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ON THE COMPUTATION OF ADEN FUNCTIONS

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Summary. The paper deals with the computation of Aden functions. It gives estimates of errors for the computation of Aden functions by downward reccurence.

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Aden functions D_n are used in the theory of light scattering on sphere particles [2, 3, 4].

Aden function $D_n(z)$ of the complex argument z is defined by upward reccurence:

(1) $$D_0(z) = \operatorname{cotg} z\,,$$

(2) $$D_{n+1}(z) = \frac{1}{\dfrac{n+1}{z} - D_n(z)} - \frac{n+1}{z}\,.$$

The computation of $D_n(z)$ by upward reccurence becomes unstable when n is large (more exactly, when $n > |z| - \frac{3}{2}$).

To compute D_n for large n, downward reccurence is used. We put

(3) $$\tilde{D}_N(z) = 0 \quad \text{for sufficiently large } N, \text{ and}$$

(4) $$\tilde{D}_n(z) = \frac{n+1}{z} - \frac{1}{\dfrac{n+1}{z} + \tilde{D}_{n+1}(z)} \quad \text{for} \quad 0 \leqq n < N\,.$$

($\tilde{D}_n$ is taken as the approximation of D_n.)

The present paper gives a method how to determine N when $D_n(z)$ must be computed with a given accuracy.

The analysis of errors is easier if relation (3) is replaced by

(5) $$\tilde{D}_N(z) = \frac{N+1}{z}\,.$$

This approximation is suggested by relations (13) and (16) below, see also [4].

1. PROPERTIES OF ADEN FUNCTIONS

Aden functions D_n are closely related to Riccati-Bessel functions, which are defined by formulas

(6) $$\psi_0(z) = \sin z\,,$$

(7) $$\psi_1(z) = \frac{\sin z}{z} - \cos z\,,$$

(8) $$\psi_{n+1}(z) = \frac{2n+1}{z}\,\psi_n(z) - \psi_{n-1}(z)\,.$$

Denote

(9) $$C_n(z) = \frac{\psi_{n+1}(z)}{\psi_n(z)}\,.$$

Then (6)–(8) imply

(10) $$C_0(z) = \frac{1}{z} - \operatorname{cotg} z\,,$$

(11) $$C_n(z) = \frac{2n+1}{z} - \frac{1}{C_{n-1}(z)}\,, \quad \text{and}$$

(12) $$C_{n-1}(z) = \frac{1}{\dfrac{2n+1}{z} - C_n(z)}\,.$$

Using induction, (1), (2), (10) and (11) it is easy to show that

(13) $$D_n(z) = \frac{n+1}{z} - C_n(z) \quad \text{for all } n\,.$$

The Riccati-Bessel function ψ_n may be expressed as the series

(14) $$\psi_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{2k+n+1}}{(2k)!!\,(2k+2n+1)!!}$$

(see [1], p. 256).

For fixed z and $n \to \infty$ (14) gives

(15) $$\psi_n(z) \sim \frac{z^{n+1}}{(2n+1)!!}\,,$$

which together with (9) yields

(16) $$C_n(z) \sim \frac{z}{2n+3}\,.$$

(The symbol $\sim$ means that the limit of the quotient of the both sides is 1.)

Relation (16) shows that $|C_n(z)| < 1$ for sufficiently large n.

Now, suppose that $n > |z| - \frac{3}{2}$ and $|C_{n+1}(z)| < 1$. Using (12) we have

$$|C_n(z)| = \frac{1}{\left|\frac{2n+3}{z} - C_{n+1}(z)\right|} \leqq \frac{1}{\left|\frac{2n+3}{z}\right| - |C_{n+1}(z)|} < \frac{1}{2-1} = 1 .$$

It means that

(17) $\quad |C_n(z)| < 1 \quad \text{for all} \quad n > |z| - \frac{3}{2} .$

2. ERROR ESTIMATES

We assume that a complex number z is fixed, and we write D_n, C_n and $\tilde{D}_n$ instead of $D_n(z)$, $C_n(z)$ and $\tilde{D}_n(z)$. Observe that the computation of D_n by formulas (5) and (4) may be replaced by

(18) $\quad \tilde{C}_N = 0 ,$

(19) $\quad \tilde{C}_n = \dfrac{1}{\dfrac{2n+3}{z} - \tilde{C}_{n+1}} \quad \text{for} \quad 0 \leqq n < N, \quad \text{and}$

(20) $\quad \tilde{D}_n = \dfrac{n+1}{z} - \tilde{C}_n .$

The following proposition characterizes the behaviour of errors when D_n are computed by downward reccurence.

