqq (x - y)(\{0\}^c)$ and so $\{(x - y)(E) > 1/2\} \subset \subset \{(x - y)(\{0)^c) > 1/2\} \subset \{c = 1/2\}$, therefore $(x - y)(E) =_F 0 = o(E)$ whenever $0 \notin E$. If $0 \in E$, then $0 \notin E^c$ and $(x - y)(E^c) =_F 0 = o(E^c)$, which implies $(x - y) . . (E) =_F 1 = o(E)$ whenever $0 \in E$. We have proved that $(x - y)(E) =_F o(E)$ for any $E \in B(R^1)$, which is equivalent to $(x - y) =_F o$ as well as to $x =_F y$. □

Definition 2.10. *We say that a fuzzy set $a \in M$ is fuzzy less or equal to $b \in M$ and write $a \leqq_F b$, iff $a \cap b =_F a$.*

Let $x, y \in O(M)$. We say that x is fuzzy less or equal to y and write $x \leqq_F y$, iff $(y - x)([0, \infty)) =_F 1$.

Let $A, B \in K(M)$. We say that A is a fuzzy subset of B and write $A \subset_F B$, iff $A \cap B =_F A$.

Let $f, g \in F(M)$. We say that f is fuzzy less or equal to g and write $f \leqq_F g$, iff there is a $c \in W_1(M)$ such that $\{\omega: f(\omega) > g(\omega)\} \subset \{c = 1/2\}$.

The sets M, $O(M)$, $K(M)$, $F(M)$ are sets partially ordered by the relation $\leqq_F$.

Lemma 2.11. *Let $a, b \in M$ and $A, B \in K(M)$ be such that (1.3) holds. Let m be an F-state. Then the following assertion hold:*

(i) *$a \leqq_F b$ if and only if $A \subset_F B$.*

(ii) *If $a \leqq_F b$, then $m(a) \leqq m(b)$.*

(iii) *If $a \leqq_F b^\perp$, then $m(a \cup b) = m(a) + m(b)$.*

Proof. (i) It is evident.

(ii) If $a \leqq_F b$ then by (i) $A \subset_F B$, which implies $A \cap B =_F A$ and Lemma 2.2 yields $P(A \cap B) = P(A)$. Then we have $m(b) = P(B) = P(A \cap B \cup A^c \cap B) = = P(A \cap B) + P(A^c \cap B) = P(A) + P(A^c \cap B) \geqq P(A) = m(a)$.

(iii) By (ii) we have $P(A \cap B^c) = P(A)$ and then $m(a \cup b) = P(A \cup B) = = P(A \cap B^c) + P(B) = P(A) + P(B) = m(a) + m(b)$. □

Proposition 2.12. *Let $x, y \in O(M)$ and $f, g \in F(M)$ be such that $x \sim f$, $y \sim g$. Then $x \leqq_F y$ if and only if $f \leqq_F g$.*

Proof. If $x \leqq_F y$ then from definition $(y - x)([0, \infty)) =_F 1$, which implies the existence of a $c \in W_1(M)$ such that $(y - x)([0, \infty))^\perp \cap c \leqq 1/2$ and $\{(y - x) . . ((-\infty, 0)) > 1/2\} \subset \{c = 1/2\}$. The assumptions of the proposition and (i) of

Theorem 1.4 give $y - x \sim g - f$, and from Theorem 1.3 we have

$$\{(y - x)((-\infty, 0)) > 1/2\} \subset (g - f)^{-1}((-\infty, 0) \subset$$
$$\subset \{(y - x)((-\infty, 0)) \geqq 1/2\} = \{(y - x)((-\infty, 0)) > 1/2\} \cup$$
$$\cup \{(y - x)((-\infty, 0)) = 1/2\} \subset \{c = 1/2\} \cup$$
$$\cup \{(y - x)(R^1) = 1/2\} = \{c \cap (y - x)(R^1) = 1/2\} .$$

Put $d = c \cap (y - x)(R^1)$, then $d \in W_1(M)$ and $\{\omega: f(\omega) > g(\omega)\} = \{\omega: (g - f) . (\omega) < 0\} = (g - f)^{-1}((-\infty, 0)) \subset \{d = 1/2\}$, which implies $f \leqq_F g$.

