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PENALTY METHOD AND EXTRAPOLATION FOR AXISYMMETRIC ELLIPTIC PROBLEMS WITH DIRICHLET BOUNDARY CONDITIONS

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Summary. A second order elliptic problem with axisymmetric data is solved in a finite element space, constructed on a triangulation with curved triangles, in such a way, that the (nonhomogeneous) boundary condition is fulfilled in the sense of a penalty. On the basis of two approximate solutions, extrapolates for both the solution and the boundary flux are defined. Some a priori error estimates are derived, provided the exact solution is regular enough. The paper extends some of the results of J. T. King [6], [7].

Keywords: finite elements, penalty method, axisymmetric problems, extrapolation.

AMS Subject class: 65N30, 73K25.

INTRODUCTION

In some cases we need to compute both the solution and the boundary flux of an elliptic second order Dirichlet problem with a considerable accuracy. For instance, in the shape optimization the sensitivity analysis sometimes leads to the conclusion that the gradient of the cost functional can be expressed as a boundary integral involving the boundary flux ([3] § 3.3.3). Then it seems to be suitable to employ the method of penalty and extrapolation, proposed by King and Serbin [6], [7], who introduced the method for second order elliptic equations with non-homogeneous Dirichlet boundary conditions in N-dimensional domain. It is the aim of the present paper to extend the method to axisymmetric elliptic problems in $\mathbb{R}^3$ and to derive also a priori error estimates.

In Section 1 we introduce some weighted Sobolev spaces, auxiliary inequalities and finite element spaces. An elliptic model problem is presented in Section 2 together with a definition of approximate solution by means of finite elements and a penalty term. In section 3 we derive some auxiliary error estimates. In Section 4 new approximations both of the solution and of the boundary flux are defined, extrapolating two approximate solutions with two different "weights" of the penalty term. Using the auxiliary error estimates, we prove a priori error estimates for them, provided the data and, consequently, the exact solution are regular enough.

1. SOME PRELIMINARY RESULTS

Let us suppose that a domain $\Omega \subset \mathbb{R}^3$ is generated by the rotation of a bounded domain D about the axis $\mathcal{O} = \{x_1 = x_2 = 0\}$. Assume that the domain Ω has a smooth boundary $\partial\Omega$, so that the boundary ∂D can be decomposed, as follows:

$$\partial D = \Gamma_0 \cup \Gamma ,$$

where $\Gamma_0 = \partial D \cap \mathcal{O}$ and $\Gamma \subset C^3$ (Fig. 1), Γ is orthogonal to the axis $\mathcal{O}$ and straight in some neighbourhood of points $\Gamma \cap \mathcal{O}$.

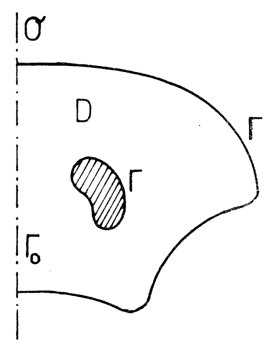

Fig. 1.

Passing to the cylindrical coordinates (r, ϑ, z), for which $x_3 = z$, we define the weighted Sobolev spaces $W_r^{k,2}(D)$ of functions $u(r, z)$, with the norm

$$\|u\|_{k,r,D} = \left(\int_D \sum_{|\alpha| \leq k} |D^\alpha u|^2 \, r \, \mathrm{d}r \, \mathrm{d}z\right)^{1/2} , \quad k = 0, 1, 2, \ldots$$

and the seminorm

$$|u|_{k,r,D} = \left(\int_D \sum_{|\alpha| = k} |D^\alpha u|^2 \, r \, \mathrm{d}r \, \mathrm{d}z\right)^{1/2} .$$

Instead of $W_r^{0,2}(D)$ we shall write $L_r^2(D)$ and define the inner product in $L_r^2(D)$ by the integral

$$(u, v)_0 = \int_D uvr \, \mathrm{d}r \, \mathrm{d}z .$$

In a similar way, we introduce the space $L_r^2(\Gamma)$ with the inner product

$$\langle u, v \rangle = \int_\Gamma uvr \, \mathrm{d}s , \quad u, v \in L_r^2(\Gamma)$$

and the associated norm $\|u\|_{0,r,\Gamma} = \langle u, u \rangle^{1/2}$.

