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ON THE REGULARITY OF SOLUTIONS OF A THERMOELASTIC SYSTEM UNDER NONCONTINUOUS HEATING REGIMES

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Summary. The continuity and boundedness of the stress to the solution of the thermoelastic system is studied first for the linear case on a strip and then for the twodimensional model involving nonlinearities, noncontinuous heating regimes and isolated boundary nonsmoothnesses of the heated body.

Keywords: Nonlinear heat equation, Lamé system, noncontinuous heating regime, isolated boundary nonsmoothness, boundedness and continuity of the stresses, Sobolev spaces, Fourier transformation.

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The research in the field indicated by the title was encouraged by the machine industry. Its aim is to heat a large body up to a prescribed temperature without damage caused possibly by thermoelastic stresses. As the fulfilment of the compatibility condition (i.e. the initial temperature satisfies the boundary value condition) can be hardly required, it is reasonable to study the influence of jumping temperature regimes in the furnace to the heated body e.g. when the body is quickly inserted into the furnace. The aim of the paper is to find sufficient conditions for continuity (thus also boundedness) of the stresses in the body. To reach the aim, a thorough investigation of the regularity of solutions both of the heat equation and of the Lamé system is necessary.

In the sequel, the heat equation is considered in the form

$$\begin{aligned}
&\beta(u)\frac{\partial u}{\partial t} = \Delta u \quad \text{on} \quad Q = (0, \mathscr{T}) \times \Omega\,,\\
(1)\qquad &\frac{\partial u}{\partial \nu} = \mathscr{g}(T) - \mathscr{g}(\Lambda u) \quad \text{on} \quad S = (0, \mathscr{T}) \times \partial\Omega\,,\\
&u(0, x) = u_0 \quad \text{on} \quad \Omega\,,
\end{aligned}$$

where $\Omega \subset R^n$ is a domain with a sufficiently smooth boundary $\partial\Omega$. $T: S \to R^1$ is the heating regime in the furnace, $\beta, \Lambda: R^1 \to R^1$ are given positive functions, Λ has a positive derivative. The model for g is e.g. $g: y \mapsto \alpha_k y + \sigma_S y^4$ with $\alpha_k > 0$, $\sigma_S \geqq 0$ constants. ν denotes the outer normal vector. Throughout the paper, all gradients and Laplacians are meant to be in the space variables only. The Lamé system is considered as follows:

$$
\begin{aligned}
&(1 - 2\sigma)\,\Delta v + \operatorname{grad}\operatorname{div} v = (2 + 2\sigma)\operatorname{grad}\gamma(u) \quad \text{on} \quad Q\,,\\
(2)\qquad &(1 - 2\sigma)\left(\frac{\partial v}{\partial \nu} + ((\nu, \operatorname{grad}_i v)_n)_i\right) + 2\sigma\nu\operatorname{div} v = (2 + 2\sigma)\,\gamma(u)\,\nu \quad \text{on } S\,,
\end{aligned}
$$

where γ is a function with a positive first derivative, $\sigma \in (0, \frac{1}{2})$ is a constant (the Poisson ratio). The stress tensor is defined as

$$
\tau_{ij} = \frac{E}{2 + 2\sigma}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} + \delta_{ij}\frac{2\sigma}{1 - 2\sigma}\sum_{k=1}^{n}\frac{\partial v_k}{\partial x_k}\right),
$$

where $E > 0$ is the Young modulus of elasticity, possibly temperature-dependent, $\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$. For details of the technical formulation of the problem and derivation of the system cf. [4].

In Section 1, a survey of some necessary facts about anisotropic Sobolev spaces will be made. In Sec. 2, the simplest case of a strip-form body and linear heating process will be treated. In Sec. 3, more general results for two-dimensional Ω will be obtained. In Sec. 4, the case of Ω with isolated boundary nonsmoothnesses is mentioned.

1. ANISOTROPIC SOBOLEV SPACES OF HILBERT TYPE WITH FRACTIONAL DERIVATIVES

The theory is dealt with e.g. in [2], [5], [7]. For handling Hilbert-type spaces on R^n, the use of the Fourier transformation is advantageous. For a multiindex $\alpha \in R^n_+$ ($R_+ \equiv \langle 0, +\infty)$) we introduce

$$
\begin{aligned}
&H^\alpha(\Omega) = \{u \in L_2(\Omega);\ \|u\|_{\alpha,\Omega} < +\infty\}\,,\\
(3)\qquad &\|u\|^2_{\alpha,\Omega} = \|u\|^2_{L_2(\Omega)} + \|u\|'^2_{\alpha,\Omega} = \|u\|^2_{L_2(\Omega)} + \sum_{i=1}^{n}\left(\chi_{\mathbb{N}\setminus\{0\}}(\alpha_i)\left\|\frac{\partial^{\alpha_i} u}{\partial x_i^{\alpha_i}}\right\|^2_{L_2(\Omega)} + \right.\\
&\left. + (1 - \chi_{\mathbb{N}}(\alpha_i))\int_\Omega\int_{R^1}\left(\frac{\Delta_1^{se_i}\dfrac{\partial^{[\alpha_i]}}{\partial x_i^{[\alpha_i]}}u(x)}{|s|^{n/1+\{\alpha_i\}}}\right)^2 \mathrm{d}s\,\mathrm{d}x\right).
\end{aligned}
$$

In (3) $\|\cdot\|_{L_2(\Omega)}$ denotes the norm in $L_2(\Omega)$, $\chi_M: x \to \begin{cases} 0, & x \notin M \\ 1, & x \in M \end{cases}$ is the characteristic function of a set M, $\mathbb{N}$ is the set of all natural numbers. For $q \in R^1$, $[q]$ denotes its integer part (the greatest integer less than or equal to q), $\{q\}$ its fractional part, i.e. $\{q\} = q - [q]$. For a function $f: R^n \to R^m$, $n, m \in \mathbb{N}$, we write $\Delta_k^h f(x) \equiv \equiv \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(x + (m-j)h)$, $k \in \mathbb{N}$, $h \in R^n$.

For $f: R^n \to R^1$ we denote by $\hat{f}$ its Fourier transform.

Lemma 1. *Let* $I_{k,\alpha}(f) \equiv \int_{R^n} \int_{R^n} \left(\frac{\Delta_k^h f(x)}{|h|^{n/2+\alpha}} \right)^2 \mathrm{d}h \, \mathrm{d}x$ *for a function* $f: R^n \to R^1$, $k \in \mathbb{N}, 0 < \alpha < k$. *Then*

(4) $$I_{k,\alpha}(f) = c_{n,k}(\alpha) \int_{R^n} |\hat{f}|^2 |\zeta|^{2\alpha} \mathrm{d}\xi \,, \quad \text{where} \quad c_{n,k}(\alpha) = c_n^0(\alpha)\, 2^{2k-2\alpha} \,. \cdot \int_{R^1} \frac{\sin^{2k} t}{|t|^{1+2\alpha}} \mathrm{d}t \,,$$

and $c_n^0(\alpha) = 1$ for $n = 1$, $c_n^0(\alpha) = \int_{R^{n-1}} \frac{\mathrm{d}s}{(1 + |s|^2)^{n/2+\alpha}}$ *for* $n \geqq 2$.

Proof. $I_{k,\alpha}(f) = \int_{R^n} \int_{R^n} \frac{|\hat{f}|^2 \, |1 - \mathrm{e}^{\mathrm{i}(n,\xi)_n}|^{2k}}{|h|^{n+2\alpha}} \mathrm{d}h \, \mathrm{d}\xi =$

$$= \int_{R^n} \int_{R^n} |\hat{f}|^2 \frac{|e^{\mathrm{i}(h,\xi)_n/2} - \mathrm{e}^{-\mathrm{i}(h,\xi)_n/2}|^{2k}}{|h|^{n+2\alpha}} \mathrm{d}h \, \mathrm{d}\xi = \int_{R^n} \int_{R^n} |\hat{f}|^2 \frac{2^{2k} \sin^{2k}((h, \xi)_n/2)}{|h|^{n+2\alpha}} \mathrm{d}h \, \mathrm{d}\xi \,.$$

To calculate

$$J_{n,k}(\alpha) = \int_{R^n} \frac{2^{2k} \sin^{2k}((h, \xi)_n/2) \, \mathrm{d}h}{|h|^{n+2\alpha}}$$

we use a rotation $\mathcal{O}$ with its center at the origin. For a couple of functions g_1, g_2: $R^1 \to R^1$ we have $\int_{R^n} g_1((x, \xi)_n) \, g_2(|x|^2) \, \mathrm{d}x = \int_{R^n} g_1((x, \mathcal{O}\xi)_n) \, g_2(|x|^2) = \int_{R^n} g_1((\mathcal{O}^*x, \xi)_n) \, g_2(|\mathcal{O}^*x|^2) \, \mathrm{d}x$. Taking $\mathcal{O}$ such that $\bar{\xi} = |\xi| \, [1, 0, \ldots, 0]$, $y = \mathcal{O}^*x$, we can see that

$$J_{n,k}(\alpha) = \int_{R^n} \frac{2^{2k} \sin^{2k}(y_1|\xi|/2)}{|y|^{n+2\alpha}} \mathrm{d}y = 2^{2k-2\alpha} |\xi|^{2\alpha} \int_{R^n} \frac{\sin^{2k} z_1}{|z|^{n+2\alpha}} \mathrm{d}z = c_{n,k}(\alpha) \, |\xi|^{2\alpha},$$

provided we use the substitutions $z = \frac{1}{2}|\xi| \, y$ and $t = z_1$, $s_i = z_{i+1}/z_1$, $i = 1, \ldots \ldots, n-1$.

Using Lemma 1 we obtain for $\Omega = R^n$

(5) $$\|u\|^2_{\alpha,R^n} = \int_{R^n} |\hat{u}|^2 \, (1 + \sum_{i=1}^{n} |\xi|^{2\alpha_i} \, c_{1,1}(\{\alpha_i\})) \, \mathrm{d}\xi \,,$$

if we define $c_{1,1}(0) = 1$. Thus the norm $(\int_{R^n} |\hat{u}|^2 \, (1 + |\xi|^{2\alpha_i}) \, \mathrm{d}\xi)^{1/2}$ is an equivalent norm in $H^\alpha(R^n)$. Hence, the dual space $H^{-\alpha}(R^n)$ can be introduced as $H^{-\alpha}(R^n) = \{u; \|u\|_{-\alpha,R^n} < +\infty\}$, where $\|u\|^2_{-\alpha,R^n} = \int_{R^n} |\hat{u}|^2 \, (1 + \sum_{i=1}^{n} (\xi_i)^{2\alpha_i})^{-1} \, \mathrm{d}\xi$.

For a Hilbert space $\mathscr{H}$ and $\alpha_0 \in (0, 1)$ we define the space $H^{\alpha_0}(a, b, \mathscr{H})$ by means of the norm

(6) $$\|f\|^2_{a,b,\mathscr{H},\alpha_0} = \int_a^b \|f(t)\|^2_{\mathscr{H}} \, dt + \int_a^b \int_a^b \frac{\|f(t) - f(s)\|^2_{\mathscr{H}}}{|t - s|^{1+2\alpha_0}} \, dt \, ds \, .$$

Let M be a closed subset of R^n. By $C_0(M)$ we denote the space of all continuous functions on M having the zero limit at infinity if M is unbounded. The space is equipped with the usual norm. The following two theorems are well-known – cf. [2], [9], [12].

Theorem 1. *If $\alpha_0 > \frac{1}{2}$, then $H^{\alpha_0}(a, b; \mathscr{H})$ si continuously imbedded into $C_0(a, b; \mathscr{H})$.*

Theorem 2. *If Ω has a sufficiently regular boundary and $\alpha \in R^n_+$ is a multi-index satisfying $\frac{1}{2}\sum_{i=1}^{n} 1/\alpha_i < 1$, then $H^\alpha(\Omega)$ is continuously imbedded into $C_0(\bar{\Omega})$.*

Of course, $\bar{\Omega} = \Omega \cup \partial\Omega$. We remark that for $\Omega = R^n$ Theorem 2 is an easy consequence of Theorem 1.

