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APPROXIMATION BY OPTIMAL PERIODIC INTERPOLATION

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0. INTRODUCTION

Babuška [1] introduced the concept of periodic Hilbert space for studying universally optimal quadrature formulas. Práger [6] continued these investigations and discovered the relationship between optimal approximations of linear functionals on periodic Hilbert spaces and in minimum norm interpolation (optimal periodic interpolation). Recently we have shown that optimal periodic interpolation is closely related to interpolation by translation [2, 3, 5]. In this paper we will study the approximation power of optimal periodic interpolation in the uniform norm.

1. OPTIMAL PERIODIC INTERPOLATION

In this section we will describe minimum norm interpolation in periodic Hilbert spaces (*optimal periodic interpolation*) as developed by Práger [6]. A periodic Hilbert space H_d is defined by an l^1-sequence $d = (d_k)$ satisfying

$$d_0 = 1\,, \quad d_{-k} = d_k > 0 \quad (k \in \mathbb{N})\,.$$

Then

$$H_d = \{f \in L^2_{2\pi};\ \sum_{k=-\infty}^{\infty} |(f, e_k)|^2/d_k < \infty\}$$

is the Hilbert space of continuous 2π-periodic functions with inner product

$$(f, g)_d = \sum_{k=-\infty}^{\infty} (f, e_k)\,(e_k, g)/d_k\,.$$

Here we use the notation $e_k(t) = \exp(\mathrm{i}kt)\,(k \in \mathbb{Z})$ and

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \operatorname{conj} g(t)\, \mathrm{d}t, \quad (\operatorname{conj} z = \bar{z})\,.$$

We denote by $A_{2\pi}$ the Wiener algebra of functions from $C_{2\pi}$ having an absolutely convergent Fourier series [4]. $A_{2\pi}$ is a Banach algebra with respect to the norm

$$\|f\|_a = \sum_{k=-\infty}^{\infty} |(f, e_k)| .$$

The periodic Hilbert space is uniquely related to the *generating function* $\psi \in A_{2\pi}$ which is defined by

$$\psi(t) = \sum_{k=-\infty}^{\infty} d_k\, e_k(t) = 1 + \sum_{k=1}^{\infty} 2d_k \cos(kt) .$$

For any $a \in \mathbb{R}$ we have $\psi(\cdot\ - a) \in H_d$. Moreover, the relation

$$f(x) = (f, \psi(\cdot\ - x))_d , \quad f \in H_d \tag{1.1}$$

holds for any $x \in \mathbb{R}$. Thus, H_d is a reproducing kernel Hilbert space with reproducing kernel function

$$K(y, x) = \psi(y - x) .$$

Minimum norm interpolation in H_d is related to the uniform mesh $t_j = 2\pi j/n$, $j \in \mathbb{Z}$. The functions

$$\psi(\cdot\ - t_j) , \quad 0 \leqq j < n ,$$

are representatives of the point evaluation functionals

$$L_j(f) = f(t_j) , \quad 0 \leqq j < n .$$

We denote by S_n the unique orthogonal projector on H_d with

$$\mathfrak{R}(S_n) = \langle \psi(\cdot\ - t_0), \ldots, \psi(\cdot\ - t_{n-1}) \rangle \tag{1.2}$$

S_n is the projector of *optimal periodic interpolation* in H_d. For given $f \in H_d$, $S_n(f)$ is the unique interpolant of f in H_d with minimum norm, i.e.,

$$\begin{aligned} &S_n(f)(t_j) = f(t_j) \quad (0 \leqq j < n) , \\ &\|S_n(f)\|_d = \min \{ \|g\|_d \colon g(t_j) = f(t_j), 0 \leqq j < n \} . \end{aligned} \tag{1.3}$$

It is easily seen that $\mathfrak{R}(S_n)$ is translation invariant with respect to t_1. Thus, the method of interpolation by translation is applicable [2, 3, 5]. It uses the discrete Fourier transform method. Let

$$c_{k,n}(f) = \frac{1}{n} \sum_{j=0}^{n-1} f(t_j)\, e_k(-t_j) , \quad k \in \mathbb{Z} .$$

Then the trigonometric polynomial

$$T_n(f) = \sum_{k=0}^{n-1} c_{k,n}(f)\, e_k$$

satisfies

$$T_n(f)(t_j) = f(t_j), \quad 0 \leqq j < n.$$

This implies the *discrete Fourier representation*

$$(1.4) \qquad S_n(f) = \sum_{k=0}^{n-1} c_{k,n}(f)\, S_n(e_k).$$

Next we will determine the Fourier series of the exponential interpolants $S_n(e_k)$. We consider the functions

$$(1.5) \qquad B_k(t) = \sum_{j=0}^{n-1} \psi(t - t_j)\, e_k(-t_j), \quad k \in \mathbb{Z},$$

which satisfy $\mathfrak{R}(S_n) = \langle B_0, \ldots, B_{n-1} \rangle$. The Fourier series of B_k is given by

$$(1.6) \qquad B_k(t) = n \sum_{r=-\infty}^{\infty} d_{k+rn} e_{k+rn}(t),$$

which implies

$$(1.7) \qquad S_n(e_k)(t) = B_k(t)/B_k(0).$$

Since

$$(1.8) \qquad c_{k,n}(\psi(\cdot - t)) = B_k(t)/n$$

we obtain the *Fourier expansions*

$$(1.9) \qquad S_n(e_k)(t) = \sum_{r=-\infty}^{\infty} c_{k,n}(\psi)^{-1}\, d_{k+rm} e_{k+rn}(t), \quad k \in \mathbb{Z}.$$