Note that a function $U(x_1, x_2, x_3)$ is axisymmetric in Ω if and only if

$$U(r\cos\vartheta, r\sin\vartheta, z) = u(r, z).$$

Then

$$U \in W^{1,2}(\Omega) \Leftrightarrow u \in W_r^{1,2}(D)$$

(see e.g. [9] – Sect. 2).

There exists a continuous mapping $G\colon W_r^{1,2}(D) \to L_r^2(\Gamma)$ such that $Gu = u|_\Gamma$ for any $u \in C^1(\bar{D})$.

(The proof can be found e.g. in [5] – Section 1.)

Moreover, we introduce the following subspace

$$V = \{v \in W_r^{1,2}(D) \mid Gv = 0\}.$$

Henceforth the "C" will denote a generic positive constant, possibility different at different places.

For any $u \in W_r^{1,2}(D)$ the following Friedrichs inequality holds

$$C\|u\|_{1,r,D}^2 \leqq |u|_{1,r,D}^2 + \|u\|_{0,r,\Gamma}^2. \tag{1.1}$$

(This is an immediate consequence of the classical Friedrichs inequality in $W^{1,2}(\Omega)$, $\Omega \subset \mathbb{R}^3$, see e.g. [8] – Thm. 1.9).

From (1.1) we conclude the following inequality

$$\|u\|_{1,r,D} \leqq C|u|_{1,r,D} \quad \forall u \in V. \tag{1.2}$$

Lemma 1.1. *Let Γ belong to the class C^2, being orthogonal to the axis $\mathcal{O}$. Then there exists a positive constant C such that for all $\varepsilon > 0$*

$$\|v\|_{0,r,\Gamma}^2 \leqq \varepsilon\|v\|_{0,r,D}^2 + C\varepsilon^{-1}\|v\|_{1,r,D}^2. \quad \forall v \in W_r^{1,2}(D). \tag{1.3}$$

Proof (see [2] – Lemma 4.1 for functions from $W^{1,2}(\Omega)$, $\Omega \subset \mathbb{R}^N$). There exists a vector function $\mathbf{f} = (f_r, f_z) \in [C^1(\bar{D})]^2$ such that $f_r = \nu_r$, $f_z = \nu_z$ on Γ (where ν_r, ν_z are components of the unit outward normal to Γ) and $f_r = 0$ on Γ_0.

Then for any positive η we may write

$$\int_\Gamma v^2 r \,\mathrm{d}s = \int_D \left[\frac{\partial}{\partial r}(f_r r v^2) + \frac{\partial}{\partial z}(f_z r v^2)\right] \mathrm{d}r\,\mathrm{d}z =$$

$$= \int_D v^2 \left(\frac{1}{r} f_r + \frac{\partial}{\partial r} f_r + \frac{\partial}{\partial z} f_z\right) r\,\mathrm{d}r\,\mathrm{d}z + 2\int_D v\left(\frac{\partial v}{\partial r} f_r + \frac{\partial v}{\partial z} f_z\right) r\,\mathrm{d}r\,\mathrm{d}z.$$

Since $f_r, f_z \in C^1(\bar{D})$ and

$$\left|\frac{1}{r} f_r(r, z)\right| = \left|\frac{\partial}{\partial r} f_r(\varrho, z)\right| \leqq \|f_r\|_{C^1}, \quad 0 < \varrho < r$$

we obtain the estimate

$$\|v\|^2_{0,r,\Gamma} \leqq 3\|\mathbf{f}\|_{C^1}\left(\|v\|^2_{0,r,D} + 2\int_D |v|\left(\left|\frac{\partial v}{\partial r}\right| + \left|\frac{\partial v}{\partial z}\right|\right) r \, dr \, dz\right) \leqq$$

$$\leqq 3\|\mathbf{f}\|_{C^1}\left(\tfrac{3}{2}\eta\|v\|^2_{0,r,D} + \eta^{-1}\|v\|^2_{1,r,D}\right) \quad \forall \eta > 0 .$$

Setting $9\|\mathbf{f}\|_{C^1}\,\eta/2 = \varepsilon$, we arrive at (1.3). Q.E.D.