For a function f of the form $k_1\chi_{\langle a,c)} + k_2\chi_{\langle c,b\rangle}$, $-\infty < a < c < b < +\infty$, the seminorm $\|\cdot\|'_{1/2-\varepsilon,(a,b)}$ defined in (3) and an arbitrary $\varepsilon \in (0, \frac{1}{2})$ we have

(7) $$\|f\|'^2_{1/2-\varepsilon,(a,b)} = \int_a^b \int_a^b \frac{|f(x) - f(y)|^2}{|x - y|^{2-2\varepsilon}} \, dx \, dy = |k_2 - k_1|^2 \frac{(b - a)^{2\varepsilon}}{\varepsilon(1 - 2\varepsilon)} \cdot$$
$$\cdot \left[\left(\frac{b - c}{b - a}\right)^{2\varepsilon} + \left(\frac{c - a}{b - a}\right)^{2\varepsilon} - 1\right] \leqq |k_1 - k_2|^2 \left(C(b - a, \varepsilon)\right)^2 ,$$
$$C(b - a, \varepsilon) = (b - a)^\varepsilon \sqrt{\left(\frac{2^{1-2\varepsilon} - 1}{\varepsilon(1 - 2\varepsilon)}\right)} \, .$$

For a nondecreasing and piecewise constant function f on (a, b) we obtain from (7)

(8) $$\|f\|'_{1/2-\varepsilon,(a,b)} \leqq |f(b) - f(a)| \, C(b - a, \varepsilon) \, .$$

For each nondecreasing function f on (a, b) there is a sequence of nondecreasing, piecewise constant functions $\{f_n\}$ tending uniformly to f. Using the Fatou lemma we verify the validity of the inequality (8) for each monotonic function on $\langle a, b\rangle$. Therefore for each function f having a bounded variation on $\langle a, b\rangle$ we have

(9) $$\|f\|'_{1/2-\varepsilon,(a,b)} \leqq 2C(b - a, \varepsilon) \operatorname{var}_{\langle a,b\rangle} f \, .$$

Hence the following lemma holds:

Lemma 2. *Let $f: \langle a, b\rangle \to R^1$ have a bounded variation on $\langle a, b\rangle$. Then $f \in \bigcap_{\varepsilon>0} H^{1/2-\varepsilon}(a, b)$.*

2. PRELIMINARY RESULTS

In this section Ω is considered to be a strip, i.e. $\Omega = R^n \times (0, r)$, $n = 1, 2$. For the sake of simplicity we choose $r = 1$. The simplest linear heat equation $\partial u/\partial t = \Delta u$ will be treated and $\gamma(u) = u$ will be supposed in the Lamé system. For the heat equation, we shall study two types of the boundary value condition, namely the Newton-Neumann type

$$(10) \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad R^{n+1} \times \{0\}\,, \quad \frac{\partial u}{\partial \nu} = u - T \quad \text{on} \quad R^{n+1} \times \{1\}\,,$$

and the Dirichlet type

$$(11) \qquad u = 0 \quad \text{on} \quad R^{n+1} \times \{0\}\,, \quad u = T \quad \text{on} \quad R^{n+1} \times \{1\}\,.$$

The first case corresponds to the heating of a symmetric strip with the axis at $R^{n+1} \times \{0\}$ without radiation. The boundary value conditions for the Lamé system will be of the type (2) on $R^{n+1} \times \{1\}$ and $v = 0$ on $R^{n+1} \times \{0\}$ both for (10) and (11). Such a modified problem (2) will be denoted by (2').

Theorem 3. *For $n = 2$, Ω described above, the linear system, the homogeneous initial condition and the boundary value condition* (10), *let $T \in \bigcap_{\varepsilon > 0} H^{1/2-\varepsilon,k,k}(R^3)$, $k > 1$. Let E and γ be constants. Then the corresponding stresses belong to $C_0(R^1 \times \bar{\Omega}; R^3)$. For $n = 1$ the situation is analogous, the sufficient condition is $k \geqq 1$.*

Remark. It seems that such results cannot be obtained for the case (11) for any k. For details cf. the remark at the end of this section.

To prove Theorem 3 we shall first prove the following proposition concerning the heat equation:

Proposition 1. *Under the assumptions of Theorem 3 let $k > 1 - \varepsilon$, $\varepsilon \in (0, \frac{1}{2})$, and let us consider the boundary value condition* (10). *Then for $n = 1, 2$ the solution u of the heat equation belongs to*

$$\bigcap_{\varepsilon>0} H^{5/4-\varepsilon/2,k+3/2,5/2-\varepsilon}(R^1 \times \Omega)\,, \quad \bigcap_{\varepsilon>0} H^{5/4-\varepsilon/2,k+3/2,k+3/2,5/2-\varepsilon}(R^1 \times \Omega)\,,$$

respectively. Considering the case (11) *we obtain*

$$u \in \bigcap_{\varepsilon>0} {}^{3/4-\varepsilon,k+1/2,1}(R^1 \times \Omega)\,, \quad \bigcap_{\varepsilon>0} H^{3/4-\varepsilon,k+1/2,k+1/2,1}(R^1 \times \Omega)\,,$$

respectively, for $k \geqq \frac{1}{2}$.

We remark that, in general, the results of Prop. 1 in time and in tangential directions can not be improved.

To prove Prop. 1 we shall use the partial Fourier transformation

$$(12) \qquad \tilde{u}(\tau_0, \xi, x_{n+1}) = (2\pi)^{-(n+1)/2} \int_{R^{n+1}} u(t, x)\, e^{-i(t\tau_0 + (\tilde{x},\xi)_n)}\, dt\, d\tilde{x}\,, \quad n = 1, 2,$$

where for a vector $x \equiv [x_1, x_2, \ldots, x_n, x_{n+1}] \in R^{n+1}$ we define $\tilde{x} \in R^n$ $\tilde{x} \equiv [x_1, x_2, \ldots, x_n]$. With the help of (12) we rewrite the problem with the boundary value condition (10) into the form

$$\text{(13)} \qquad \frac{\partial^2 \tilde{u}}{\partial x_3^2} = d^2 \tilde{u} \quad \text{on} \quad Q = R^1 \times \Omega \quad \text{with} \quad d = \sqrt{(\mathrm{i}\tau_0 + |\xi|^2)} =$$

$$= 2^{-1/2}\left(\sqrt{\left(\sqrt{(\tau_0^2 + |\xi|^4)} + |\xi|^2\right)} + \mathrm{i}\,\mathrm{sign}\,\tau_0 \sqrt{\left(\sqrt{(\tau_0^2 + |\xi|^4)} - |\xi|^2\right)}\right),$$

$$\frac{\partial \tilde{u}}{\partial x_3}(\tau_0, \xi, 1) = \hat{T}(\tau_0, \xi) - \tilde{u}(\tau_0, \xi, 1),$$

$$\frac{\partial \tilde{u}}{\partial x_3}(\tau_0, \xi, 0) = 0, \quad [\tau_0, \xi] \in R^{n+1}.$$

The solution of (13), which is the partial Fourier transform of the solution of the heat equation with the boundary value condition (10) and the homogeneous initial condition at $-\infty$, has the form

$$\text{(14)} \qquad \tilde{u}(\tau_0, \xi, x_{n+1}) = \frac{\hat{T}\,\mathrm{ch}\,(d x_{n+1})}{d\,\mathrm{sh}\,d + \mathrm{ch}\,d},$$

and we shall estimate $\|u\|_{1/2+\eta,1+\vartheta,1,Q}$ or $\|u\|_{1/2+\eta,1+\vartheta,1+\vartheta,1,Q}$ for $\eta, \vartheta \geqq 0$. From (3), (5) and taking the equivalent norm (putting $c_{n,k}(\alpha) = 1$, $n = 1, 2, \ldots, k = 0, 1, \ldots$ $\ldots, \alpha \in (0, k)$) we obtain (denoting the norm by $\|\cdot\|_{1/2+\eta,(1+\vartheta)_n,1,Q}$, $n = 1, 2$)

$$\text{(15)} \qquad \|u\|^2_{1/2+\eta,(1+\vartheta)_n,1,Q} =$$

$$= \int_{R^{n+1}} \int_0^1 \frac{|\hat{T}|^2}{|d\,\mathrm{sh}\,d + \mathrm{ch}\,d|^2} \left[\left(|\tau_0|^{1+2\eta} + |\xi|^{2+2\vartheta} + 1\right) |\mathrm{ch}\,(d x_{n+1})|^2 + \right.$$

$$\left. + |d|^2 |\mathrm{sh}\,(d x_{n+1})|^2\right] \mathrm{d}x_{n+1}\, \mathrm{d}\tau_0\, \mathrm{d}\xi, \quad n = 1, 2.$$

Now we calculate $|d|^2 = \sqrt{(\tau_0^2 + |\xi|^4)}$, $|\mathrm{ch}\,(d x_{n+1})|^2 = \frac{1}{2}\mathrm{ch}\,(2\,\mathrm{Re}\,d x_{n+1}) + \frac{1}{2}\cos(2\,\mathrm{Im}\,d x_{n+1})$, $|\mathrm{sh}\,(d x_{n+1})|^2 = \frac{1}{2}(\mathrm{ch}\,(2\mathrm{Re}\,d x_{n+1}) - \cos(2\,\mathrm{Im}\,d x_{n+1}))$. As $|d\,\mathrm{sh}\,d + \mathrm{ch}\,d|^2 = \frac{1}{4}[\mathrm{e}^{2\mathrm{Re}\,d}(|d|^2 + 2\,\mathrm{Re}\,d + 1) + \mathrm{e}^{-2\mathrm{Re}\,d}(|d|^2 - 2\,\mathrm{Re}\,d + 1)] + \frac{1}{2}[(1 - |d|^2)\cos(2\,\mathrm{Im}\,d) - 2\,\mathrm{Im}\,d \sin(2\,\mathrm{Im}\,d)]$ has a positive infimum as a function of d and

$$\text{(16)} \qquad \int_0^1 \left(|\tau_0|^{1+2\eta} + |\xi|^{2+2\vartheta} + 1\right) |\mathrm{ch}\,(d x_{n+1})|^2 + |d|^2 |\mathrm{sh}\,(d x_{n+1})|^2\, \mathrm{d}x_{n+1} =$$

$$= \frac{\left(\mathrm{e}^{2\mathrm{Re}\,d} - \mathrm{e}^{-2\mathrm{Re}\,d}\right)\left(|\tau_0|^{1+2\eta} + |\xi|^{2+2\vartheta} + \sqrt{(\tau_0^2 + |\xi|^4)} + 1\right)}{4\sqrt{2}\sqrt{\left(\sqrt{(\tau_0^2 + |\xi|^4)} + |\xi|^2\right)}} +$$

$$+ \frac{\left(|\tau_0|^{1+2\eta} + |\xi|^{2+2\vartheta} + 1 - \sqrt{(\tau_0^2 + |\xi|^4)}\right) \sin\left(\sqrt{2}\sqrt{\left(\sqrt{(\tau_0^2 + |\xi|^4)} - |\xi|^2\right)}\right)}{2\sqrt{2}\sqrt{\left(\sqrt{(\tau_0^2 + |\xi|^4)} - |\xi|^2\right)}},$$

we can easily derive, inserting these results into (15), that there exist finite positive constants k_1, k_2 independent of T such that the inequality

$$\text{(17)} \qquad k_1 \|T\|_{-1/4+\eta,(-1/2+\vartheta)_n,R^{n+1}} \leqq \|u\|_{1/2+\eta,(1+\vartheta)_n 1,Q} \leqq$$

$$\leqq k_2 \|T\|_{-1/4+\eta,(-1/2+\vartheta)_n,R^{n+1}}, \quad \eta \geqq \tfrac{1}{4}, \vartheta > \tfrac{1}{2},$$

holds, because the resulting coefficient of $|\hat{T}|^2$ is bounded in a neighbourhood of $0 \in R^{n+1}$. Thus for $T \in \bigcap_{\varepsilon>0} H^{1/2-\varepsilon,k}(R^2)$, $\bigcap_{\varepsilon>0} H^{1/2-\varepsilon,k,k}(R^2)$ we get the solution u of (13) in $\bigcap_{\varepsilon>0} H^{5/4-\varepsilon,k+3/2,1}(Q)$, $\bigcap_{\varepsilon>0} H^{5/4-\varepsilon,k+3/2,k+3/2,1}(Q)$, respectively. We remark that the restriction $n = 1$ or 2 is only formal.

To obtain the regularity in the normal direction we shall calculate for $\varepsilon \in (0, \frac{1}{2})$

$$I^2(u) = \frac{1}{c} \int_{R^{n+1}} \int_0^1 \int_0^1 \frac{\left(\frac{\partial^2 \tilde{u}}{\partial x_{n+1}^2}(y) - \frac{\partial^2 \tilde{u}}{\partial x_{n+1}^2}(z) \right)^2}{|y-z|^{2-2\varepsilon}} \, d\tau_0 \, d\xi \, dy \, dz = \tag{18}$$

$$= \int_{R^{n+1}} \int_0^1 \int_0^1 \frac{|\hat{T}|^2 \, |\mathscr{d}|^4}{|\mathscr{d} \operatorname{sh} \mathscr{d} + \operatorname{ch} \mathscr{d}|^2} \frac{|\operatorname{ch}(\mathscr{d} y) - \operatorname{ch}(\mathscr{d} z)|^2}{|y-z|^{2-2\varepsilon}} \, d\tau_0 \, d\xi \, dy \, dz =$$

$$= \int_{R^{n+1}} \frac{|\hat{T}|^2 \, |\mathscr{d}|^4 \, d\tau_0 \, d\xi}{|\mathscr{d} \operatorname{sh} \mathscr{d} + \operatorname{ch} \mathscr{d}|} J^2(u) \, .$$

Using the decomposition $\mathscr{d} = |\mathscr{d}| \, e$ we obtain

$$J^2(u) = \frac{1}{|\mathscr{d}|^{2\varepsilon}} \int_0^{|\mathscr{d}|} \int_0^{|\mathscr{d}|} \frac{|\operatorname{ch}(ey) - \operatorname{ch}(ez)|^2}{|y-z|^{2-2\varepsilon}} \, dy \, dz \, .$$

As $\langle 0, |\mathscr{d}| \rangle \times \langle 0, |\mathscr{d}| \rangle \subset M_1 \cup M_2 \cup M_3$, where $M_1 = \langle 1, |\mathscr{d}| \rangle \times \langle 0, y-1 \rangle$, $M_2 = \langle 0, |\mathscr{d}| - 1 \rangle \times \langle y+1, |\mathscr{d}| \rangle$, $M_3 = \langle 0, |\mathscr{d}| \rangle \times \langle y-1, y+1 \rangle$, and we can estimate

$$\int_{M_1 \cup M_2} \frac{|\operatorname{ch}(ey) - \operatorname{ch}(ez)|^2}{|y-z|^{2-2\varepsilon}} \, dy \, dz \leqq 4 \int_{M_1 \cup M_2} \frac{|\operatorname{ch}(ey)|^2}{|y-z|^{2-2\varepsilon}} \, dy \, dz \leqq \tag{19}$$

$$\leqq \frac{8}{1-2\varepsilon} \int_0^{|\mathscr{d}|} |\operatorname{ch}(ey)|^2 \, dy =$$

$$= \frac{2}{1-2\varepsilon} \left[\frac{1}{\operatorname{Re} e} \operatorname{sh}(2 \operatorname{Re} \mathscr{d}) + \frac{1}{\operatorname{Im} e} \sin(2 \operatorname{Im} \mathscr{d}) \right].$$