Proposition 1. *If* $N > n > |z| - \frac{3}{2}$ *then*

(21) $\quad |D_n - \tilde{D}_n| < |D_{n+1} - \tilde{D}_{n+1}| .$

Proof. Relations (13) and (20) imply

(22) $\quad D_n - \tilde{D}_n = \tilde{C}_n - C_n .$

Therefore, it is sufficient to prove

(23) $\quad |\tilde{C}_n - C_n| < |C_{n+1} - \tilde{C}_{n+1}| .$

Relations (19) and (12) give

$$\tilde{C}_n - C_n = \frac{1}{\frac{2n+3}{z} - \tilde{C}_{n+1}} - \frac{1}{\frac{2n+3}{z} - C_{n+1}} =$$

$$= \frac{\tilde{C}_{n+1} - C_{n+1}}{\left(\frac{2n+3}{z} - \tilde{C}_{n+1}\right)\left(\frac{2n+3}{z} - C_{n+1}\right)} = C_n\tilde{C}_n(\tilde{C}_{n+1} - C_{n+1}) .$$

We obtain

(24) $$|\tilde{C}_n - C_n| = |C_n| \, . \, |\tilde{C}_n| \, . \, |\tilde{C}_{n+1} - C_{n+1}| \, .$$

The inequality

(25) $$|\tilde{C}_n| < 1 \quad \text{for} \quad N \geqq n > |z| - \tfrac{3}{2}$$

may be proved in the same way as (17). Now, (23) follows from (24), (17) and (25).

Remark. We have also the following inequality for relative errors of C_n:

$$\left|\frac{\tilde{C}_n}{C_n} - 1\right| = \frac{1}{|C_n|} |\tilde{C}_n - C_n| = \frac{1}{|C_n|} |C_n| \, |\tilde{C}_n| \, |\tilde{C}_{n+1} - C_{n+1}| =$$

$$= |\tilde{C}_n| \, |C_{n+1}| \left|\frac{\tilde{C}_{n+1}}{C_n} - 1\right| < \left|\frac{\tilde{C}_{n+1}}{C_{n+1}} - 1\right|$$

whenever $N > n > |z| - {}^3/_2$.

However, the behaviour of relative errors of D_n is somewhat complicated and we consider only absolute errors.

Now, suppose that for a fixed $n_0 > |z| - 3/2$ we want to obtain $\tilde{D}_{n_0}$ such that $|\tilde{D}_{n_0} - D_{n_0}| < \Delta$, where Δ is prescribed. The question is, how large must N be chosen.

Let $N \geqq n_0$ be fixed. Denote

(26) $$k = N - n_0 \, .$$

Relations (12) and (19) give

$$C_{n_0} = \cfrac{1}{\cfrac{2n_0 + 3}{z} - \cfrac{1}{\cfrac{2n_0 + 5}{z} - \cfrac{1}{\ddots \; \cfrac{2n_0 + 2k + 1}{z} - C_{n_0+k}}}}$$

and

$$\tilde{C}_{n_0} = \cfrac{1}{\cfrac{2n_0 + 3}{z} - \cfrac{1}{\cfrac{2n_0 + 5}{z} - \cfrac{1}{\ddots \; \cfrac{2n_0 + 2k + 1}{z} - \tilde{C}_{n_0+k}}}} \, .$$

Note that $\tilde{C}_{n_0+k} = 0$ by (18) and (26).

Theory of continued fractions (see [5], pp. 39—49) shows that

(27) $\tilde{C}_{n_0} = \frac{P_k}{Q_k}$ and

(28) $C_{n_0} = \frac{R_k}{S_k}$,

where the numbers P_k, Q_k, R_k and S_k are defined by

(29.a) $P_0 = 0$, (30.a) $Q_0 = 1$,

(29.b) $P_1 = 1$, (30.b) $Q_1 = \frac{2n_0 + 3}{z}$,

(29.c) $P_j = \frac{2n_0 + 2j + 1}{z} P_{j-1} - P_j$,

(30.c) $Q_j = \frac{2n_0 + 2j + 1}{z} Q_{j-1} - Q_{j-2}$

for $j > 1$,

(31) $R_k = \left(\frac{2n_0 + 2k + 1}{z} - C_{n_0+k}\right) P_{k-1} - P_{k-2}$,

(32) $S_k = \left(\frac{2n_0 + 2k + 1}{z} - C_{n_0+k}\right) Q_{k-1} - Q_{k-2}$.