Assume that a finite element space Σ_h is available, where h is a (small) parameter, such that $\Sigma_h \subset W^{1,2}_r(D)$ and there exists a constant C, independent of u and h, and a function $u_I \subset \Sigma_h$ such that

$$(1.4) \qquad \|u - u_I\|_{0,r,D} + h\|u - u_I\|_{1,r,D} \leqq Ch^s\|u\|_{s,r,D}$$

holds for any function $u \in W^{s,2}_r(D)$, where $s = 2, 3$ and for any $h \in (0, 1]$.

For instance, we can employ spaces of piecewise smooth functions, proposed by Zlámal [10]. Let the domain D be carved into triangles K, which may have one curved side, if K is adjacent to an arc of Γ. Assume that all triangles, for which $K \cap \mathcal{O} \neq \emptyset$, are straight (recall that in some neighbourhood of points $\Gamma \cap \mathcal{O}$ the boundary Γ coincides with a straight segment). Assume moreover, that the family of triangulations $\{\mathcal{T}_h\}$, $h \in (0, 1]$, is regular in the following sense: there exists a positive ϑ_0 independent of h, such that the interior angles of all triangles $K \in \mathcal{T}_h$ are not less than ϑ_0. (Here the angles of a curved triangle are measured as if the arc were replaced by the chord).

Let us sketch the proof of (1.4).

1^0 Let U_Q be the union of triangles having one vertex at a point $Q \in \Gamma_0$ and let $\Pi_{U_Q}u$ be the piecewise quadratic interpolate of u over the union U_Q. Since all the triangles $K \subset U_Q$ are straight, we can use Lemma 6.1 of [9] to obtain

$$(1.5) \qquad |u - \Pi_{U_Q}u|_{1,r,U_Q} \leqq Ch^2|u|_{3,r,U_Q} \quad \forall u \in W^{3,2}_r(U_Q) .$$

By the same argument, however, one can derive the upper bound $Ch|u|_{2,r,U_Q}$ for all $u \in W^{2,2}_r(U_Q)$. It is easy to prove that

$$(1.6) \qquad \|u - \Pi_{U_Q}u\|_{0,r,U_Q} \leqq Ch^s|u|_{s,r,U_Q} , \quad s = 2, 3$$

following a similar line of thoughts.

2^0 Next let us consider a triangle (possibly curved) K, such that $K \cap \mathcal{O} = \emptyset$. Modifying slightly the proof of Lemma 5.2 in [4], we obtain the following assertion: there exists a positive constant C, independent of K and such that

$$r_0 = \min_{(r,z)\in K} r \geqq Ch_K ,$$

where h_K is the maximal side of the "straightened" triangle K (with the same vertices). Since the boundary $\Gamma \subset C^3$, we have

$$R_0 = \max_{(r,z)\in K} r \leqq r_0 + h_K + C_1h_K^2 \leqq r_0 + C_2h_K ,$$

so that

(1.7) $$R_0/r_0 \leqq 1 + C_2 h_K/r_0 \leqq C_3 .$$

Let $\Pi_K u$ be the function corresponding with the quadratic interpolate on the reference "unit" triangle – in the sense of Zlámal [10]. Then we have (cf. [10] – Thm. 1 and the proof)

$$\|u - \Pi_K u\|_{j,K} \leqq C h_K^{s-j} \|u\|_{s,K} , \quad s = 2, 3 ; \quad j = 0, 1 ,$$

where $\|\cdot\|_{j,K}$ denotes the norm in $W^{j,2}(K)$. Obviously, we have

$$\|u\|_{s,K} \leqq r_0^{-1/2} \|u\|_{s,r,K} ,$$

so that we may write

(1.8) $$\|u - \Pi_K u\|_{j,r,K} \leqq R_0^{1/2} \|u - \Pi_K u\|_{j,K} \leqq C h_K^{s-j} R_0^{1/2} r_0^{-1/2} \|u\|_{s,r,K} \leqq \leqq C_4 h_K^{s-j} \|u\|_{s,r,K} ,$$

using (1.7). Combining (1.5), (1.6) and (1.8), we arrive at the condition (1.4).