We need to estimate

$$\int_0^{|\mathscr{d}|} \int_{y-1}^{y+1} \frac{|\operatorname{ch}(ey) - \operatorname{ch}(ez)|^2}{|y-z|^{2-2\varepsilon}} \, dy \, dz = \int_0^{|\mathscr{d}|} \int_{y-1}^{y+1} \frac{|e \int_y^z \operatorname{sh}(es) \, ds|^2}{|y-z|^{2-2\varepsilon}} \, dy \, dz \leqq \tag{20}$$

$$\leqq \int_0^{|\mathscr{d}|} \int_{y-1}^{y+1} \int_{y-1}^{y+1} \frac{|\operatorname{sh}(es)|^2 \, ds}{|y-z|^{1-2\varepsilon}} \, dy \, dz \leqq \int_0^{|\mathscr{d}|} \frac{1}{\varepsilon} \int_{y-1}^{y+1} |\operatorname{sh}(es)|^2 \, ds \, dy \leqq$$

$$\leqq \frac{1}{4\varepsilon} \left| \frac{e^{2\operatorname{Re} e} - e^{-2\operatorname{Re} e}}{(2 \operatorname{Re} e)^2} (e^{2\operatorname{Re}\mathscr{d}} - e^{-2\operatorname{Re}\mathscr{d}}) \right| +$$

$$+ \frac{1}{4\varepsilon (\operatorname{Im} e)^2} \left| \cos(2 \operatorname{Im} e(|\mathscr{d}| + 1)) - \cos(2 \operatorname{Im} e(|\mathscr{d}| - 1)) \right| .$$

By virtue of (19) and (20), the seminorm $I(u)$ can be estimated by means of $\|T\|_{(1-\varepsilon)/2,(1-\varepsilon)_n,R^{n+1}}$, $n = 1, 2$, $\varepsilon \in (0, \frac{1}{2})$, and therefore for $k \geqq 1$ the assertion of Prop. 1 for the case (10) is verified.

In the case of the boundary value condition (11) we use the same technique. Applying the partial Fourier transformation, we obtain the problem

$$
(21) \qquad \frac{\partial^2 \tilde{u}}{\partial x_{n+1}^2} = d^2 \tilde{u} \quad \text{on} \quad Q\,, \quad d = \sqrt{(\mathrm{i}\tau_0 + |\xi|^2)} =
$$
$$
+ 2^{-1/2}(\sqrt{(\sqrt{(\tau_0^2 + |\xi|^4)} + |\xi|^2)} + \mathrm{i}\,\mathrm{sign}\,\tau_0 \sqrt{(\sqrt{(\tau_0^2 + |\xi|^4)} - |\xi|^2)})\,,
$$
$$
\tilde{u}(\tau_0, \xi, 0) = 0\,, \quad \tilde{u}(\tau_0 \xi, 1) = \hat{T}(\tau_0, \xi)\,, \quad [\tau_0, \xi] \in R^{n+1}\,, \quad n = 1, 2\,,
$$

with the solution

$$
(22) \qquad \tilde{u}(\tau_0, \xi, x_{n+1}) = \frac{\hat{T}\,\mathrm{sh}\,(d x_{n+1})}{\mathrm{sh}\,d}\,.
$$

We calculate

$$
(23) \qquad \|u\|_{1/2+\eta,(1+\vartheta)_n,1,\Omega} = \int_{R^{n+1}} |\hat{T}|^2\,.
$$
$$
\cdot \Bigg[\frac{(|\tau_0|^{1+2\eta} + |\xi|^{2+2\vartheta} + 1 + \sqrt{(|\tau_0|^2 + |\xi|^4)})\,(1 - \mathrm{e}^{-4\mathrm{Re}\,d})}{\sqrt{2}\sqrt{(\sqrt{(\tau_0^2 + |\xi|^4)} + |\xi|^2)}\,(1 + \mathrm{e}^{-4\mathrm{Re}\,d} - 2\,\mathrm{e}^{-2\mathrm{Re}\,d}\cos(\mathrm{Im}\,d))} +
$$
$$
+ \frac{(\sqrt{(|\tau_0|^2 + |\xi|^4)} - 1 - |\tau_0|^{1+2\eta} - |\xi|^{2+2\vartheta})\mathrm{e}^{-2\mathrm{Re}\,d}\sqrt{2}\sin(2\,\mathrm{Im}\,d)}{\sqrt{(\sqrt{(\tau_0^2 + |\xi|^4)} - |\xi|^2)}\,(1 + \mathrm{e}^{-4\mathrm{Re}\,d} - 2\mathrm{e}^{-2\mathrm{Re}\,d}\cos(2\,\mathrm{Im}\,d))} \Bigg]\,.
$$
$$
.\,\mathrm{d}\tau_0\,\mathrm{d}\xi, \quad \eta, \vartheta \geqq 0\,,
$$

and thus there exist positive constants k_1', k_2' independent of T such that for $\eta \geqq 0$, $\vartheta \geqq 0$

$$
(24) \qquad k_1' \|T\|_{1/4+\eta,(1+\vartheta)_n,R^{n+1}} \leqq \|u\|_{1/2+\eta,(1+\vartheta)_n,1,Q} \leqq k_2' \|T\|_{1/4+\eta,(1+\vartheta)_n,R^{n+1}}\,.
$$

Proposition 1 is proved.

Remark. For $k \geqq 1$ it is possible to prove the normal regularity of u of the $\frac{3}{2} - \varepsilon -$ type by means of a slight modification of (18)–(20). The result is that for $T \in H^{1/2-\varepsilon,(k)_n}(R^{n+1})$, $\varepsilon \in \langle 0, \frac{1}{4} \rangle$, $k > 1$, u belongs to $H^{3/4-\varepsilon,(k+1/2)_n,3/2-2\varepsilon}(R^1 \times \Omega)$.

Proposition 2. *Let* $u \in C_0(R^1; H^\alpha(\Omega))$, $\alpha = [\alpha_i] \in R_+^{n+1}$, $\alpha_1, \ldots, \alpha_n \geqq \alpha_{n+1} \geqq 0$. *Let* $\gamma \equiv 1$, *let the Poisson ratio* $\sigma \in (0, \frac{1}{2})$ *and the Young modulus of elasticity be constants independent of* u. *Then the stress tensor to the solution of* (2′) *satisfies* $\tau \in C_0(R^1; H^\alpha(\Omega; R^{(n+1)^2}))$. *Particularly, for* $\alpha_1, \ldots, \alpha_n \geqq \alpha_{n+1} \geqq 0$ *and* $\sum_{i=1}^{n+1} \alpha_i^{-1} < 2$ *we have* $\tau \in C_0(R^1 \times \Omega; R^{(n+1)^2})$.

Proof. Let us define $\mathscr{W} := \{w \in H^1(\Omega; R^{n+1});\ w = 0 \text{ on } R^n \times \{0\}\}$. The variational formulation of the problem (2′) has for $\gamma = 1$ the following form: Look for $v \in \mathscr{W}$ such that

$$(25)\qquad \int_\Omega (1 - 2\sigma) \sum_{i=1}^{n+1} (\operatorname{grad} v_i, \operatorname{grad} w_i)_{n+1} + \operatorname{div} v \operatorname{div} w - (2 + 2\sigma)\, u \operatorname{div} w\,.$$

$$.\, dx = 0\,, \quad i = 1, \ldots, n + 1\,, \quad \text{for every} \quad w \in \mathscr{W}\,, \quad t \in R^1\,.$$

Let us denote by $(25)_{-h_k}$ the equality

$$(25)_{-h_k}\qquad \int_\Omega (1 - 2\sigma) \sum_{i=1}^{n+1} (\operatorname{grad} (v_i)_{-h_k} \operatorname{grad} (w_i)_{-h_k})_{n+1} + \operatorname{div} v_{-h_k} \operatorname{div} w_{-h_k} -$$

$$- (2 + 2\sigma)\, u_{-h_k} \operatorname{div} w_{-h_k}\,, \quad i = 1, \ldots, n + 1\,, \quad k \in \{1, \ldots, n\}\,,$$

where for a function f we write $f_{-h_k} = f(x + h_k)$, and $h_k = \hbar e_k$, $\hbar \in R^1$ and $(e_k)_{k=1}^{n+1}$ is the canonical basis of R^{n+1}. Summing up $\sum_{j=0}^{[\alpha_k]+1} (26)_{-jh_k} \binom{[\alpha_k] + 1}{j} (-1)^j$ with $w = w_{-h_k} = \ldots = w_{-h_k(1+[\alpha_k])} = \Delta^{\hbar e_k}_{[\alpha_k]+1} v$, v solves (25), multiplying the resulting sum by $|\hbar|^{-1-2\alpha_k}$ and integrating it over R^1 in the variable $\hbar$, we obtain by means of the Korn inequality, carrying out the process for $k = 1, \ldots, n$,

$$(26)\qquad \left\| \frac{\partial v_i}{\partial x_j} \right\|_{\alpha_1, \ldots, \alpha_n, 0, \Omega} \leqq \text{const } \|u\|_{\alpha, \Omega}\,, \quad i, j = 1, \ldots, n + 1\,, \quad t \in R^1\,.$$

As $\Delta_1^t u$ could be inserted for u, where t is an arbitrary shift in time, we finally obtain

$$(27)\qquad \frac{\partial v_i}{\partial x_j} \in C_0(R^1; H^{\alpha_1, \ldots, \alpha_n, 0}(\Omega; R^{n+1}))\,, \quad i, j = 1, \ldots, n + 1\,.$$

We rewrite the equation (2) into the form

$$(28)\qquad (1 - 2\sigma) \frac{\partial^2 v_i}{\partial x_{n+1}^2} = (2 + 2\sigma) \frac{\partial u}{\partial x_i} - (1 - 2\sigma) \sum_{j=1}^{n} \frac{\partial^2 v_i}{\partial x_j^2} - \frac{\partial}{\partial x_i} \operatorname{div} v\,,$$

$$i = 1, \ldots, n\,,$$

$$(2 - 2\sigma) \frac{\partial^2 v_{n+1}}{\partial x_{n+1}^2} = (2 + 2\sigma) \frac{\partial u}{\partial x_{n+1}} - (1 - 2\sigma) \sum_{j=1}^{n} \frac{\partial^2 v_{n+1}}{\partial x_j^2} -$$

$$- \frac{\partial}{\partial x_{n+1}} \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}\,,$$

and we have to prove the normal regularity (in the direction e_{n+1}). First, let us suppose α_{n+1} to be an integer. If $\alpha_{n+1} \geqq 1$, (28) immediately yields that $\partial^2 v_i / \partial x_{n+1}^2 \in C_0(R^1; L_2(\Omega))$, $i = 1, \ldots, n + 1$. Suppose that $\partial^{|\beta|} v_i / \partial x^\beta \in C_0(R^1; L_2(\Omega))$, $i = 1, \ldots, n + 1$ for $1 + \alpha_{n+1} > k \in \mathbb{N}$, where $\tilde{\beta}$ is an arbitrary $(n + 1)$-dimensional multiindex of the rank $|\tilde{\beta}| \leqq \alpha_{n+1} + 1$ such that $\tilde{\beta}_{n+1} = k$. We shall prove that the same assertion is true for such $\tilde{\beta}$ with $\tilde{\beta}_{n+1} = k + 1$. Let us apply an arbitrary differential operator $D^{\tilde{\beta}^{\sim}}$ to (28) with $|\tilde{\beta}^{\sim}| \leqq \alpha_{n+1} - 1$, $\tilde{\beta}^{\sim}_{n+1} = k - 1$. By the induction hypothesis and the supposition put on u, the right-hand side of (28) belongs to $C_0(R^1; L_2(\Omega))$. Thus the same holds for the left hand side. Thus the proposition is proved for α_{n+1} an integer. For α_{n+1} non-integer, we obtain the result using an appropriate interpolation theorem – cf. e.g. [12], Thms. 5.1 and 9.6.

Proof of Theorem 3. As the space $H^{1/2+\varkappa}(R^1; H^a(R^{n+1}))$ for $\varkappa > 0$ and $a \in R^{n+1}_+$ has $|||\cdot|||$ as its norm, where

$$(29)\qquad |||u|||^2 \equiv \int_{R^{n+2}} |\hat{u}|^2 \left(1 + |\tau_0|^{1+2\varkappa}\right) \left(1 + \sum_{i=1}^{n+1} |\xi_i|^{2\alpha_i}\right) \mathrm{d}\xi \, \mathrm{d}\tau_0 ,$$

and because the Hölder inequality

$$(30)\qquad |\xi_i|^{2a_i} |\tau_0|^{1+2\varkappa} \leqq \frac{1 + 2\varkappa}{2\alpha_0} |\tau_0|^{2\alpha_0} + \frac{a_i}{\alpha_i} |\xi_i|^{2\alpha_i} , \quad i = 1, \dots, n + 1$$

holds for $\varkappa \geqq 0$, $\alpha_i > 0$, $i = 0, \dots, n + 1$, $a_i \geqq 0$, $i = 1, \dots, n + 1$, satisfying the equality $(1 + 2\varkappa)/2\alpha_0 + a_i/\alpha_i = 1$, we can prove the implication

$$(31)\qquad U \in \bigcap_{\varepsilon>0} H^{5/4-3/2,(k+3/2)_n,5/2-\varepsilon}(R^{n+2}) \equiv \mathscr{H} \Rightarrow$$

$$\Rightarrow U \in \bigcap_{\varepsilon>0} C_0(R^1; H^{(3/5k+0,9-\varepsilon)_n,3/2-\varepsilon}(R^{n+1})) \equiv \mathscr{H}_0 .$$

If $\Omega \subsetneqq R^{n+1}$, it is possible to extend $u \in \bigcap_{\varepsilon>0} H^{1/2+\varkappa}(R^1;\ H^{5/4-\varepsilon,(k+3/2)_n,5/2-2\varepsilon}(\Omega))$ to $U \in \mathscr{H}$ by means of the construction described e.g. in [12]. Then $U \in \mathscr{H}_0$ and $u = U/_{R^1\times\Omega} \in \bigcap_{\varepsilon>0} C_0(R^1; H^{(3/5k+0,9-\varepsilon)_n,3/2-\varepsilon}(\Omega))$. If $n = 2, k \geqq 1$ or $n = 1, k \geqq 1$, then the necessary conditions of Prop. 2 are satisfied so that $\tau \in C_0(R^1 \times \Omega; R^{(n+1)^2})$.