In view of

$$(1.10) \qquad \operatorname{conj} c_{k,n}(f) = c_{-k,n}(\operatorname{conj} f)$$

we obtain

$$(1.11) \qquad S_n(e_{-k}) = \operatorname{conj} S_n(e_k), \quad k \in \mathbb{Z}.$$

2. CONVERGENCE IN H_d

Let $n = 2m + 1$ and

$$D_n = \sum_{r=1}^{\infty} d_{rm}.$$

We assume that

$$d_k \geqq d_{k+1} > 0 \quad (k \in \mathbb{Z}_+).$$

Then we have

$$(2.1) \qquad c_{k,n}(\psi) - d_k \leqq D_n(|k| \leqq m).$$

This result is used to establish

Proposition 2.1. *Let $n = 2m + 1$ and $|k| \leqq m$. Then we have*

$$\|e_k - S_n(e_k)\|_d = O(D_n^{1/2}) \quad (n \to \infty) . \tag{2.2}$$

Proof. Parseval's relation yields

$$(e_k - S_n(e_k), e_k - S_n(e_k))_d = (1 - d_k c_{k,n}(\psi)^{-1})^2/d_k + \sum_{r \neq 0} d_{k+rn} c_{k,n}(\psi)^{-2} \leqq$$
$$\leqq D_n^2 d_k^{-3} + D_n d_k^{-2} = O(D_n) ,$$

which completes the proof.

Recall that the projector of optimal periodic interpolation is an orthogonal projector on H_d. Moreover, the algebra of trigonometric polynomials

$$\tau = \langle\{e_k \colon k \in \mathbb{Z}\}\rangle$$

is dense in H_d. Taking into account Proposition 2.1 and the Banach-Steinhaus theorem we obtain

Proposition 2.2. *For any $f \in H_d$ we have*

$$\lim_{n \to \infty} \|f - S_n(f)\|_d = 0 .$$

Remark. Práger [6] proved the convergence of the optimal approximations Φ_n of the bounded linear functional $\Phi \in H_d^*$ in the norm topology of H_d^*. Moreover, he showed that

$$\Phi_n(f) = \Phi(S_n(f)) \quad (f \in H_d) .$$

Thus, Proposition 2.2 follows also from Práger's results in view of the duality theory for Hilbert spaces.

3. CONVERGENCE IN $A_{2\pi}$

In this section we will extend the convergence result of Proposition 2.2 from (the periodic Hilbert space) H_d to the Wiener algebra $A_{2\pi}$ [4]. Note first that

$$\|f\|_\infty \leqq \|f\|_a \quad (f \in A_{2\pi}) . \tag{3.1}$$

Proposition 3.1. *For $f \in A_{2\pi}$ the estimate*

$$\|S_n(f)\|_\infty \leqq \|f\|_a \tag{3.2}$$

holds.

Proof. Recall that

$$S_n(e_k)(t) = B_k(t)/B_k(0) .$$

Since $|B_k(t)| \leqq B_k(0)$ we have

$$\|S_n(e_k)\|_\infty = 1 . \tag{3.3}$$

Using the discrete Fourier representation and the relation

$$c_{k,n}(f) = \sum_{r=-\infty}^{\infty} (f, e_{k+rn})$$

we can conclude

$$|S_n(f)(t)| \leqq \sum_{k=0}^{n-1} |c_{k,n}(f)| \leqq \sum_{k=0}^{n-1} \sum_{r=-\infty}^{\infty} |(f, e_{k+rn})| = \|f\|_a ,$$

which completes the proof.

Proposition 3.2. For any $f \in A_{2\pi}$ we have

$$\lim_{n\to\infty} \|f - S_n(f)\|_\infty = 0 .$$

Proof. It follows from Proposition 3.1 that the sequence of linear operators

$$S_n: A_{2\pi} \to C_{2\pi} \quad (n = 2m + 1, m \geqq 0)$$

is uniformly bounded. Moreover, the estimate

$$\|f\|_\infty \leqq \sqrt{[\psi(0)]}\ \|f\|_d \quad (f \in H_d) \tag{3.4}$$

holds. Taking into account Proposition 2.2 we obtain

$$\lim_{n\to\infty} \|f - S_n(f)\|_\infty = 0 \quad (f \in H_d) .$$

Since H_d is dense in $A_{2\pi}$ again an application of the Banach-Steinhaus theorem completes the proof.