2. MODEL PROBLEM AND THE PENALTY METHOD

We shall consider the following elliptic boundary value problem

(2.1) $$-\left[\frac{\partial}{\partial r}\left(a_r \frac{\partial u}{\partial r}\right) + \frac{1}{r} a_r \frac{\partial u}{\partial r} + \frac{\partial}{\partial z}\left(a_z \frac{\partial u}{\partial z}\right)\right] = f \quad \text{in} \quad D , \quad u = g \quad \text{on} \quad \Gamma ,$$

where the coefficients a_r and a_z belong to $C^2(\bar{D})$, $\partial a_r/\partial r = \partial a_z/\partial r = 0$ for $r = 0$, and a constant $a_0 > 0$ exists such that

$$a_r \geqq a_0 , \quad a_z \geqq a_0 \quad \text{in} \quad D ,$$

$f \in W_r^{1,2}(D)$, $g \in G(X^3(D))$, where

$$X^3(D) = \left\{ v \in W_r^{3,2}(D) \middle| \frac{1}{r}\frac{\partial v}{\partial r} \in W_r^{1,2}(D) \right\} .$$

We introduce the following bilinear form

$$a(u, v) = \int_D \left(a_r \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + a_z \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} \right) r \, dr \, dz , \quad u, v \in W_r^{1,2}(D) .$$

Let $u_0 \in W_r^{1,2}(D)$ be such that $Gu_0 = g$. We say that $u \in W_r^{1,2}(D)$ is a *weak solution* of the problem (2.1) if $u - u_0 \in V$ and

$$a(u, v) = (f, v)_0 \quad \forall v \in V .$$

Since

$$a(v, v) \geqq a_0 |v|_{1,r,D}^2 \geqq C a_0 \|v\|_{1,r,D}^2 \quad \forall v \in V$$

follows from the Friedrichs inequality (1.2), there exists a unique weak solution of (2.1).

If $w \in W_r^{2,2}(D)$, we define

$$\frac{\partial w}{\partial v_A} = a_r \frac{\partial w}{\partial r} v_r + a_z \frac{\partial w}{\partial z} v_z$$

and $\partial w/\partial v_A \in L_r^2(\Gamma)$ follows, since both $G(\partial w/\partial r)$ and $G(\partial w/\partial z)$ belong to $L_r^2(\Gamma)$.

Henceforth, we assume that the weak solution u of (2.1) is such that

$$(2.2) \qquad u \in W_r^{3,2}(D) \quad \text{and} \quad \frac{1}{r}\frac{\partial^2 u}{\partial r\, \partial z} \in L_r^2(D).$$

Remark 2.1. Defining for any $u(r, z)$, $(r, z) \in D$ the axisymmetric function

$$\tilde{u}(x_1, x_2, x_3) = u((x_1^2 + x_2^2)^{1/2}, x_3), \quad (x_1, x_2, x_3) \in \Omega,$$

we have the following relations (cf. [9] – Lemma 2.1, 2.2):

$$\tilde{u} \in W^{2,2}(\Omega) \Leftrightarrow u \in W_r^{2,2}(D) \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial r} \in L_r^2(D),$$

$$\tilde{u} \in W^{3,3}(\Omega) \Leftrightarrow u \in W_r^{3,2}(D) \quad \text{and}$$

$$\frac{1}{r}\frac{\partial u}{\partial r} \in W_r^{1,2}(D) \Leftrightarrow u \in X^3(D).$$

Remark 2.2. Sufficient conditions for the regularity in (2.2) are e.g.: $a_r, a_z \in C^2(\bar{D})$, $\partial a_r/\partial r = \partial a_z/\partial r = 0$ for $r = 0$, $f \in W_r^{1,2}(D)$, $g \in G(X^3(D))$ and the boundary can be described by means of functions from $C^{(4),1}$. This follows from Theorem 4.2.2 in [8], if we pass to the Cartesian coordinate system and use Remark 2.1.