Remark. For $\Omega \subset R^2$, a sufficient condition for stresses to be continuous is $T \in H^{1/4+\varepsilon_0,1/2+2\varepsilon_0}(S)$, $\varepsilon_0 > 0$ arbitrary small, and in such a way we are able to improve Theorem 3. For the boundary value condition (10) and $\Omega \subset R^2$ or R^3, from the assumption $u \in \bigcap_{\varepsilon>0} H^{3/4-\varepsilon,(k+1/2)_n,3/2-2\varepsilon}(Q)$ we obtain only $u \in \bigcap_{\varepsilon>0} C_0(R^1; H^{(1/3k+1/6)_n,1/2-\varepsilon}(\Omega))$ which does not satisfy the sufficient condition of Prop. 2.

3. RESULTS FOR SMOOTH BODIES

In this section we shall study the regularity results for (1), (2) for a body $\Omega \subset R^2$ which is bounded and has an at least $C_{1,1}$-smooth boundary, i.e. the first derivatives of functions locally describing the boundary are Lipschitz. The heating regimes are supposed to be positive, non-decreasing and bounded. In the sequel, we shall suppose $u_0 = 0$, $T(0, \cdot) = 0$ and extend T onto $\mathfrak{A} := (-\infty, 4\mathscr{T}) \times \Omega$ by $T(t, x) = 0$,

$t < 0$, $x \in \Omega$, $T(t, x) = T(\mathscr{T}, x)$, $t \in \langle \mathscr{T}, 4\mathscr{T} \rangle$, $x \in \Omega$. We shall exploit the following variational formulation of the problem (1):

$$\int_{-\infty}^{t_0} \int_\Omega \beta(u) \frac{\partial u}{\partial t} w \,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^{t_0} \int_\Omega (\operatorname{grad} u, \operatorname{grad} w)_n \,\mathrm{d}x\,\mathrm{d}t + \tag{32}$$
$$+ \int_{-\infty}^{t_0} \int_{\partial\Omega} (g \circ \Lambda \circ u)\, w \,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{t_0} \int_{\partial\Omega} (g \circ T)\, w \,\mathrm{d}x\,\mathrm{d}t\,, \quad t_0 \leqq 4\mathscr{T}.$$

As a solution of (32) we shall consider every function $u \in \mathscr{L} \cap C_0(-\infty, 4\mathscr{T}; L_2(\Omega))$, where $\mathscr{L} \equiv L_2(-\infty, 4\mathscr{T}; H^1(\Omega))$, such that $\partial u/\partial t \in \mathscr{L}^*$ and (32) holds for every $w \in \mathscr{L}$. In [11] the existence of such a solution with the property $u \in \langle 0, B \rangle$ a.e. in $\mathfrak{A}$ is proved, where $B = \max_{[t,x] \in Q} \Lambda^{-1}(T(t, x))$. Such result, however, does not suffice to prove the continuity of the stress tensor. Put $\tilde{B} = \max\{1, B\}$.

Theorem 4. *Let Ω be a bounded domain in R^2 with a $C_{1,1}$-smooth boundary Γ. Let g, Λ be non-negative functions with non-negative continuous first derivatives on $\langle 0, +\infty)$ and such that $g(0) = \Lambda(0) = 0$. Moreover, let there be a constant $\lambda_0 > 0$ such that $\Lambda' > \lambda_0$ on $\langle 0, +\infty)$. Let β be an analytical function on $\{x \in \mathbb{C};\ |x| \leqq 2\tilde{B}\}$, where $\tilde{B} = \max\{1, B\}$, let β' be non-negative on $\langle 0, B \rangle$ and $\beta \geqq \beta_0 > 0$ on the same interval, where β_0 is constant. Let T be left continuous at $t = \mathscr{T}$ and nondecreasing in time for each $x \in \partial\Omega$, bounded on S by ΛB, having $T(0) = 0$ and let it belong to $L_2(0, \mathscr{T}; H^\alpha(\Omega))$, $\alpha \in \langle 0, \frac{1}{2} \rangle$. Then there is a unique solution of (32) belonging to $\bigcap_{\varepsilon > 0} C_0(0, \mathscr{T}; H^{1+\tilde{\alpha}-\varepsilon}(\Omega))$, where $\tilde{\alpha} \equiv \tilde{\alpha}(\alpha) = (1 + 2\alpha)^2/(14 + 12\alpha)$.*

Remark 1. The uniqueness of u is e.g. a consequence of essential boundedness of all solutions of (32), cf. [11], [16], and of considerations similar to (39)–(43) below. To prove it, we put u_1, u_2 instead of u, u_{-h} there, in the terms involving time derivatives we take mollifiers which are arbitrarily close to $\beta(u_i)\, \partial u_i/\partial t$ in the $\mathscr{L}^*$ norm.

2. For $\beta \equiv \beta_0$ the condition of monotonicity of T in time can be replaced by the condition of bounded variation in time uniformly with respect to $x \in \partial\Omega$ - cf. Proposition 3 below, where under such a linearity condition we can avoid the term whose estimation depends on the non-negativity of $\partial u/\partial t$.

We shall suppose that for $t > 4\mathscr{T}$ and $\mathscr{F} = T, u$ we have $\mathscr{F}(t, x) = \omega(t)\, \mathscr{F}(4\mathscr{T}, x)$, where $\omega(4\mathscr{T}) = 1$, $\omega = 0$ on $\langle 5\mathscr{T}, +\infty)$, $\omega'(t) \leqq 0$ on $\langle 4\mathscr{T}, +\infty)$.

Proposition 3. *Let $\Omega \subset R^2$ or R^3 be bounded, let $\partial\Omega$ be of the class $C_{1,1}$ and let $T: S \to R^1$ be a heating regime non-decreasing in time or with a bounded variation in time uniformly with respect to $x \in \partial\Omega$, and such that $T(0, \cdot) = 0$. Let β, g, Λ fulfil the conditions of Theorem 4. For the non-monotonous case let, moreover, $\beta(u) \equiv \beta_0 > 0$, $u \in R^1$. Then for the solution u of (1) we have $\partial u/\partial t \in L_2(Q)$. Extending T in the above described manner we obtain $\partial u/\partial t \in L_2((-\infty, 4\mathscr{T}) \times \Omega)$.*

Proof. Suppose T is non-decreasing. As $\mathscr{g}$ is monotonic, $G = \mathscr{g} \circ T$ is also non-decreasing in time on $\langle 0, \mathscr{T} \rangle$ for every $x \in \partial\Omega$. Using the extension of T and the mollifier technique – cf. [14] – we construct a sequence of regular functions $\{G_m\}_{m \geqq m_0}$ $(m_0 = 2[\mathscr{T}^{-1}] + 2)$ tending to G in e.g. $\bigcap_{\varepsilon > 0} H^{1/2-\varepsilon,1}(R^1 \times \partial\Omega)$ and, moreover, such that every G_m is monotonic on $I_1^m \times \partial\Omega$ and $I_2^m \times \partial\Omega$, where

$$I_1^m = \left(-\infty, \frac{m+1}{m} \mathscr{T}\right\rangle, \quad I_2^m = \left\langle \frac{4m-1}{m} \mathscr{T}, +\infty \right), \quad G_m = 0$$

$$\text{on} \quad \left(-\infty, -\frac{1}{m}\right) \times \partial\Omega \quad \text{and}$$

$$G_m(t, x) = G(\mathscr{T}, x) \quad \text{on} \quad \left\langle \frac{m+1}{m} \mathscr{T}, \; \frac{4m-1}{m} \mathscr{T} \right\rangle.$$

Of course, $G_m(t, x) \leqq \mathscr{g} \circ A(B) \equiv \mathscr{B}$ for every $[t, x] \in R^1 \times \partial\Omega$. Taking the variational formulation for G_m, denoting the corresponding solution by u_m and putting $w = \partial u_m / \partial t$ we obtain

$$\text{(33)} \qquad \int_{-\infty}^{t_0} \int_\Omega \beta(u_m) \left(\frac{\partial u_m}{\partial t}\right)^2 \mathrm{d}x \, \mathrm{d}t + \tfrac{1}{2} \int_\Omega |\text{grad } u_m(t_0, x)|_n^2 \, \mathrm{d}x +$$

$$+ \int_{\partial\Omega}^{t_0} (\mathscr{g} \circ A \circ u_m) \frac{\partial u_m}{\partial t} \mathrm{d}x \, \mathrm{d}t =$$

$$= \int_{\partial\Omega} G_m \, u_m(t_0, x) \, \mathrm{d}x - \int_{-\infty}^{t_0} \int_{\partial\Omega} \frac{\partial G_m}{\partial t} u_m \, \mathrm{d}x \, \mathrm{d}t, \quad t_0 \in R^1, \quad m \geqq m_0.$$

As $\mathscr{g} \circ A(z) \geqq \mathscr{g}_0 z - \mathscr{g}_1$ for $z \in \langle 0, B \rangle$ and suitable constants $\mathscr{g}_0, \mathscr{g}_1 > 0$, we have $\mathscr{G}(z) \equiv \int_0^z \mathscr{g} \circ A(s) \, \mathrm{d}s \geqq \frac{1}{2} \mathscr{g}_0 z^2 - \mathscr{g}_1 z$, thus

$$\text{(34)} \qquad \int_{-\infty}^{t_0} \int_{\partial\Omega} \mathscr{g} \circ A(u_m) \frac{\partial u_m}{\partial t} \mathrm{d}x \, \mathrm{d}t = \int_{\partial\Omega} \mathscr{G}(u_m)(t_0, x) \, \mathrm{d}x \geqq$$

$$\geqq \tfrac{1}{2} \mathscr{g}_0 \int_{\partial\Omega} u_m^2(t_0, x) \, \mathrm{d}x - \mathscr{g}_1 \int_{\partial\Omega} u_m(t_0, x) \, \mathrm{d}x$$

$$\text{for a.e.} \quad t_0 \in R^1 \quad \text{for all} \quad m \geqq m_0.$$

From the comparison and the trace theorems we conclude that $u_m \in \langle 0, B \rangle$ almost everywhere in $R^1 \times \partial\Omega$, $m \geqq m_0$, and clearly $|\int_{\partial\Omega} (G_m \, u_m(t_0, x) \, \mathrm{d}x| \leqq \varepsilon_1 \int_{\partial\Omega} u_m^2(t_0, x)$. . $\mathrm{d}x + (1/\varepsilon_1) \int_{\partial\Omega} G_m^2(t_0, x) \, \mathrm{d}x$, $|\int_{\partial\Omega} u_m(t_0, x) \, \mathrm{d}x| \leqq (1/\varepsilon_1) \text{ mes } \partial\Omega + \varepsilon_1 \int_{\partial\Omega} u_m^2(t_0, x) \, \mathrm{d}x$, where $\varepsilon_1 \in (0, 1)$ will be taken sufficiently small. Thus we obtain the inequality

$$\text{(35)} \qquad \beta_0 \int_{-\infty}^{+\infty} \int_\Omega \left(\frac{\partial u_m}{\partial t}\right)^2 \mathrm{d}t \, \mathrm{d}x \leqq \mathscr{K}_0(B^2 + \mathscr{K}_1) \text{ mes } \partial\Omega +$$

$$+ B \int_{-\infty}^{+\infty} \int \left|\frac{\partial G_m}{\partial t}\right| \mathrm{d}t \, \mathrm{d}x, \quad \mathscr{K}_0, \mathscr{K}_1 > 0 \quad \text{independent of } m.$$

As the last integral in (35) can be estimated by $2B \operatorname{mes} \partial\Omega$, $\{\|\partial u_m/\partial t\|_{L_2(\mathfrak{A})}\}$ is a bounded sequence and the same is clearly true for $\{\|\operatorname{grad} u_m\|_{L_2(\mathfrak{A})}\}$. Thus u, the weak limit of $\{u_m\}$, belongs to $H^1(\mathfrak{A})$ and due to the compact imbedding theorem and the uniform essential boundedness of u_m, we can carry out the limit procedure in (32) and prove that u is the unique solution of (32) for G.