Using (29.c), (31), (30.c) and (32) we have

(33) $R_k = P_k - C_{n_0+k}P_{k-1}$,

(34) $S_k = Q_k - C_{n_0+k}Q_{k-1}$.

Relations (22), (27), (28), (33) and (34) imply

$$\tilde{D}_{n_0} - D_{n_0} = C_{n_0} - \tilde{C}_{n_0} = \frac{R_k}{S_k} - \frac{P_k}{Q_k} =$$

$$= \frac{Q_k(P_k - C_{n_0+k}P_{k-1}) - P_k(Q_k - C_{n_0+k}Q_{k-1})}{(Q_k - C_{n_0+k}Q_{k-1})\, Q_k} =$$

$$= \frac{C_{n_0+k}(P_kQ_{k-1} - P_{k-1}Q_k)}{Q_k(Q_k - C_{n_0+k}Q_{k-1})}.$$

The equality

$$P_kQ_{k-1} - P_{k-1}Q_k = 1$$

follows from (29), (30) by induction.

Hence

(35) $$|\tilde{D}_{n_0} - D_{n_0}| = \frac{|C_{n_0+k}|}{|Q_k|\,|Q_k - C_{n_0+k}Q_{k-1}|}.$$

From (30) it is easy to obtain

$$(36) \qquad |Q_j| \geqq \left(\frac{2n_0 + 2j + 1}{|z|} - 1\right) |Q_{j-1}| > |Q_{j-1}| \text{ and}$$

$$(37) \qquad |Q_{j+1}| - |Q_j| > |Q_j| - |Q_{j-1}|$$

whenever $n_0 > |z| - \frac{3}{2}$ and $j \geqq 1$.

Using (35), (36) and (17) we have

$$(38) \qquad |\tilde{D}_{n_0} - D_{n_0}| < \frac{1}{|Q_k| (|Q_k| - |Q_{k-1}|)} .$$

Theorem 1. *Let a complex number z and a natural number $n_0 > |z| - \frac{3}{2}$ be fixed. For any positive Δ there exists a natural number k such that*

$$(39) \qquad \frac{1}{|Q_k| (|Q_k| - |Q_{k-1}|)} < \Delta .$$

If the computation of $\tilde{D}_n$ by (4) and (5) is started from $N = n_0 + k$, then

$$(40) \qquad |\tilde{D}_n - D_n| < \Delta \quad \text{whenever} \quad n_0 \geqq n > |z| - \tfrac{3}{2} .$$

Proof. The existence of k such that (39) holds, follows from (36) and (37). (To find this natural number k it is necessary to compute Q_k by relations (30.a)–(30.c) until relation (39) is satisfied.) Relations (21), (38) and (39) give (40).

The method presented here was tested on EC 1033 by the authors. For a given complex number z and a natural number $n_0 > |z| - \frac{3}{2}$ we have found N such that $|\tilde{D}_n - D_n| < 10^{-13}$ whenever $n_0 \geqq n > |z| - \frac{3}{2}$. The value $\tilde{D}_0$ was compared with cotg z. No rounding error was observed. Partial results are summarized in the table.

Table

z	n_0	N
$1 + 0{,}1\,i$	3	9
$1 + i$	5	11
$1 + 10\,i$	15	26
$10 + i$	15	26
$10 + 10\,i$	20	32
$10 + 100\,i$	150	163
$100 + 10\,i$	150	165
$100 + 100\,i$	200	214
$100 + 1000\,i$	1200	1215
$1000 + 10\,i$	1100	1132
$1000 + 100\,i$	1200	1224
$1000 + 1000\,i$	1800	1816

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Súhrn

VÝPOČET ADENOVÝCH FUNKCIÍ

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Článok sa zaoberá výpočtom Adenových funkcií spätnou rekurziou. Sú v ňom odhadnuté chyby pri numerických výpočtoch.

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