Lemma 2.1. *Assume that the boundary Γ is straight in some neighbourhood of the points $\Gamma \cap \mathcal{O}$.*

Let w_1 be the solution of (2.1) with $f = 0$ and $g = -\partial u/\partial v_A$. Then $w_1 \in W_r^{2,2}(D)$ and

$$(2.3) \qquad \|w_1\|_{2,r,D} \leqq C\|u\|'_{3,r,D},$$

where

$$\|u\|'_{3,r,D} = \left(\|u\|_{3,r,D}^2 + \left\|\frac{1}{r}\frac{\partial^2 u}{\partial r\, \partial z}\right\|_{0,r,D}^2\right)^{1/2}.$$

Proof. 1[0] We can show that a function $\omega \in W_r^{2,2}(D)$ exists such that $G(\omega) = \partial u/\partial v_A$, the corresponding function $\tilde{\omega}$ (see Remark 2.1) belongs to $W^{2,2}(\Omega)$ and

$$(2.4) \qquad \|\omega\|_{2,r,D} \leqq C\|u\|_{3,r,D}.$$

To this end, we first decompose u in the following way. Let the part $\Gamma \cap \{(r, z) \mid r < d\}$ consists of straight segments, orthogonal to the axis $\mathcal{O}$. Let

$\phi \in C^\infty([0, \infty))$ be a function such that $\phi(r) = 1$ for $r \in [0, d/2]$ and $\phi(r) = 0$ for $r \geqq d$. We denote $u_1 = u\phi$, $u_2 = (1 - \phi)\, u$,

$$P_\varepsilon = \{(r, z) \mid r \leqq \varepsilon\}\,.$$

Then $u_2 = 0$ in $P_{d/2} \cap \bar{D}$, so that $u_2 \in W^{3,2}(D)$ and $\partial u_2/\partial v_A$ can be extended into D by a function

$$\omega_2 = a_r f_r \frac{\partial u_2}{\partial r} + a_z f_z \frac{\partial u_2}{\partial z}\,,$$

where f_r, f_z are functions from $C^2(\bar{D})$ such that $f_r = v_r, f_z = v_z$ on Γ (cf. Lemma 1.1). Then $\omega_2 \in W_r^{2,2}(D)$ and $G(\omega_2) = \partial u_2/\partial v_A$. Moreover, we have

$$\text{(2.5)} \qquad \|\omega_2\|_{2,r,D} \leqq C\|u\|_{3,r,D}$$

$$\left\| \frac{1}{r}\frac{\partial \omega_2}{\partial r} \right\|^2_{0,r,D} \leqq \frac{2}{d} \left\| \frac{\partial \omega_2}{\partial r} \right\|^2_{0,D} \leqq C\|u\|^2_{2,r,D}\,.$$

Making use of Remark 2.1, we conclude that the corresponding function $\tilde{\omega}_2 \in W^{2,2}(\Omega)$.