Suppose now that $f \leqq_F g$. Then there is a $c \in W_1(M)$ such that $\{\omega: f(\omega) > g(\omega)\} = (g - f)^{-1}((-\infty, 0)) \subset \{c = 1/2\}$. From Theorem 1.3 we have $\{(y - x) . ((-\infty, 0)) > 1/2\} \subset (g - f)^{-1}((-\infty, 0)) \subset \{c = 1/2\} = \{c \cup c^{\perp} = 1/2\}$. Put $d = c \cup c^{\perp}$. It is evident that $d \in W_1(M)$ and $\{(y - x)((-\infty, 0)) > 1/2\} \subset \{d = 1/2\}$, therefore, $(y - x)((-\infty, 0)) \cap d \leqq 1/2$ and also $(y - x)([0, \infty)) \cap 1^{\perp} \cap d = 0 \leqq \leqq 1/2$, which implies $(y - x)([0, \infty)) =_F 1$. □

3. CONVERGENCES OF F-OBSERVABLES

Let x be an F-observable. Then the mean value of x in an F-state m is the expression $m(x)$ defined by

$$m(x) = \int_{R^1} t \, \mathrm{d}m_x(t) , \tag{3.1}$$

(if the right-hand side exists and is finite), where m_x is a probability measure on $B(R^1)$ defined via $m_x(E) = m(x(E))$, $E \in B(R^1)$, and we say that the F-observable x is integrable and write $m(x) = \int x \, \mathrm{d}m$. Moreover, if f is a Borel measurable function, then $m(f \circ x) = \int_{R^1} f(t) \, \mathrm{d}m_x(t)$, in the sense that if one side exists, then the other exists, and both are equal.

Motivated by many types of convergences for the observables in quantum logics [1] we introduce the following notions.

Definition 3.1. *We say that a sequence* $\{x_n\}_{n \in N} \subset O(M)$ $(\{f_n\}_{n \in N} \subset F(M))$ *converges to* $x \in O(M)$ $(f \in F(M))$:

(1) *fuzzy everywhere, if, for every* $\varepsilon > 0$,

$$\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (x - x_n)([-\varepsilon, \varepsilon]) =_F 1 \, (\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (f - f_n)^{-1}([-\varepsilon, \varepsilon]) =_F \Omega;$$

(2) *almost everywhere in an F-state m (in a measure P), if, for every* $\varepsilon > 0$,

$$m(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (x - x_n)([-\varepsilon, \varepsilon])) = 1$$

$$(P(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (f - f_n)^{-1}([-\varepsilon, \varepsilon])) = 1);$$

(3) *fuzzy uniformly, if, for every* $\varepsilon > 0$, *there is an integer* k *such that* $(x - x_n) \cdot ([-\varepsilon, \varepsilon]) =_F 1$

$((f - f_n)^{-1}([-\varepsilon, \varepsilon]) =_F \Omega)$ *for all* $n \geqq k$;

(4) *uniformly almost everywhere in an F-state* m *(in a measure* P*), if, for every* $\varepsilon > 0$, *there is an integer* k *such that* $m((x - x_n)([-\varepsilon, \varepsilon])) = 1$ $(P((f - f_n)^{-1}([-\varepsilon, \varepsilon])) = 1)$ *for all* $n \geqq k$;

(5) *fuzzy almost uniformly in an F-state* m *(in a measure* P*), if, for every* $\varepsilon > 0$ *and* $\delta > 0$ *there are* $a \in M (A \in K(M))$ *such that* $m(a) < \delta (P(A) < \delta)$ *and an integer* k *such that* $a^\perp \leqq_F (x - x_n)([-\varepsilon, \varepsilon])$ $(A^c \subset_F (f - f_n)^{-1}([-\varepsilon, \varepsilon]))$ *for all* $n \geqq k$;