If G is not monotonic, we find a non-decreasing $\bar{G}$ such that $G(t, x) = \operatorname{var}_{(-\infty,t\rangle} \cdot G(\cdot, x) - \bar{G}(t, x)$, $t \in (-\infty, 3\mathcal{T}\rangle$. Thus $G_m = H_m - \bar{G}_m$ on $(-\infty, 3\mathcal{T}\rangle$, where H_m is the mollifier of $\operatorname{var}_{(-\infty,t\rangle} G$ and $\int_{\partial\Omega} \int_{-\infty}^{2\mathcal{T}} |\partial G_m/\partial t| \, dx \, dt \leqq \operatorname{mes} \partial\Omega \, 2 \sup_{x\in\partial\Omega} \operatorname{var}_{(-\infty,\mathcal{T})} \cdot G(t, x)$, $m \geqq m_0$. Of course, $\int_{\partial\Omega} \int_{2\mathcal{T}}^{+\infty} |\partial G_m/\partial t| \, dt \, dx \leqq B \operatorname{mes} \partial\Omega$ and the final conclusion is the same as in the preceding case. $\square$

As we have proved that $f_0 \equiv \beta(u) \, \partial u/\partial t \in L_2(Q)$ and $g_0 \equiv \mathscr{g}(T) - \mathscr{g}(\Lambda u) \in L_2(S)$ (by virtue of the trace theorem u and $\mathscr{g}(\Lambda u)$ belong to $L_2(0, \mathcal{T}; H^{1/2}(\partial\Omega))$), $u(t, \cdot)$ solves the problem

$$(36) \qquad \Delta u = f_0 \quad \text{on} \quad \Omega\,, \quad \frac{\partial u}{\partial \nu} = g_0 \quad \text{on} \quad \partial\Omega$$

for almost every $t \in \langle 0, 4\mathcal{T}\rangle$. Due to the interpolation theorem (cf. e.g. [12]) $u(t, \cdot) \in H^{3/2}(\Omega)$ for almost every $t \in \langle 0, \mathcal{T}\rangle$ and the inequality

$$(37) \qquad \int_0^{\mathcal{T}} \|u(t, \cdot)\|^2_{H^{3/2}(\Omega)} \, dt \leqq \text{const} \int_0^{\mathcal{T}} \|f(t, \cdot)\|^2_{L_2(\Omega)} + \|g(t, \cdot)\|^2_{L_2(\partial\Omega)} \, dt$$

holds. Consequently $u \in L_2(0, \mathcal{T}; H^{3/2}(\Omega))$ and its trace on S belongs to $L_2(0, 4\mathcal{T}; H^{1-\varepsilon}(\partial\Omega))$ for every $\varepsilon > 0$ because of the usual trace theorem. If T is in $L_2(0, \mathcal{T}; H^{\alpha}(\partial\Omega))$, $\alpha \in \langle 0, \frac{1}{2}\rangle$, and T satisfies all the other suppositions of Theorem 4, then we can use the same arguments to prove $u \in L_2(0, 4\mathcal{T}; H^{3/2+\alpha}(\Omega))$, $u/s \in L_2(0, \mathcal{T}; H^{1+\alpha}(\partial\Omega))$. Thus we have proved the following proposition:

Proposition 4. *Under the suppositions of Theorem* 4 *the solution of* (1) *belongs to* $H^{1,3/2+\alpha,3/2+\alpha}(Q)$.

To prove Theorem 4, we apply the shift technique to the time variable. Let ϱ_0 be a sufficiently smooth nonincreasing function on R^1 such that $\varrho_0/_{(-\infty,2\mathcal{T})} = 1$, $\varrho_0/_{(3\mathcal{T},+\infty)} = 0$. Taking the variational formulation (32), its shifted version $(32)_{-\ell}$ with the time shift ℓ and putting $w = w_{-\ell} = \varrho_0^2(u_{-\ell} - u)$ into $(32)_{-\ell}$–(32), we obtain

$$(38) \qquad \int_{-\infty}^{t_0} \int_{\Omega} \left(\beta(u_{-\ell}) \frac{\partial u_{-\ell}}{\partial t} - \beta(u) \frac{\partial u}{\partial t}\right) \varrho_0^2(u_{-\ell} - u) +$$
$$+ (\operatorname{grad}(u_{-\ell} - u), \operatorname{grad} \varrho_0^2(u_{-\ell} - u))_2 \, dx \, dt +$$
$$+ \int_{-\infty}^{t_0} \int_{\partial\Omega} (\mathscr{g} \circ \Lambda \circ u_{-\ell} - \mathscr{g} \circ \Lambda \circ u) \, \varrho_0^2(u_{-\ell} - u) \, dx \, dt =$$
$$= \int_{-\infty}^{t_0} \int_{\partial\Omega} \varrho_0^2(\mathscr{g} \circ T_{-\ell} - \mathscr{g} \circ T)(u_{-\ell} - u) \, dx \, dt\,, \quad t_0 \in R^1\,, \quad |\ell| \leqq \mathcal{T}\,,$$

where we use the above defined extensions of u, T to $R^1 \times \Omega$. For $\ell \in R^1$ we obtain

(a)
$$\int_{-\infty}^{t_0}\int_\Omega (\operatorname{grad} \Delta_1^\ell u, \operatorname{grad}(\varrho_0^2 \Delta_1^\ell u))_2 \,dx\,dt =$$
$$= \int_{-\infty}^{t_0}\int_\Omega (\operatorname{grad}(\Delta_1^\ell(\varrho_0 u)), \operatorname{grad}(\Delta_1^\ell(\varrho_0 u)))_2 \,dx\,dt +$$
$$+ \int_{-\infty}^{t_0}\int_\Omega (\operatorname{grad}(u_{-\ell}\Delta_1^\ell \varrho_0), \operatorname{grad}(u_{-\ell}\Delta_1^\ell \varrho_0))_2 -$$
$$- 2(\operatorname{grad}(u_{-\ell}\Delta_1^\ell\varrho_0), \operatorname{grad}(\Delta_1^\ell(\varrho_0 u)))_2 - (\Delta_1^\ell u)^2 \,|\operatorname{grad}\varrho_0|^2 \,dx\,dt\,.$$

Moreover,

(b)
$$\int_{-\infty}^{t_0}\int_\Omega \varrho_0^2[(g \circ A \circ u_{-\ell}) - (g \circ A \circ u)](u_{-\ell} - u)\,dx\,dt > 0\,, \quad \ell \in R^1\,,$$

(c)
$$\int_{-\infty}^{t_0}\int_\Omega \varrho_0^2 \Delta_1^\ell g(T)\, \Delta_1^\ell u \,dx\,dt =$$
$$= \int_{-\infty}^{t_0}\int_{\partial\Omega} \Delta_1^\ell(\varrho_0 g(T))\, \Delta_1^\ell(\varrho_0 u)\,dx\,dt + \int_{-\infty}^{t_0}\int_{\partial\Omega} (g(T)\,u)_{-\ell}\,(\Delta_1^\ell\varrho_0)^2 -$$
$$- u_{-\ell}\Delta_1^\ell(\varrho_0 g(T))\,\Delta_1^\ell\varrho_0 - (g(T))_{-\ell}\,\Delta_1^\ell(\varrho_0 u)\,\Delta_1^\ell\varrho_0 \,dx\,dt\,, \quad \ell \in R^1\,.$$

The term containing the time derivatives will be estimated as follows: (For β constant, it can be written as $\partial/\partial t[(\Delta_1^\ell(\varrho_0 u))^2 - 2u_{-\ell}\Delta_1^\ell\varrho_0\Delta_1^\ell(\varrho_0 u) + u_{-\ell}^2(\Delta_1^\ell\varrho_0)^2] - 2(\Delta_1^\ell u)^2 \cdot \varrho_0(\partial\varrho_0/\partial t)$.) For the general case we use the equality

(39)
$$\varrho_0^2\left(\beta(u_{-\ell})\frac{\partial u_{-\ell}}{\partial t} - \beta(u)\frac{\partial u}{\partial t}\right)(u_{-\ell} - u) =$$
$$= \varrho_0^2 \frac{\partial}{\partial t}\left(\sum_{n=0}^{m} \frac{\beta^{(n)}(u)}{(n+2)!}(u_{-\ell} - u)^{n+2}\right) +$$
$$+ \varrho_0^2 \frac{\partial u_{-\ell}}{\partial t}\left(\beta(u_{-\ell}) - \sum_{n=0}^{m+1}\frac{\beta^{(n)}(u)}{(n+1)!}(u_{-\ell} - u)^n\right)(u_{-\ell} - u) +$$
$$+ \varrho_0^2 \frac{\beta^{(m+1)}(u)}{(m+2)!}\left(\frac{\partial u_{-\ell}}{\partial t} - \frac{\partial u}{\partial t}\right)(u_{-\ell} - u)^{m+2}\,.$$

We prove it by means of mathematical induction. For $m = 0$ (39) can be easily verified. Let (39) hold for $m_0 \geqq 0$. The last term of (39) can be rewritten as

(40)
$$\varrho_0^2 \frac{\beta^{(m_0+1)}(u)}{(m_0+2)!}\left(\frac{\partial u_{-\ell}}{\partial t} - \frac{\partial u}{\partial t}\right)(u_{-\ell} - u)^{m_0+2} =$$
$$= \varrho_0^2 \frac{\partial}{\partial t}\left(\frac{\beta^{(m_0+1)}(u)}{(m_0+2)!}\frac{(u_{-\ell} - u)^{m_0+3}}{m_0+3}\right) -$$
$$- \varrho_0^2 \frac{\partial u_{-\ell}}{\partial t}\frac{\beta^{(m_0+2)}(u)}{(m_0+3)!}(u_{-\ell} - u)^{m_0+3} +$$
$$+ \varrho_0^2 \frac{\beta^{(m_0+2)}(u)}{(m_0+3)!}\left(\frac{\partial u_{-\ell}}{\partial t} - \frac{\partial u}{\partial t}\right)(u_{-\ell} - u)^{m_0+3}\,.$$

Adding the appropriate terms on the right-hand side of (40) to the corresponding terms on the right-hand side of (39), we obtain (39) for $m_0 + 1$.

Under the suppositions of Theorem 4 we have $\beta(u + h) = \sum_{k=0}^{+\infty} (\beta^{(k)}(u)/k!)\, h^k$, u, $u + h \in \langle 0, B\rangle$. Let us define

$$(41) \qquad p_u(h) = h \sum_{n=0}^{+\infty} \left(\frac{\beta^{(n)}u}{n!} h^n - \frac{\beta^{(n)}(u)}{(n+1)!} h^n \right) = \sum_{n=1}^{+\infty} \frac{n\beta^{(n)}(u)}{(n+1)!} h^{n+1} ;$$

$$r_u(h) = \sum_{n=0}^{+\infty} \frac{\beta^{(n)}(u)}{(n+2)!} h^{n+2} .$$

Since $p_u(0) = r_u(0) = r'_u(0) = 0$, we can see (differentiating the series p_u) that $\beta'(u + h) = p'_u(h)/h$ and (after the second order differentiation of r_u) $r''_u(h) = \beta(u + h)$. Thus, after the appropriate integration, we obtain

$$(42) \qquad p_u(h) = \int_0^h z\beta'(u + z)\, dz, \quad r_u(h) = \int_0^h (h - z)\, \beta(u + z)\, dz .$$

Because of the non-negativity of β' and the strong positivity of β we obtain on $\langle 0, B\rangle$

$$(43) \qquad p_u(h) \geqq 0 , \quad r_u(h) \geqq \beta_0 \tfrac{1}{2} h^2 \quad \forall u, u + h \in \langle 0, B\rangle .$$

Using (39) and (43) for $h = u_{-\ell} - u$, we obtain the following assertion:

Proposition 5. *Under the supposition of Theorem* 4, grad $u \in \bigcap_{\varepsilon > 0} H^{(7+6\alpha)/(12+8\alpha)-\varepsilon,0,0}(Q; R^2)$.

Proof. First, we suppose that the heating regime is sufficiently regular (e.g. it is a mollifier of the original regime – cf. e.g. [14]). We put (a), (c) and (39) into (38), multiply it by $|\ell|^{-1-2\alpha_0}$, $\alpha_0 \in (0, 1)$, and integrate the resulting equality in ℓ over $(-\mathcal{T}, \mathcal{T})$. Using the Lipschitz continuity of g, Λ, ϱ_0, the monotonicity of g, Λ (cf. (b)) and (for the case of nonconstant β) also (43) and the nonnegativity of $\partial u/\partial t$ (it holds due to the monotonicity of T in time – cf. [16]), and finally the obvious inequality $\int_{R^1} \int_{|\ell| > \mathcal{T}} ((\Delta_1^\ell f)^2/|\ell|^{1+2\alpha})\, d\ell\, dt \leqq (4/\mathcal{T}^{1+2\alpha}) \|f\|^2_{L_2(R^1)}$, $f \in L_2(R^1)$, we obtain the inequality

$$(44) \qquad \int_{R^1} |\ell|^{-1-2\alpha_0} \int_\Omega \varrho_0^2 \sum_{k=0}^{m} \frac{\beta^{(k)}(u)}{(k+2)!} (u_{-\ell} - u)^{k+2} (t_0, x)\, dx\, d\ell +$$

$$+ (1 - \eta) \int_{R^1} |\ell|^{-1-2\alpha_0} \int_{R^1 \times \Omega} (\text{grad}\, (\Delta_1^\ell(\varrho_0 u)))^2\, dt\, dx\, d\ell +$$

$$+ \int_{R^1} |\ell|^{-1-2\alpha_0} \int_{R^1 \times \Omega} \varrho_0^2 \frac{\partial u_{-\ell}}{\partial t} (\beta(u_{-\ell}) -$$

$$- \sum_{k=0}^{m+1} \frac{\beta^{(k)}(u)}{(k+1)!} (u_{-\ell} - u)^k) (u_{-\ell} - u)\, dt\, dx\, d\ell \leqq$$

$$\leqq (1 + \eta) \int_{R^1} |\ell|^{-1-2\alpha_0} \int_{R^1 \times \partial\Omega} (\Delta_1^\ell(\varrho_0 \mathscr{g}(T)) \, \Delta_1^\ell(\varrho_0 u) \, dt \, dx \, d\ell +$$

$$+ \int_{R^1} |\ell|^{-1-2\alpha_0} \int_{R^1 \times \Omega} \varrho_0^2 \frac{\beta^{(m+1)}(u)}{(m+2)!} \left(\frac{\partial u_{-\ell}}{\partial t} - \frac{\partial u}{\partial t} \right) (u_{-\ell} - u)^{m+2} \, dt \, dx \, d\ell +$$

$$+ K_0(\|u\|_{H^1(Q)}, \|\varrho_0\|_{C^1(R^1)}, B, \|\mathscr{g}\|_{C^1(0,AB)}, \|T\|_{L_2(\mathfrak{A})}, \|A\|_{C^1(0,B)}),$$

$\eta > 0$ arbitrarily small, $m \in \mathbb{N}$ arbitrary, $\alpha_0 \in (0, 1)$.