Next consider u_1. Obviously, $\operatorname{supp} u_1 \subset P_d \cap \bar{D}$, $\partial u_1/\partial v_A = \pm a_z(\partial u_1/\partial z)$. Then the latter derivative can be extended into D by a function

$$\omega_1 = a_z f_z \frac{\partial u_1}{\partial z}\,,$$

which belongs to $W_r^{2,2}(D)$ and $G(\omega_1) = \partial u_1/\partial v_A$. Moreover,

$$\text{(2.6)} \qquad \|\omega_1\|_{2,r,D} \leqq C\|u\|_{3,r,D}\,,$$

$$\left\| \frac{1}{r}\frac{\partial \omega_1}{\partial r} \right\|_{0,r,D} = \left(\int_{D \cap P_d} \left[\frac{1}{r} \frac{\partial(a_z f_z \phi)}{\partial r} \frac{\partial u}{\partial z} + \right.\right.$$

$$\left.\left. + a_z f_z \phi \frac{1}{r} \frac{\partial^2 u}{\partial r\, \partial z} \right]^2 r \,\mathrm{d}r \,\mathrm{d}z \right)^{1/2} \leqq C\|u\|'_{3,r,D}\,,$$

since $\partial(f_z\phi)/\partial r = 0$ in $P_{d/2}$ and $1/r\, |\partial a_z/\partial r| \leqq C$ can be deduced from the assumptions.

Then the corresponding function $\tilde{\omega}_1 \in W^{2,2}(\Omega)$. For the sum $\omega = \omega_1 + \omega_2$, we obtain $\omega \in W_r^{2,2}(D)$,

$$G(\omega) = G(\omega_1) + G(\omega_2) = \frac{\partial u_1}{\partial v_A} + \frac{\partial u_2}{\partial v_A} = \frac{\partial u}{\partial v_A}\,,$$

$$\|\omega\|_{2,r,D} \leqq \|\omega_1\|_{2,r,D} + \|\omega_2\|_{2,r,D} \leqq C\|u\|_{3,r,D}\,,$$

combining (2.5) and (2.6), $\tilde{\omega} = \tilde{\omega}_1 + \tilde{\omega}_2 \in W^{2,2}(\Omega)$.

2^0 Instead of (2.1) with $f = 0$ and $g = -\partial u/\partial v_A$, let us solve the corresponding Dirichlet problem in Ω. Since the boundary condition is given by the trace of $(-\tilde{\omega}) \in W^{2,2}(\Omega)$, the solution $U \in W^{2,2}(\Omega)$ and

$$\|U\|_{2,\Omega} \leqq C\|\tilde{\omega}\|_{2,\Omega}$$

(see [8] – Thm. 2.2.1). Passing to the cylindrical coordinates, we may set

$$U(r\cos\vartheta, r\sin\vartheta, z) = w_1(r, z)\,.$$

Using again [9] – Lemma 2.1, 2.2, we obtain

$$\|w_1\|_{2,r,D} \leqq (2\pi)^{-1}\,\|U\|_{2,\Omega} \leqq C_1\|\tilde{\omega}\|_{2,\Omega} =$$
$$= C_2\left(\|\omega\|^2_{2,r,D} + \left\|\frac{1}{r}\frac{\partial\omega}{\partial r}\right\|^2_{0,r,D}\right)^{1/2} \leqq$$
$$\leqq C_3(\|u\|_{3,r,D} + \|u\|'_{3,r,D}) \leqq C_4\|u\|'_{3,r,D}\,,$$

by virtue of (2.5) and (2.6). Q.E.D.

Lemma 2.2. *Let $p, \phi \in W^{1,2}_r(D)$, $y \in W^{s,2}_r(D)$, $s \geqq 2$, $h \in (0, 1]$ and $\gamma \geqq 1$. We define*

$$H_\gamma(p) = (|p|^2_{1,r,D} + \gamma h^{-1}\|p\|^2_{0,r,\Gamma})^{1/2}\,,$$
$$G_\gamma(\phi; y) = \left(|\phi - y|^2_{1,r,D} + \gamma h^{-1}\left\|\phi - y + \gamma^{-1}h\frac{\partial y}{\partial v_A}\right\|^2_{0,r,\Gamma}\right)^{1/2}.$$

Then

$$H_\gamma(y - y_I) \leqq Ch^{s-1}\|y\|_{s,r,D}\,, \tag{2.7}$$

where y_I is the element of Σ_h, approximating y in the sense of (1.4) and

$$\inf_{\phi\in\Sigma_h} G_\gamma(\phi; y) \leqq Ch(\|y\|_{s,r,D} + \|u\|_{2,r,D})\,. \tag{2.8}$$