(6) *in an F-state* m *(in a measure* P*), if, for every* $\varepsilon > 0$ $\lim_{n\to\infty} m((x - x_n)([-\varepsilon, \varepsilon])) = 1$ $(\lim_{n\to\infty} P((f - f_n)^{-1}([-\varepsilon, \varepsilon])) = 1)$;

(7) *in mean* p, *where* $1 \leqq p < \infty$, *if*

$$\lim_{n\to\infty} \int |x - x_n|^p \, \mathrm{d}m = 0 \quad (\lim_{n\to\infty} \int_\Omega |f - f_n|^p \, \mathrm{d}P = 0).$$

We say that a sequence $\{x_n\}_{n\in N} \subset O(M)$ $(\{f_n\}_{n\in N} \subset F(M))$ *is*

(8) *fuzzy fundamental everywhere, if, for every* $\varepsilon > 0$,

$$\bigcup_{k=1}^{\infty} \bigcap_{n,s=k}^{\infty} (x_n - x_s)([-\varepsilon, \varepsilon]) =_F 1$$

$$\bigcup_{k=1}^{\infty} \bigcap_{n,s=k}^{\infty} (f_n - f_s)^{-1}([-\varepsilon, \varepsilon]) =_F \Omega);$$

(9) *fundamental almost everywhere in an F-state* m *(in a measure* P*), if, for every* $\varepsilon > 0$,

$$m(\bigcup_{k=1}^{\infty} \bigcap_{n,s=k}^{\infty} (x_n - x_s)([-\varepsilon, \varepsilon])) = 1$$

$$(P(\bigcup_{k=1}^{\infty} \bigcap_{n,s=k}^{\infty} (f_n - f_s)^{-1}([-\varepsilon, \varepsilon])) = 1);$$

(10) *fuzzy fundamental uniform, if, for every* $\varepsilon > 0$, *there is an integer* k *such that* $(x_n - x_s)([-\varepsilon, \varepsilon]) =_F 1$

$((f_n - f_s)^{-1}([-\varepsilon, \varepsilon]) =_F \Omega)$ *for all* $n, s \geqq k$;

(11) *fundamental uniform almost everywhere in an F-state* m *(in a measure* P*), if, for every* $\varepsilon > 0$, *there is an integer* k *such that* $m((x_n - x_s)([-\varepsilon, \varepsilon])) = 1$

$(P((f_n - f_s)^{-1}([-\varepsilon, \varepsilon])) = 1)$ *for all* $n, s \geqq k$;

(12) *fuzzy fundamental almost uniform in an F-state m (in a measure P), if, for every* $\delta > 0$, *there is an* $a \in M$ $(A \in K(M))$ *such that* $m(a) < \delta(P(A) < \delta)$, *and for every* $\varepsilon > 0$ *there is an integer k such that*

$$a^\perp \leqq_F (x_n - x_s)([-\varepsilon, \varepsilon])\,(A^c \subset_F (f_n - f_s)^{-1}([-\varepsilon, \varepsilon])) \text{ for all } n, s \geqq k;$$

(13) *fundamental in an F-state m (in a measure P), if, for every* $\varepsilon > 0$,

$$\lim_{n,s\to\infty} m(x_n - x_s)([-\varepsilon, \varepsilon])) = 1$$

$$(\lim_{n,s\to\infty} P((f_n - f_s)^{-1}([-\varepsilon, \varepsilon])) = 1);$$

(14) *fundamental in mean p, where* $1 \leqq p < \infty$, *if*

$$\lim_{n,s\to\infty} \int |x_n - x_s|^p \, dm = 0 \;(\lim_{n,s\to\infty} \int_\Omega |f_n - f_s|^p \, dP = 0).$$