As

$$\text{(45)} \qquad \int_{R^1} |\ell|^{-1-2\alpha_0} \int_{R^1 \times \partial\Omega} \Delta_1^\ell(\varrho_0 \mathscr{g}(T)) \, \Delta_1^\ell(\varrho_0 u) \, dt \, dx \, d\ell \leqq$$

$$\leqq K(\|\mathscr{g}\|_{C^1(0,AB)}, \|\varrho_0\|_{C^1(R^1)}, \|T\|_{1/2-\varepsilon,0,S}) +$$

$$+ \int_{R^1} |\ell|^{-4\alpha_0 - 2\varepsilon} \int_{R^1 \times \partial\Omega} (\Delta_1^\ell(\varrho_0 u))^2 \, dt \, dx \, d\ell, \quad \varepsilon > 0 \quad \text{arbitrarily small},$$

and by Theorem 1 the last term in (45) can be estimated by $\|\varrho_0 u\|_{H^{2\alpha_0 - 1/2 + \varepsilon}} \cdot (0, 4\mathscr{T}; H^{1/2+\vartheta}(\Omega))$, $\vartheta > 0$ arbitrarily small, we can apply (considering the extension procedure from Ω to R^2 – cf. [12]) Prop. 3 and Prop. 4 and the Hölder inequality

$$\text{(46)} \qquad |\tau|^{(4+4\alpha)/(3+2\alpha) - \tilde{\varepsilon}(\tilde{\vartheta})} |\xi|^{1+\tilde{\vartheta}} \leqq \text{const} (\tau^2 + |\xi|^{3+2\alpha}), \quad \tilde{\varepsilon}(\tilde{\vartheta}) \searrow 0 \quad \text{for} \quad \tilde{\vartheta} \searrow 0,$$

and we obtain for $4\alpha_0 - 1 < (4+4\alpha)/(3+2\alpha)$, i.e. for $\alpha_0 < (7+6\alpha)/(12+8\alpha)$ that $\|\varrho_0 u\|_{H^{2\alpha_0 - 1/2 + \varepsilon}(0, 4\mathscr{T}, H^{1/2+\vartheta}(\Omega))} \leqq \text{const} (\|u\|_{1,3/2+\alpha,3/2+\alpha} + K_1(\varrho_0))$ for suitably small $\varepsilon, \vartheta > 0$. Moreover,

$$\text{(47)} \qquad \int_{R^1} |\ell|^{-1-2\alpha_0} \int_{R^1 \times \Omega} \varrho_0^2 \frac{\beta^{(m+1)}(u)}{(m+2)!} \left(\frac{\partial u_{-\ell}}{\partial t} - \frac{\partial u}{\partial t} \right) (\Delta_1^\ell u)^{m+2} \, dt \, dx \, d\ell \leqq$$

$$\leqq \left\| \frac{\partial u}{\partial t} \right\|_{L_2(0,\mathscr{T})}^2 \text{const} \, \tilde{B}^{m+1} \sup_{y \in \langle 0, \tilde{B} \rangle} \frac{|\beta^{m+1}(y)|}{(m+2)!} .$$

$$(\int_{R^1} |\ell|^{-1-4\alpha_0-\varepsilon} \int_{R^1 \times \Omega} (\Delta_1^\ell(\varrho_0 u)^4 \, dt \, dx \, dt)^{1/2} + K_2(\|\varrho_0\|_{C^1(R^1)}, B).$$

From the supposition concerning the convergence radius of the analytic function β, it follows that $(\tilde{B}^{m+1}/(m+2)!) \sup_{y \in \langle 0. \tilde{B} \rangle} |\beta^{(m+1)}(y)|$ tends to 0 for $m \to +\infty$. Due to the regularity of T, the integral term at the right-hand side of (47) is also bounded, cf. e.g. [2], [11]. For such regular T, we can put (45) and (47) into (44) and passing with m to $+\infty$, we get

$$\text{(48)} \qquad \sup_{t_0 \in R^1} \tfrac{1}{2} \beta_0 \int_{R^1} |\ell|^{-1-2\alpha_0} \int_\Omega (\Delta_1^\ell(\varrho_0 u))^2 (t_0, x) \, dx \, d\ell +$$

$$+ (1 - 2\eta) \int_{R^1} |\ell|^{-1-2\alpha_0} \int_{R^1 \times \Omega} (\text{grad} \, \Delta_1^\ell(\varrho_0 u))^2 \, dt \, dx \, d\ell \leqq$$

$$\leqq K(\|u\|_{1,3/2+\alpha,3/2+\alpha,0}, \|\mathscr{g}\|_{C_1(0,AB)}, \|\varrho_0\|_{C_1(R^1)} \|T\|_{1/2-\varepsilon,0,Q}),$$

$\alpha_0 < (7+6\alpha)/(12+8\alpha)$, $\eta > 0$ arbitrarily small for sufficiently small $\varepsilon > 0$. Passing to the limit irregular temperature regime T, we prove (48) also for such T.

Prop. 5 is proved. Now we have for τ, $\xi > 0$:

$$(49) \qquad \tau^{1+\vartheta}\xi^{2+2\tilde{\alpha}-\varepsilon(\eta,\vartheta)} \leqq \text{const}\left(\tau^{(7+6\alpha)/(4+4\alpha)-\eta}\xi^2 + \xi^{3+2\alpha}\right), \quad \varepsilon(\eta, \vartheta) \searrow 0$$

$$\text{for} \quad \eta, \vartheta \searrow 0\,, \quad \tilde{\alpha} \equiv \tilde{\alpha}(\alpha) = \frac{(1 + 2\alpha)^2}{14 + 12\alpha}\,.$$

and Theorem 1 yields that $u \in \bigcap_{\varepsilon>0} C_0(0, \mathscr{T}; H^{1+\tilde{\alpha}-\varepsilon}(\Omega))$. Theorem 4 is proved. We remark that $\tilde{\alpha}(0) = \frac{1}{14}$ and $\tilde{\alpha}(\frac{1}{2}) = \frac{1}{5}$.

Theorem 5. *Under the suppositions of Theorem* 4, *let $\partial\Omega$ be of the class, $C_{11/5}$, $\gamma \in C_2(0, B)$ and let E be a constant. Then the stress tensor satisfies $\tau \in \bigcap_{\varepsilon>0} C_0 \,.$ $.\, (0, \mathscr{T}; H^{1+\tilde{\alpha}-\varepsilon}(\Omega; R^4))$. Particularly, $\tau \in C_0(\bar{Q}; R^4)$. The last assertion is also true for smoothly temperature-dependent E.*

Proof. As $\operatorname{grad}(\gamma(u_{-h}) - \gamma(u)) = \gamma'(u_{-h}) \operatorname{grad}(u_{-h} - u) + (\gamma'(u_{-h}) - \gamma'(u))\,.$ $.\,\operatorname{grad} u$ for $x, x + h \in \Omega$, $t \in \langle 0, \mathscr{T}\rangle$, we have

$$(50) \qquad \|\gamma(u)\|^2_{C_0(0,\mathscr{T};H^{1+\alpha_1}(\Omega))} \leqq \text{const}\left(\|\gamma'\|^2_{C_0(0,B)} \|u\|^2_{C_0(0,\mathscr{T};H^{1+\alpha_1}(\Omega))} + \right.$$

$$+ \|\gamma''\|_{C_0(0,B)} \sup_{t\in\langle 0,\mathscr{T}\rangle} \int_\Omega \int_\Omega (\operatorname{grad} u)^2 (t, x) |x - y|^{-2-2\alpha_1} (u(t, x) -$$

$$\left. - u(t, y))^2 \,\mathrm{d}x\, \mathrm{d}y\right), \quad \alpha_1 = (0, \tilde{\alpha})\,.$$

The first term on the right-hand side of (50) is bounded due to Theorem 4. To the second term we apply the Hölder inequality and the following theorem (cf. [2]) which is a generalization of Theorem 2:

Theorem 2′. *Let Ω be a domain in R^N with a sufficiently smooth boundary $\partial\Omega$. Let $a = [a_1, \ldots, a_N] \in R^N_+$ and $\alpha = [\alpha_1, \ldots, \alpha_N] \in R^N_+$,* let $f \in H^\alpha(\Omega)$, $p \in (1, +\infty)$. *Then* $\mathrm{D}^a f \in L_p(\Omega)$ if $\sum_{i=1}^{N} (1/\alpha_i)\, (\frac{1}{2} - (1/p) + a_i) < 1$.

As $(2/(1 + \tilde{\alpha}))\,(\frac{1}{2} - 1/2p) + \alpha_1/(1 + \tilde{\alpha}) < 1$ for $\alpha_1 \in (0, \tilde{\alpha})$, $p > 1/\tilde{\alpha}$, the Hölder inequality, Theorem 2′ and Theorem 4 also yield the boundedness of the second term of (50). Thus $\gamma(u) \in \bigcap_{\varepsilon>0} C_0(0, \mathscr{T}; H^{1+\tilde{\alpha}-\varepsilon}(\Omega))$.

Now, we are practically in the same situation as in Prop. 2. To prove the assertion concerning τ_{ij}, $i, j = 1, 2$ it is necessary to follow its proof in combination with the local straightening of the boundary for the appropriate variational inequality — cf. our Sec. 4 or [5]. Such a procedure inflicts only small technical complexities. As the idea is quite clear, we omit the proof. □

4. RESULTS FOR BODIES WITH ISOLATED BOUNDARY NONSMOOTHNESSES

In this section we deal with the case when the bounded domain $\Omega \subset R^2$ has $C_{2+\varepsilon}$-smooth boundary $\partial\Omega$ ($\varepsilon > 0$ small) with the exception of a finite set $M_0 \subset \partial\Omega$, but $\partial\Omega$ is globally of the class $C_{0,1}$. All the remaining assumptions of Thms. 4 and 5 will be satisfied. We remark that all proofs concerning the Lamé system were done independently by the author, the close connections of some of them with the results of [4], [17] were found only when the copies of these papers were obtained. Due to the connections, some parts of the proofs concerning the Lamé system are given without details.

Under the above mentioned assumptions, Prop. 3 remains valid. For further considerations we need to use the localization technique. We denote $\mathscr{B}_n(y, r) := := \{x \in R^n; |x - y| < r\}$ for $y \in R^n$ and due to the assumptions concerning $\partial\Omega$ there exists a constant φ_0 such that for every $x_0 \in \partial\Omega$, $x_0 \equiv [x_{0,1}, x_{0,2}]$ there is a function φ: $\mathscr{B}_1(0, \varphi_0) \to R^2$ such that $\varphi(0) = x_0$, $\varphi(z) = [z + x_{0,1}, \varphi_1(z)] \in \partial\Omega$ for $z \in \mathscr{B}_1(0, \varphi_0)$ and $\Omega \cap \mathscr{B}_2(x_0, \varphi_0) = \{x \in \mathscr{B}_2(x_0, \varphi_0); x_2 > \varphi_1(x_1 - x_{0,1})\}$. For each $x_0 \in \partial\Omega$ we can suppose Ω shifted in such a way that $x_0 = 0$. Let there be a sufficiently smooth (e.g. $C_{11/5}$) partition of unity on Ω (denoted by $\mathscr{R}$) such that diam supp $\varrho < \varphi_0$ for every $\varrho \in \mathscr{R}$, and for every $x_0 \in M_0$ there is $\varrho \equiv \varrho_{x_0} \in \mathscr{R}$ and a constant $\delta_0 > 0$ such that $\varrho \equiv 1$ on $\mathscr{B}_2(x_0, \delta_0)$. Denote $\mathscr{R}_0 := \{\varrho \in \mathscr{R}; \exists x_0 \in M_0 . . (\varrho \equiv \varrho_{x_0})\}$. For the sake of simplicity we make an additional assumption

$$(51) \qquad \exists \tilde{\varphi}_0 \in (0, \varphi_0\rangle \quad \forall x_0 \in M_0 \quad \exists \upsilon \in R^1 \quad \text{such that} \quad \varphi(x) = [x, \upsilon|x|]\,, \quad x \in \mathscr{B}_1(0, \tilde{\varphi}_0)\,,$$

for x_0 shifted into 0 and after a suitable rotation. Such a supposition is not necessary (cf. e.g. [8]), but useful for exploiting the technique of [10]. Moreover, we suppose supp $\varrho_{x_0} \subset \mathscr{B}_2(x_0, \tilde{\varphi}_0)$, $x_0 \in M_0$.