Proof. Making use of Lemma 1.1 with $\varepsilon = h^{-1}$ and (1.4), we may write

$$H^2_\gamma(y - y_I) \leqq$$
$$\leqq |y - y_I|^2_{1,r,D} + \gamma h^{-1}(h^{-1}\|y - y_I\|^2_{0,r,D} + Ch\|y - y_I\|^2_{1,r,D}) \leqq$$
$$\leqq (1 + \gamma C)\,\|y - y_I\|^2_{1,r,D} + \gamma h^{-2}\|y - y_I\|^2_{0,r,D} \leqq$$
$$\leqq (C_1 h^{2s-2} + \gamma h^{2s-2})\,\|y\|^2_{s,r,D}\,,$$

so that (2.7) is verified.

By means of (2.7) and (1.4) we obtain

$$G^2_\gamma(y_I + \gamma^{-1}hw_{1I}; y) = |y_I + \gamma^{-1}hw_{1I} - \gamma^{-1}hw_1 + \gamma^{-1}hw_1 - y|^2_{1,r,D} +$$
$$+ \gamma h^{-1}\left\|y_I + \gamma^{-1}hw_{1I} - y + \gamma^{-1}h\frac{\partial u}{\partial v_A} - \gamma^{-1}h\frac{\partial u}{\partial v_A} + \gamma^{-1}h\frac{\partial y}{\partial v_A}\right\|^2_{0,r,\Gamma} \leqq$$
$$\leqq 3|y - y_I|^2_{1,r,D} + 3(\gamma^{-1}h)^2\,(|w_{1I} - w_1|^2_{1,r,D} + \|w_1\|^2_{1,r,D}) +$$
$$+ 3\gamma h^{-1}\left\{\|y - y_I\|^2_{0,r,\Gamma} + (\gamma^{-1}h)^2\,\|w_{1I} - w_1\|^2_{0,r,\Gamma} +\right.$$
$$\left. + 2(\gamma^{-1}h)^2\left(\left\|\frac{\partial u}{\partial v_A}\right\|^2_{0,r,\Gamma} + \left\|\frac{\partial y}{\partial v_A}\right\|^2_{0,r,\Gamma}\right)\right\} \leqq$$
$$\leqq 3H^2_\gamma(y - y_I) + 3(\gamma^{-1}h)^2\,H^2_\gamma(w_1 - w_{1I}) +$$

$$+ C\gamma^{-2}h^2(\|w_1\|^2_{1,r,D} + \|u\|^2_{2,r,D} + \|y\|^2_{2,r,D}) \leqq$$
$$\leqq C_1[(h^{2s-2} + h^2)\,\|y\|^2_{s,r,D} + h^2\|w_1\|^2_{2,r,D} + h^2\|u\|^2_{2,r,D}] \leqq$$
$$\leqq Ch^2(\|y\|^2_{s,r,D} + \|u\|^2_{2,r,D}) .$$

Since

$$\inf_{\phi\in\Sigma_h} G_\gamma(\phi; y) \leqq G_\gamma(y_I + \gamma^{-1}hw_{1I}; y) ,$$

the estimate (2.8) follows. Q.E.D.

The approximate solution by penalty method is defined as $v(\gamma) \in \Sigma_h$ such that

$$(2.9) \qquad a(v(\gamma), \Phi) + \gamma h^{-1} \langle v(\gamma) - g, \Phi\rangle = (f, \Phi)_0 \quad \forall\Phi \in \Sigma_h .$$

Remark. Note that (2.9) corresponds with the Ritz-Galerkin approximation of the boundary value problem (2.1), where the Dirichlet boundary condition is replaced by the following one

$$\gamma^{-1}h \frac{\partial u}{\partial v_A} + u = g \quad \text{on} \quad \Gamma .$$

3. SOME ERROR ESTIMATES

In the present section, we shall derive some auxiliary error estimates, which involve also the solution w_1 of the auxiliary problem (2.1) with $f = 0$ and $g = -\partial u/\partial v_A$.