Theorem 3.2. *Let* $x, x_n \in O(M)$ *and* $f, f_n \in F(M)$, *for any* $n \geqq 1$, *be such that* $x \sim f$, $x_n \sim f_n$, $n \geqq 1$. *The sequence* $\{x_n\}_{n\in N}$ *converges to x in an arbitrary sense from* (1) *through* (14) *if and only if the sequence* $\{f_n\}_{n\in N}$ *converges to f in the corresponding sense.*

Proof. Suppose that the sequence $\{x_n\}_{n\in N}$ converges to x fuzzy everywhere. Let $\varepsilon > 0$ and denote

$$a = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (x - x_n)([-\varepsilon, \varepsilon]) \quad \text{and} \quad A = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (f - f_n)^{-1}([-\varepsilon, \varepsilon]).$$

Then $a \in M$ and $A \in K(M)$, and by Lemma 1.2 they fulfil the condition (1.3). By the assumption we have $a =_F 1$, hence there is a $c \in W_1(M)$ such that $a^\perp \cap c \leqq 1/2$, which is equivalent to $\{a^\perp > 1/2\} \subset \{c = 1/2\}$. Then $\{a^\perp > 1/2\} \subset A^c \subset \subset \{a^\perp \geqq 1/2\} = \{a^\perp > 1/2\} \cup \{a^\perp = 1/2\} \subset \{c = 1/2\} \cup \{a \cup a^\perp = 1/2\} = = \{c \cap (a \cup a^\perp) = 1/2\}$. Put $d = c \cap (a \cup a^\perp)$, then evidently $d \in W_1(M)$ and $A \cap \Omega^c = \emptyset \subset \{d = 1/2\}$ and $A^c \cap \Omega = A^c \subset \{d = 1/2\}$, too, therefore $A \Delta\, \Omega \subset \subset \{d = 1/2\}$, which implies $A =_F \Omega$ i.e. the sequence $\{f_n\}_{n\in N}$ converges to f fuzzy everywhere.

Now we prove the converse assertion. Suppose that $A =_F \Omega$ where $A = = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (f - f_n)^{-1}([-\varepsilon, \varepsilon])$ for some $\varepsilon > 0$. Then there is a $c \in W_1(M)$ such that $A^c = A^c \cap \Omega \subset \{c = 1/2\}$. In view of the above ,we have $x_a \sim I_A$ where $a = = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (x - x_n)([-\varepsilon, \varepsilon])$. Therefore $\{a^\perp > 1/2\} \subset A^c \subset \{c = 1/2\}$, which gives $a^\perp \cap c \leqq 1/2$ or $a =_F 1$.

Other types of convergences may be proved in an analogous way. □

Proposition 3.3. *If a sequence* $\{x_n\}_{n\in N} \subset O(M)$ *converges to* $x \in O(M)$ *fuzzy uniformly then the sequence* $\{x_n\}_{n\in N}$ *is fuzzy fundamental uniform. Conversely, if* $\{x_n\}_{n\in N}$ *is fuzzy fundamental uniform, then there is an* $x \in O(M)$ *such that* $\{x_n\}_{n\in N}$ *converges to* x *fuzzy uniformly.*

Proof. Let $f, f_n \in F(M)$ for any $n \geqq 1$ be such that $x \sim f$ and $x_n \sim f_n$. By Theorem 3.2 the sequence $\{f_n\}_{n\in N}$ converges to f fuzzy uniformly. For every $\varepsilon > 0$ there is an integer k such that $(f - f_n)^{-1}([-\varepsilon, \varepsilon]) =_F \Omega$ for all $n \geqq k$. By Lemma 2.4 there is a $c \in W_1(M)$ such that $(f - f_n)^{-1}([-\varepsilon, \varepsilon]^c) \subset \{c = 1/2\}$ for every $\varepsilon > 0$ and for all $n \geqq k$. Denote $A = \Omega - \{c = 1/2\}$. It is evident that $A \in K(M)$ and $A =_F \Omega$. The sequence $\{f_n\}_{n\in N}$ converges to f uniformly on the set A, which is equivalent to the assertion that, for every $\varepsilon > 0$, there is an integer $k = k(\varepsilon)$ such that for all $n, s \geqq k$, $\Omega =_F A \subset (f_s - f_n)^{-1}([-\varepsilon, \varepsilon])$. This gives that the sequence $\{f_n\}_{n\in N}$ is fuzzy fundamental uniform and applying Theorem 3.2, we obtain that the sequence $\{x_n\}_{n\in N}$ is fuzzy fundamental uniform, too.