For $\varrho \in \mathscr{R}$ we can easily see that

$$(52) \qquad 0 = \varrho\left(\Delta u - \beta(u)\frac{\partial u}{\partial t}\right) = \Delta\varrho u - f\,, \quad f = u\,\Delta\varrho + 2(\operatorname{grad} u, \operatorname{grad} \varrho)_2 + + \varrho\,\beta(u)\frac{\partial u}{\partial t} \in L_2(\Omega) \quad \text{for a.e.} \quad t \in \langle 0, \mathscr{T}\rangle$$

$$0 = \varrho\left(\frac{\partial u}{\partial v} + g(\Lambda u) - g(T)\right) = \frac{\partial(\varrho u)}{\partial v} - g\,,$$

$$g = u\frac{\partial \varrho}{\partial v} - \varrho g(\Lambda u) + \varrho g(T) \quad \text{for a.e.} \quad t \in \langle 0, \mathscr{T}\rangle\,.$$

Evidently, the regularity of g depends on the regularity of T analogously to Sec. 3. Using [3] or [4], we can prove $\varrho\, u \in H^{1,3/2,3/2}(Q)$ for every $T \in L_2(S)$ satisfying the suppositions of Thm. 4. Such a result does not depend on $\upsilon \in R^1$. If T is more regular in the space variables we obtain better spacial regularity of ϱu, but it depends also on v. The above proved regularity of ϱu, however, is sufficient for the further considerations. As for $\varrho \in \mathscr{R} \setminus \mathscr{R}_0$ the proof of regularity of ϱu is easy (as in Sec. 3), we have proved $u \in H^{1,3/2,3/2}(Q)$. Using the technique of the proof of Prop. 5 and Thms. 4, 5 we prove that u and $\gamma(u)$ belong to $C_0(0, \mathscr{T}; H^{15/14-\varepsilon}(\Omega))$ for every $\varepsilon \in (0, \frac{15}{14})$. The above mentioned technique includes the possibility to extend u from $H^{1,3/2,3/2}(Q)$ to $H^{1,3/2,3/2}(R^3)$, which can be done as in Sec. 3 in time and via [3] Thm. 1.4.3.1 in the space variable.

The localization by means of $\mathscr{R}$ will be used in the proof of regularity of solutions of the Lamé system, too. For $x_0 \in M_0$, the corresponding $\varrho \equiv \varrho_{x_0} \in \mathscr{R}_0$ and the solution v of (2), ϱv solves the Lamé system on an infinite angle V with the right hand side

$$(1 - 2\sigma)\left((2 \operatorname{grad} \varrho, \operatorname{grad} v)_2 + v\, \Delta\varrho\right) + \operatorname{grad} \varrho \operatorname{div} v + \\ + \operatorname{grad}\left((v, \operatorname{grad} \varrho)_2\right) + \varrho(2 + 2\sigma) \operatorname{grad} \gamma(u) \quad \text{on} \quad Q \tag{53}$$

and the Neumann boundary condition (cf. (2)) with the right hand side

$$(1 - 2\sigma)\left(v \frac{\partial \varrho}{\partial \nu} + (\nu, v)_2 \operatorname{grad} \varrho\right) + 2\sigma\nu(v, \operatorname{grad} \varrho)_2 + \\ + (2 + 2\sigma)\, \varrho\, \gamma(u) \quad \text{on} \quad S. \tag{54}$$

The expression (53) clearly belongs at least to $C_0(0, \mathscr{T}; L_2(\Omega))$, the term in (54) to $C_0(0, \mathscr{T}; H^{1/2}(\partial\Omega))$. If we were able to prove better regularity of v – it depends on the magnitude of the angle – say e.g. $v \in C_0(0, \mathscr{T}; H^2(\Omega))$, the right hand side of (53), (54) would make it possible to prove, again in dependence on the angle, $\tau \in C_0(\bar{Q}; R^4)$. Thus we can restrict ourselves to the problem of regularity of the solution of the Lamé system on an infinite angle $V = \{[x_1, x_2] \in R^2;\ x_1 > |x_2| (\operatorname{tg} \frac{1}{2}\omega_0)^{-1}\}$. In polar coordinates $V = \{[r, \omega];\ r \in (0, +\infty), |\omega| < \frac{1}{2}\omega_0\}$, $\omega_0 \in (0, 2\pi)$, and the Lamé system has the following form for polar components of $v = [v_r, v_\omega]$, $v_r(r, \omega) \equiv v_1(r, \omega) \cos \omega + v_2(r, \omega) \sin \omega$, $v_\omega(r, \omega) = v_2(r, \omega) \cos \omega - v_1(r, \omega) \sin \omega$:

$$(2 - 2\sigma)\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2}\right) + \frac{(1 - 2\sigma)}{r^2}\frac{\partial^2 v_r}{\partial \omega^2} + \\ + \frac{1}{r}\frac{\partial^2 v_\omega}{\partial r\, \partial \omega} - \frac{3 - 4\sigma}{r}\frac{\partial v_\omega}{\partial \omega} = (2 + 2\sigma)\frac{\partial \gamma(u)}{\partial r}, \\ (1 - 2\sigma)\left(\frac{\partial^2 v_\omega}{\partial r^2} + \frac{1}{r}\frac{\partial v_\omega}{\partial r} - \frac{v_\omega}{r^2}\right) + \frac{(2 - 2\sigma)}{r^2}\frac{\partial^2 v_\omega}{\partial \omega^2} + \\ + \frac{1}{r}\frac{\partial^2 v_r}{\partial \omega\, \partial r} + \frac{3 - 4\sigma}{r}\frac{\partial v_r}{\partial \omega} = \frac{(2 + 2\sigma)}{r}\frac{\partial \gamma(u)}{\partial \omega} \quad \text{on} \quad V. \tag{55}$$

The boundary value conditions have the form

$$(56)\qquad (1-2\sigma)\left(\frac{\partial v_\omega}{\partial r}-\frac{v_\omega}{r}+\frac{1}{r}\frac{\partial v_r}{\partial \omega}\right)=0\,,\quad \frac{2-2\sigma}{r}\left(\frac{\partial v_\omega}{\partial_\omega}+v_r\right)+$$
$$+\,2\sigma\frac{\partial v_r}{\partial r}=(2+2\sigma)\,\gamma(u)\,,\quad r\in\langle 0,+\infty)\,,\quad \omega=\pm\frac{\omega_0}{2}\,.$$

Using the usual transformation $r = e^s$ and the Fourier transformation with respect to the variable s (cf. [10]) and putting $\jmath = 3 - 4\sigma$ we obtain the following system of ordinary differential equations with the parameter $n = i\lambda$ (λ is the dual variable in the Fourier sense), where the tilde denotes the corresponding Fourier transform and $q_n(\omega)$ stands for the term $(\widetilde{\gamma(u)})\,(i\,(1-n),\,\omega)$:

$$(57)\qquad \left.\begin{aligned}&(\jmath-1)\,\tilde v_r''+(\jmath+1)(n^2-1)\,\tilde v_r+2(n-\jmath)\,\tilde v_\omega'=(7-\jmath)(n-1)\,q_n\,,\\&(\jmath+1)\,\tilde v_\omega''+(\jmath-1)(n^2-1)\,\tilde v_\omega+2(n+\jmath)\tilde v_r'=(7-\jmath)\,q_n'\,,\end{aligned}\right\}$$
$$\omega\in\left(-\frac{\omega_0}{2},\frac{\omega_0}{2}\right),$$
$$(\jmath-1)\,(\tilde v_r'+(n-1)\,\tilde v_\omega)=0\,,\quad (\jmath+1)\,(\tilde v_\omega'+\tilde v_r)+(3-\jmath)\,n\tilde v_r=$$
$$=(7-\jmath)\,q_n\,,\quad \omega=\pm\tfrac{1}{2}\omega_0\,.$$

The homogeneous system corresponding to (57) has nontrivial solutions for every solution of the equation

$$(58)\qquad n\sin\omega_0=\pm\sin(n\omega_0)\,\cdot$$

If n does not fulfil (58), the system (57) has the following solution, where $q_n(\omega) = (\widetilde{\gamma(u)})\,(i(1-n),\,\omega)$:

$$(59)\qquad \tilde v_r(-in,\omega)=P_r(n,\omega)+A_1\sin((n+1)\,\omega)+A_2(n-\jmath)\sin((n-1)\,\omega)+$$
$$+\,A_3\cos((n+1)\,\omega)+A_4(\jmath-n)\cos((n-1)\,\omega)\,,$$
$$\tilde v_\omega(-in,\omega)=P_\omega(n,\omega)+A_1\cos((n+1)\,\omega)+A_2(\jmath+n)\cos((n-1)\,\omega)-$$
$$-\,A_3\sin((n+1)\,\omega)+A_4(\jmath+n)\sin((n-1)\,\omega)\,,$$
$$P_r(n,\omega)=c\int_0^\omega\sin((n+1)(\omega-\zeta))\,q_n(\zeta)\,d\zeta\,,$$
$$P_\omega(n,\omega)=c\int_0^\omega\cos((n+1)(\omega-\zeta))\,q_n(\zeta)\,d\zeta\,,$$
$$A_1=c\int_0^{\omega_0/2}\frac{n\sin(\omega_0-(n+1)\,\zeta)-\sin(n\omega_0-(n+1)\,\zeta)}{2(\sin(n\omega_0)-n\sin\omega_0)}\,.$$
$$.\,(q_n(\zeta)-q_n(-\zeta))\,d\zeta\,,$$
$$A_2=c\int_0^{\omega_0/2}\frac{\sin((n+1)\,\zeta)\,(q_n(\zeta)-q_n(-\zeta))}{2(\sin(n\omega_0)-n\sin\omega_0)}\,d\zeta\,,$$

$$A_3 = c \int_0^{\omega_0/2} \frac{n \cos(\omega_0 - (n+1)\zeta) + \cos(n\omega_0 - (n+1)\zeta)}{2(\sin(n\omega_0) + n \sin \omega_0)} \,.$$

$$. (q_n(\zeta) + q_n(-\zeta))\, d\zeta ,$$

$$A_4 = c \int_0^{\omega_0/2} \frac{\cos((n+1)\zeta)(q_n(\zeta) + q_n(-\zeta))}{2(\sin(n\omega_0) + n \sin \omega_0)}\, d\zeta , \quad c = \frac{7 - \sigma}{1 + \sigma} = \frac{1 + \sigma}{1 - \sigma}$$

Let us mention the general case with a general function $2F = 2[F_r, F_\omega]$ on the right hand side of (55) and $2K = 2[K_r, K_\omega]$ on the right hand side of (56). The Fourier transform of the solution has in this case the same structure as in (59) with P, A_i, $i = 1, 2, 3, 4$, defined as follows:

(60)
$$P_r(n, \omega) = \frac{1}{n(\sigma^2 - 1)} \int_0^\omega \tilde{F}_r(i(1-n), \zeta) [(n + \sigma) \sin((n+1)(\omega - \zeta)) +$$
$$+ (\sigma - n) \sin((n-1)(\omega - \zeta))] +$$
$$+ \tilde{F}_\omega(i(1-n), \zeta) [(n - \sigma)(\cos((n+1)(\omega - \zeta)) -$$
$$- \cos((n-1)(\omega - \zeta)))]\, d\zeta ,$$

$$P_\omega(n, \omega) = \frac{1}{n(\sigma^2 - 1)} \int_0^\omega \tilde{F}_r(i(1-n), \zeta) [(n + \sigma) \,.$$
$$. (\cos((n+1)(\omega - \zeta)) - \cos((n-1)(\omega - \zeta)))] + \tilde{F}_\omega(i(1-n), \zeta) \,.$$
$$. [(\sigma - n) \sin((n+1)(\omega - \zeta)) + (\sigma + n) \sin((n-1)(\omega - \zeta))]\, d\zeta ,$$

$$A_1 = \frac{(n+1)(Z_1 + Z_2) \sin((n-1)\tfrac{1}{2}\omega_0) + (n-1)(Z_3 - Z_4) \cos((n-1)\tfrac{1}{2}\omega_0)}{2(n \sin \omega_0 - \sin(n\omega_0))} ,$$

$$A_2 = \frac{-(Z_1 + Z_2) \sin((n+1)\tfrac{1}{2}\omega_0) + (Z_4 - Z_3) \cos((n+1)\tfrac{1}{2}\omega_0)}{2(n \sin \omega_0 - \sin(n\omega_0))} ,$$

$$A_3 = \frac{(n+1)(Z_1 - Z_2) \cos((n-1)\tfrac{1}{2}\omega_0) - (n-1)(Z_3 + Z_4) \sin((n-1)\tfrac{1}{2}\omega_0)}{2(n \sin \omega_0 + \sin(n\omega_0))} ,$$

$$A_4 = \frac{(Z_1 - Z_2) \cos((n+1)\tfrac{1}{2}\omega_0) - (Z_3 + Z_4) \sin((n+1)\tfrac{1}{2}\omega_0)}{2(n \sin \omega_0 + \sin(n\omega_0))} ,$$

$$Z_{1,2} = \frac{1}{n(\sigma - 1)} (\varrho_r(n, \pm \tfrac{1}{2}\omega_0) - \tilde{K}_r(i(1-n), \pm \tfrac{1}{2}\omega_0)) ,$$

$$Z_{3,4} = \frac{1}{n(\sigma - 1)} (\varrho_\omega(n, \pm \tfrac{1}{2}\omega_0) - \tilde{K}_\omega(i(1-n), \pm \tfrac{1}{2}\omega_0)) ,$$

$$\varrho_r(n, \omega) = \frac{1}{\sigma + 1} \int_0^\omega \tilde{F}_r(i(1-n), \zeta) [(n + \sigma) \cos((n+1)(\omega - \zeta)) -$$
$$- (n-1) \cos((n-1)(\omega - \zeta))] + \tilde{F}_\omega(i(1-n), \zeta) \,.$$
$$. [(\sigma - n) \sin((n+1)(\omega - \zeta)) + (n-1) \sin((n-1)(\omega - \zeta))]\, d\zeta ,$$

$$\sigma_\omega(n, \omega) = \frac{1}{\sigma + 1} \int_0^\omega \tilde{F}_r(\mathrm{i}(1 - n), \zeta) [-(n + \sigma) \sin((n + 1)(\omega - \zeta)) +$$

$$+ (n + 1) \sin((n - 1)(\omega - \zeta))] + \tilde{F}_\omega(\mathrm{i}(1 - n), \zeta) .$$

$$. [(\sigma - n) \cos((n + 1)(\omega - \zeta)) + (n + 1) \cos((n - 1)(\omega - \zeta))] \, \mathrm{d}\zeta .$$