4. QUANTITATIVE ERROR BOUNDS

The generating sequence $d = (d_k)$ of H_d defines the linear operator A in $L^2_{2\pi}$ by

$$Af = \sum_{k=-\infty}^{\infty} (f, e_k)\, d_k^{-1} e_k \tag{4.1}$$

with

$$\operatorname{dom}(A) = \{\psi * g : g \in L^2_{2\pi}\} . \tag{4.2}$$

Proposition 4.1. *Let $n = 2m + 1$ and $|k| \leqq m$. Then we have*

$$\|e_k - S_n(e_k)\|_\infty \leqq 2D_n \|Ae_k\|_a . \tag{4.3}$$

Proof. Taking into account the Fourier series of $S_n(e_k)$ we get

$$|e_k(t) - S_n(e_k)| \leqq |1 - d_k\, c_{k,n}(\psi)^{-1}| + \sum_{r \neq 0} d_{k+rn}\, c_{k,n}(\psi)^{-1} =$$
$$= 2(c_{k,n}(\psi) - d_k)/c_{k,n}(\psi) \leqq 2D_n/d_k = 2D_n\|Ae_k\|_a .$$

This completes the proof.

We will extend the error estimate (4.3) for the exponentials e_k to a larger class of functions.

Proposition 4.2. *Suppose that* $f \in \operatorname{dom}(A)$ *satisfies* $\|Af\|_a < \infty$. *Then we have for* $n = 2m + 1$:

$$\|f - S_n(f)\|_\infty \leqq 2D_n\|Af\|_a . \tag{4.4}$$

Proof. We denote by F_m the Fourier partial sum projector:

$$F_m(f) = \sum_{k=-m}^{m} (f, e_k)\, e_k .$$

Then we have

$$\|f - F_m(f)\|_\infty \leqq D_n\|A(f - F_m(f))\|_a . \tag{4.5}$$

Now we can conclude

$$\|f - S_n(f)\|_\infty \leqq \|f - F_m(f)\|_\infty + \|F_m(f) - S_n(F_m(f))\|_\infty +$$
$$+ \|S_n(F_m(f) - f)\|_\infty \leqq 2D_n\|A(f - F_m(f))\|_a +$$
$$+ \sum_{k=-m}^{m} |(f, e_k)|\, \|e_k - S_n(e_k)\|_\infty \leqq 2D_n\|A(f - F_m(f))\|_a +$$
$$+ 2D_n \sum_{k=-m}^{m} |(f, e_k)|/d_k = 2D_n\|Af\|_a ,$$

which completes the proof of Proposition 4.2.

5. EXAMPLES

Let $r \in \mathbb{N}$ and

$$d_0 = 1\,, \quad d_k = k^{-2r} \quad (k \neq 0)\,.$$

The periodic Hilbert space is the periodic Sobolev space $W^r_{2\pi}$. The generating function ψ is given by

$$\psi(t) = 1 + (-1)^r B_{2r}(t) \tag{5.1}$$

where $B_q(t)$ is the Bernoulli function of degree q:

$$B_q(t) = \sum_{k \neq 0} (ik)^{-q} \exp(ikt)\,. \tag{5.2}$$

It is easily seen that $\mathfrak{N}(S_n)$ is a space of *periodic monosplines* of degree $2r$ [2, 5].

Suppose that $f \in C_{2\pi}^{2r}$ satisfies $D^{2r}f \in A_{2\pi}$. Then an application of Proposition 4.2 yields the asymptotic error estimate

$$(5.3) \qquad \|f - S_n(f)\|_\infty = O(m^{-2r}) \quad (n = 2m + 1 \to \infty)\,.$$

Next we consider the sequence

$$d_k = \exp(-|k|\, b) \quad (k \in \mathbb{Z})$$

where b is a positive real number. The generating function ψ is a Poisson kernel given by

$$(5.4) \qquad \psi(t) = \sinh(b)/(\cosh(b) - \cos(t))\,.$$

The periodic Hilbert space consists of functions being holomorphic in $|\mathrm{Im}(z)| < b$ and satisfying the growth condition

$$(5.5) \qquad \sum_{k=-\infty}^{\infty} |(f, e_k)|^2 \exp(b|k|) < \infty\,.$$

The space $\mathfrak{R}(S_n)$ consists of *rational trigonometric functions.*

Suppose that

$$f = \psi * g\,, \quad g \in A_{2\pi}\,.$$

Then an application of Proposition 4.2 yields the asymptotic error estimate

$$(5.6) \qquad \|f - S_n(f)\|_\infty = O(e^{-bm}) \quad (n = 2m + 1 \to \infty)\,.$$

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Souhrn

APPROXIMACE OPTIMÁLNÍ PERIODICKOU INTERPOLACÍ

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V článku se studuje rychlost konvergence optimálních aproximací v periodických Hilbertových prostorech v návaznosti na dřívější výsledky I. Babušky a M. Prágera. Hlavním výsledkem je konvergenční věta a odhad chyby ve stejnoměrné normě. Tento odhad je ilustrován na dvou příkladech.

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