Suppose now that $\{x_n\}_{n\in N}$ is fuzzy fundamental uniform. Due to Theorem 3.2, $\{f_n\}_{n\in N}$ is fuzzy fundamental uniform. Hence, for any integer $i \geqq 1$ there is an integer $k = k(i)$ such that $A_{n,s,i} := (f_n - f_s)^{-1}([-1/i, 1/i]) =_F \Omega$ for any $n, s \geqq k$. Put $A_0 = \bigcap_{i=1}^{\infty} \bigcap_{n,s=k(i)}^{\infty} A_{n,s,i}$. Then $A_0 \in K(M)$ and $A_0 =_F \Omega$. For any $\varepsilon > 0$ we find an integer i such that $\varepsilon > 1/i > 0$, which entails that for any $n, s \geqq k(i)$ we have $\Omega =_F A_0 \subset (f_n - f_s)^{-1}([-\varepsilon, \varepsilon])$, in other words, $\{f_n\}_{n\in N}$ is fundamental uniform on A_0. In view of a classical result, there is a $K(M)$-measurable function f such that $\{f_n\}_{n\in N}$ converges uniformly to f on A_0, that is $\{f_n\}_{n\in N}$ converges to f fuzzy uniformly, too. According to the Theorem 1.3, there is an F-observable x such that $x \sim f$, which in view of Theorem 3.2 yields that $\{x_n\}_{n\in N}$ converges fuzzy uniformly to x. □

Analogously we can prove similar results on the relationship of the fundamentality of a given type of convergence and the existence of the limit-observable of the given type for F-observables.

It is clear that:

(i) the convergence (fundamental) almost everywhere follows from the convergence fuzzy (fundamental) everywhere, as well as from the convergence (fundamental) uniform almost everywhere;

(ii) the convergences fuzzy (fundamental) everywhere, (fundamental) uniform almost everywhere and fuzzy (fundamental) almost uniform follow from the convergence fuzzy (fundamental) uniform;

(iii) the convergence (fundamental) in an F-state m (in a measure P) follows from the convergence (fundamental) uniform almost everywhere;

(iv) the convergence fuzzy (fundamental) almost uniform follows from the convergence (fundamental) uniform almost everywhere.

Theorem 3.4. *Let $x, x_n \in O(M)$ for any $n \in N$. The following statements are equivalent*:

(i) *A sequence $\{x_n\}_{n\in N}$ converges to x fuzzy almost uniformly in an F-state m.*

(ii) *A sequence $\{x_n\}_{n\in N}$ converges to x almost everywhere in an F-state m.*

Proof. Suppose (i). Let $\varepsilon > 0$ and put $\delta = 1/i$, $i = 1, 2, \ldots$. Then there is an $a_i \in M$ such that $m(a_i) < 1/i$ and there is an integer $k = k(i)$ such that $a_i^\perp \leqq_F \leqq_F (x - x_n)([-\varepsilon, \varepsilon])$ for all $n \geqq k$, as well as $a_i^\perp \leqq_F \bigcap_{n=k}^{\infty} (x - x_n)([-\varepsilon, \varepsilon])$. Put $a = \bigcap_{i=1}^{\infty} a_i$. We have $0 \leqq m(a) \leqq m(a_i) < 1/i$ for $i = 1, 2, \ldots$, which gives $m(a) = 0$ or $m(a^\perp) = 1$. Then $a^\perp = \bigcup_{i=1}^{\infty} a_i^\perp \leqq_F \bigcup_{i=1}^{\infty} \bigcap_{n=k}^{\infty} (x - x_n)([-\varepsilon, \varepsilon]) \leqq \bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} (x - x_n) \,. \, ([\varepsilon, \varepsilon])$, which implies $1 = m(a^\perp) \leqq m(\bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} (x - x_n)([-\varepsilon, \varepsilon]) \leqq 1$.