For n_0 solving (58) the Green operators corresponding to (59) or (60) have poles. The eigenfunction corresponding to a single pole has the following form (in the Cartesian components):

(61)
$$v_{n_0} = r^{n_0}[(n_0 \cos \omega_0 - \cos(n_0\omega_0) + \sigma) \sin(n_0\omega) - n_0 \sin((n_0 - 2)\omega),$$
$$(n_0 \cos \omega_0 - \cos(n_0\omega_0) - \sigma) \cos(n_0\omega) - n_0 \cos((n_0 - 2)\omega)] , \quad \text{or}$$
$$v_{n_0} = r^{n_0}[(\cos(n_0\omega_0) + n_0 \cos \omega_0 + \sigma) \cos(n_0\omega) - n_0 \cos((n_0 - 2)\omega),$$
$$(\sigma - \cos(n_0\omega_0) - n_0 \cos \omega_0) \sin(n_0\omega) + n_0 \sin((n_0 - 2)\omega)] ,$$

where the first expression corresponds to the plus, the second one to the minus sign in (58). There are only single or double poles. The double poles must fulfil the conditions

(62)
$$\text{(i) } n_0 \in R^1 , \quad \text{(ii) } \cos n_0\omega_0 = \pm \frac{\sin \omega_0}{\omega_0} \quad \text{for} \quad n_0 \neq 0 \quad \text{or} \quad n_0 = 0 ,$$
$$\text{(iii) } n_0\omega_0 = \operatorname{tg} n_0\omega_0 , \quad \left(\text{i.e.} \quad n_0^2 = \frac{1}{\sin^2 \omega_0} - \frac{1}{\omega_0^2}\right),$$

where the sign in (58) and (62) (ii) must be the same. The first eigenfunction for double poles has the same form as for the single ones. The second eigenfunction for $n_0 \neq 0$ has the form $r^{n_0} \ln r w_{n_0}^1$, where $w_{n_0}^1$ is a solution of the system (57) with the right hand side of the equation equal to $2r^{-n_0}[n_0(\sigma + 1) v_{n_0,r} + v'_{n_0,\omega}, n_0(\sigma - 1) v_{n_0,\omega} + v'_{n_0,r}]$ and the right hand side of the boundary condition $r^{-n_0}[(\sigma - 1) v_{n_0,\omega}, (3 - \sigma) . v_{n_0,r}]$, where $v_{n_0,r}, v_{n_0,\omega}$ are the polar components of the first eigenfunction. For $n_0 = 0$ both eigenfunctions correspond to (61) and have the form $[0, 1]$, $[1, 0]$ in the Cartesian components (i.e. they are shifts). For $n_0 = 1$ (which cannot be a double pole if $\omega_0 \neq \operatorname{tg} \omega_0$) the corresponding eigenfunction has the form $[x_1, x_2] \to [x_2, -x_1]$ (the rotation).

Now we introduce the weighted Sobolev space on V in such a way that $H_\ell^\alpha(V)$ is the space of all functions for which the norm $\|\cdot\|_{\alpha,\ell,V}$ is finite, where

(63)
$$\|f\|_{\alpha,\ell,V}^2 = \|\mathrm{D}^{[\alpha]}f\|'^2_{\alpha-[\alpha],\ell,V} + \sum_{k=0}^{[\alpha]} \|\mathrm{D}^k f\|^2_{0,k-\alpha+\ell,V} ,$$
$$\|f\|_{0,\ell,V} \equiv \|r^\ell f\|_{L_2(V)} , \quad \ell \in R^1 ,$$

and the part of the norm corresponding to the noninteger part of α, i.e. $\alpha - [\alpha]$' is defined as in (3) with the weight $r^{2\ell}$. As grad $\gamma(u) \in C_0(0, \mathscr{T}; H^\eta(V; R^2))$ for any $\eta \in (0, \frac{1}{14})$ and due to the Hardy inequality (cf. [3], Thm 1.4.4.4), we have grad $\gamma(u) \in$

$\in C_0(0, \mathscr{T}; H_0^\eta(V; R^2))$. To have the right hand side of the boundary value condition in $C_0(0, \mathscr{T}; H_0^{1/2+\eta}(V))$, $\eta \in (0, \frac{1}{14})$, we define an auxiliary displacement v^0:

(64) $$v_\omega^0 = 0\,, \quad v_r^0 \equiv v_r^0(r) \quad \text{on} \quad V\,, \quad v_r^0(0) = 0\,,$$
$$2\sigma \frac{\mathrm{d}v_r^0}{\mathrm{d}r} + (2 - 2\sigma)\frac{v_r^0}{r} = (2 + 2\sigma)\,\gamma_0\,,$$
$$\text{i.e.} \quad v_r^0(r) = \frac{r}{\sigma}(1 + \sigma)\int_0^1 \gamma_0(r\zeta)\,\zeta^{1/\sigma - 1}\,\mathrm{d}\zeta\,,$$

where γ_0 is a sufficiently smooth function depending only on r and the parameter t, having a bounded support and fulfilling $\gamma_0(0) = \gamma(u)(t, 0)$. v^0 fulfils (56) with the right hand side $(2 + 2\sigma) K_0$, where $K_{0,r} = ((1/\sigma) - 1)\int_0^1 (3\gamma_0'(r\zeta) + \zeta\gamma_0''(r\zeta))\,\zeta^{1/\sigma}\,\mathrm{d}\zeta$, $K_{0,\omega} = 0$, and (56) with the right hand side $(2 + 2\sigma)\,[0, \gamma_0]$. Then for $v - v^0$ the right hand side of (55) is in $H_0^\eta(V; R^2)$ and the right hand side of (56) is in $H_0^{1/2+\eta}(V; R^2)$ for every $t \in \langle 0, \mathscr{T}\rangle$ and $\eta \in (0, \frac{1}{14})$.

We concentrate a little more on the weighted spaces with fractional derivatives for $\alpha \in (0, 1)$. We restrict ourselves to $\omega_0 \in (0, \pi)$. We use again the polar coordinates and transform $r = \mathrm{e}^s$. After some calculation, it is possible to prove that the seminorm $\|\cdot\|'_{\alpha,\ell,V}$ is equivalent to the seminorm

(65) $$\left(\left({}_\omega\|\cdot\|''_{\alpha,\ell,V}\right)^2 + \left({}_s\|\cdot\|''_{\alpha,\ell,V}\right)^2\right)^{1/2}, \quad {}_\omega\|f\|''^2_{\alpha,\ell,V} =$$
$$= \int_{R^1}\int_{-\omega_0/2}^{\omega_0/2}\int_{-\omega_0/2}^{\omega_0/2} \frac{(f(s, \omega_1) - f(s, \omega_2))^2}{|\omega_1 - \omega_2|^{1+2\alpha}}\,\mathrm{e}^{(2-2\alpha+2\ell)s}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}s\,,$$
$${}_s\|f\|''^2_{\alpha,\ell,V} = \int_{R^1}\int_0^{+\infty}\int_{-\omega_0/2}^{\omega_0/2} \mathrm{e}^{(2-2\alpha+2\ell)}\,\mathrm{e}^h\,\frac{(f(s + h, \omega) - f(s, \omega))^2}{(\mathrm{e}^h - 1)^{1+2\alpha}}\,.$$
$$.\,\mathrm{d}\omega\,\mathrm{d}h\,\mathrm{d}s\,,$$

cf. [7]. For $\ell < 1$ or defining $\|f\|'_{\alpha,\ell,V} = \|\mathrm{e}^\ell f\|'_{\alpha,V}$ (cf. (3)), we can replace ${}_s\|f\|''_{\alpha,\ell,V}$ by

(66) $$\left(\int_{R^1}\int_0^{+\infty}\int_{-\omega_0/2}^{\omega_0/2} \frac{(f_0(s + h, \omega) - f_0(s, \omega))^2}{|h|^{1+2\alpha}}\,\mathrm{d}\omega\,\mathrm{d}h\,\mathrm{d}s\right)^{1/2},$$
$$f_0(s, \omega) \equiv f(s, \omega)\,\mathrm{e}^{(1-\alpha+\ell)s}\,.$$

Due to the imbedding $H_\ell^\alpha(V) \hookrightarrow H_{\ell-j}^{\alpha-j}$, $j \in (0, \alpha + 1)$, which is a consequence of the Fourier transformation, we can use the same calculations as in Lemma 1. (These calculations yield e.g. the trace theorems etc.) Particularly they allow us to use the theory of Kondratěv (cf. [8], [10] Secs. 7–10) for the weighted spaces with fractional derivatives.

As we restrict ourselves to the case $\omega_0 \in (0, \pi)$, the only solutions of (58) with real parts in $(-1, 1\rangle$ are 0 and 1 (cf. [13]), with the eigenfunctions being either shifts or rotations. Using the technique of [8] (cf. [17]) we prove $\varrho_1(v - v_0) \in H_{1+\eta_0}^2\,\cdot$ $\cdot\,(V; R^2)$, where $\varrho_1 \in C_3(R^1; \langle 0, 1\rangle)$ is a "cut of" function (i.e. $\varrho_1/\mathscr{B}_2(0, 1) \equiv 1$,

$\varrho_1/R^2 \setminus \mathscr{B}_2(0, 2) \equiv 0)$, $\eta_0 > 0$ suitably small, and the existence of a solution $v_1 \in H^2_\varepsilon(V; R^2)$, $\varepsilon \in (0, 1)$, of the problem (53), (54) for $\varrho \equiv \varrho_1$. Via the shift and interpolation technique of the previous sections we find $(1 - \varrho_1)(v - v^0) \in H^2(V; R^2) \cap H^2_0(V; R^2)$. Using Thms. 7.3 and 10.2 of [10] based on the Cauchy residuum theorem, we obtain $v - v^0 = c_1 v_{0,1} + c_2 v_{0,2} + c_3 v_1 + v_1 + (1 - \varrho_1)(v - v_0)$, $v_{0,i}$, $i = 1, 2$ and v_1 from (61). Due to the well-known properties of the weak solution of the Neumann boundary value problem, we can suppose $c_i = 0$, $i = 1, 2, 3$. Thus $v - v^0 \in H^{2-\varepsilon}(V; R^2)$, $\varepsilon \in (0, 1)$. Using (53), (54), and the "Cauchy residuum" technique again, we can show that $v - v^0 \in H_0^{2+\eta}(V; R^2)$ for some small $\eta > 0$ (dependent on ω_0 such that there are no solutions n_0 of (58) having Re n_0 in $(1, 1 + 2\eta)$). Therefore $v - v^0 \in H^{2+\eta}(V; R^2)$ for every $t \in \langle 0, \mathscr{T} \rangle$. As v^0 is smooth and due to the possibility of choosing $\gamma_0 \equiv (\gamma_0)_t$ such that $\|\gamma_0\|_{C_0(0,\mathscr{T};C^3(V))} \leqq \|\gamma(u)\|_{C_0(0,\mathscr{T};H^{1+\eta_0}(V))}$, $\eta_0 \in (0, \frac{1}{14})$, we have $\|v(t, \cdot)\|_{H^{2+\eta}(V;R^2)} \leqq \|\gamma(u)\|_{C_0(0,\mathscr{T};H^{1+\eta}(V))}$ for all $t \in \langle 0, \mathscr{T} \rangle$. Due to the continuity of $t \to v(t, \cdot)$ in $H^1(V; R^2)$ we have $v \in C_0(0, \mathscr{T}; H^{2+\eta-\varepsilon}(V; R^2))$ for every $\varepsilon \in (0, \eta)$.

As the regularity of the solution along the regular parts of the boundary is proved (cf. Sec. 3), we have proved the following theorem:

Theorem 6. *Let all assumptions of Thms. 4 and 5 be satisfied with the exception of the regularity of $\partial\Omega$. Let $\partial\Omega$ be of the class $C_{2+\varepsilon}$ ($\varepsilon > 0$ arbitrarily small) except for a finite set M_0, where (51) holds with $v > 0$ (i.e. the angle $\omega_0 \in (0, \pi)$). Then the corresponding stress satisfies $\tau \in C_0(\bar{Q}; R^4)$.*

5. CONCLUSION

For the nonlinear twodimensional model we have proved that the jumpes in time and even discontinuities in the space variable of the heating regime and isolated nonsmoothnesses of the boundary having the convex character give a continuous stress on $\bar{Q}$. Therefore for such cases the thermoelastic model is reasonable for the technical practice. However, if the isolated singularity of the boundary has not the convex character there is little hope to obtain such a regularity (cf. Thm. 12.5 of [10]).

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Souhrn

REGULARITA ŘEŠENÍ TERMOELASTICKÉHO SYSTÉMU S NESPOJITÝMI REŽIMY OHŘEVU

Jiří Jarušek

Omezenost a spojitost napětí příslušného k řešení termoelastického systému se zkoumá zpočátku pro lineární systém na páse a pak pro systém s nelineární rovnicí vedení tepla, nelineárními okrajovými podmínkami a nespojitým režimem ohřevu pro dvourozměrný model tělesa s izolovanými nehladkostmi hranice.

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