Let now (ii) hold. By Theorem 3.2, a sequence $\{f_n\}_{n\in N}$ converges to f almost everywhere in a measure P, where $f, f_n \in F(M)$ are such that $x \sim f$, $x_n \sim f_n$ for any $n \in N$, and by the Jegorov theorem [9], the sequence $\{f_n\}_{n\in N}$ converges to f almost uniformly in a measure P, which gives that this sequence converges to f fuzzy almost uniformly in a measure P, too, and again applying Theorem 3.2 we obtain that the sequence $\{x_n\}_{n\in N}$ converges to x fuzzy almost uniformly in an F-state m.

Proposition 3.5. *Let a sequence $\{x_n\}_{n\in N} \subset O(M)$ converge to $x \in O(M)$ in an F-state m. Then this sequence is fundamental in the F-state m.*

Proof. Let $\varepsilon > 0$ and let $x \sim f$, $x_n \sim f_n$ for all $n \geqq 1$, where $f, f_n \in F(M)$. By the assumption and by Theorem 3.2 we have

$$\lim_{n\to\infty} P((f - f_n)^{-1}([-\varepsilon/2, \varepsilon/2])) = 1 \quad \text{or}$$

$$\lim_{n\to\infty} P((f - f_n)^{-1}([-\varepsilon/2, \varepsilon/2]^c)) = 0 \,.$$

Because $(f_n - f_s)^{-1}([-\varepsilon, \varepsilon]^c) \subset (f_n - f)^{-1}([-\varepsilon/2, \varepsilon/2]^c) \cup \cup (f - f_s)^{-1}([-\varepsilon/2, \varepsilon/2]^c)$, we have $P((f_n - f_s)^{-1}([-\varepsilon, \varepsilon]^c)) \leqq \leqq P((f_n - f)^{-1}([-\varepsilon/2, \varepsilon/2]^c)) + P((f - f_s)^{-1}([-\varepsilon/2, \varepsilon/2]^c))$ and hence $0 \leqq \leqq \lim_{n,s\to\infty} P((f_n - f_s)^{-1}([-\varepsilon, \varepsilon]^c)) \leqq 0$, which gives $\lim_{n,s\to\infty} P((f_n - f_s)^{-1}([-\varepsilon, \varepsilon])) = 1$. □

Proposition 3.6. *If a sequence $\{x_n\}_{n\in N} \subset O(M)$ is fundamental in an F-state m, then there is an F-observable x such that the sequence $\{x_n\}_{n\in N}$ converges to x in the F-state m.*

Proof. Let $\{f_n\}_{n\in N} \subset F(M)$ be such that $x_n \sim f_n$ for all $n \in N$. By Theorem 3.2 the sequence $\{f_n\}_{n\in N}$ is fundamental in a measure P and by Theorem 6.44 in [9] there is a $K(M)$-measurable function f such that the sequence $\{f_n\}_{n\in N}$ convergers to f in the measure P. According to Theorem 1.3, there is an F-observable x such that $x \sim f$ and by Theorem 3.2 the sequence $\{x_n\}_{n\in N}$ converges to x in the F-state m. □

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Súhrn

FUZZY ROVNOSŤ A KONVERGENCIE F-POZOROVATEĽNÝCH V F-KVANTOVÝCH PRIESTOROCH

Ferdinand Chovanec, František Kôpka

V F-kvantovom priestore zavedením relácií fuzzy rovnosti a fuzzy nerovnosti sa definují rôzne typy konvergencií pre postupnosti F-pozorovateľných a využitím reprezentácie F-pozorovateľných bodovými funkciami definovanými vo vhodnom merateľnom priestore sa dokážu niektoré konvergenčné vety.

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