Theorem 3.1. *Let the solution of* (2.1) *satisfy the assumption* (2.2) *and let* $w = \gamma^{-1}hw_1$. *Then a positive constant C exists such that*

$$(3.1) \qquad C\|v(\gamma) - u - w\|_{1,r,D} \leqq K_h(u, w) + \gamma^{-2}h^2\|u\|'_{3,r,D} ,$$

$$(3.2) \qquad C \left\|v(\gamma) - g - w + \gamma^{-1}h \frac{\partial w}{\partial v_A}\right\|_{0,r,\Gamma} \leqq (\gamma^{-1}h)^{1/2} K_h(u, w) ,$$

holds for all $h \in (0, 1/2]$, *where*

$$K_h(u, w) = \inf_{\psi\in\Sigma_h} \left\{|\psi - u - w|^2_{1,r,D} + \right.$$
$$\left. + \gamma h^{-1} \left\|\psi - g - w + \gamma^{-1} h \frac{\partial w}{\partial v_A}\right\|^2_{0,r,\Gamma}\right\}^{1/2} .$$

Proof. It is easy to see that for all $\Phi \in W^{1,2}_r(D)$

$$a(u, \Phi) = (f, \Phi)_0 + \left\langle \frac{\partial u}{\partial v_A}, \Phi \right\rangle ,$$

$$a(w, \Phi) = \gamma^{-1}ha(w_1, \Phi) = \gamma^{-1}h \left\langle \frac{\partial w_1}{\partial v_A}, \Phi \right\rangle .$$

By means of these relations we derive for $e = v(\gamma) - u$

$$a(e - w, \Phi) + \gamma h^{-1} \left\langle e - w + \gamma^{-1} h \frac{\partial w}{\partial v_A}, \Phi \right\rangle = 0 \quad \forall \Phi \in \Sigma_h .$$

Substituting

$$\Phi = e + u - \psi = (e - w) + (w + u - \psi)$$

and

$$\Phi = \left(e - w + \gamma^{-1} h \frac{\partial w}{\partial v_A}\right) + \left(w + u - \psi - \gamma^{-1} h \frac{\partial w}{\partial v_A}\right),$$

respectively, we obtain

$$a(e - w, e - w) + \gamma h^{-1} \left\| e - w + \gamma^{-1} h \frac{\partial w}{\partial v_A} \right\|^2_{0,r,\Gamma} =$$

$$= a(e - w, \psi - u - w) +$$

$$+ \gamma h^{-1} \left\langle e - w + \gamma^{-1} h \frac{\partial w}{\partial v_A}, \psi - u - w + \gamma^{-1} h \frac{\partial w}{\partial v_A} \right\rangle .$$

Denoting for brevity $\gamma^{-1} h \, \partial w / \partial v_A = B$, we may write

$$a_0 |e - w|^2_{1,r,D} + \gamma h^{-1} \|e - w + B\|^2_{0,r,\Gamma} \leqq$$

$$C |e - w|_{1,r,D} \, |\psi - u - w|_{1,r,D} +$$

$$+ \gamma h^{-1} \|e - w + B\|_{0,r,\Gamma} \, \|\psi - u - w + B\|_{0,r,\Gamma} \leqq$$

$$\leqq C_1 \{a_0 |e - w|^2_{1,r,D} + \gamma h^{-1} \|e - w + B\|^2_{0,r,\Gamma}\}^{1/2} \times$$

$$\times \{|\psi - u - w|^2_{1,r,D} + \gamma h^{-1} \|\psi - u - w + B\|^2_{0,r,\Gamma}\}^{1/2} .$$

Cancelling, we obtain

(3.3)
$$a_0 |e - w|^2_{1,r,D} + \gamma h^{-1} \|e - w + B\|^2_{0,r,\Gamma} \leqq$$
$$\leqq C \inf_{\psi \in \Sigma_h} \{|\psi - u - w|^2_{1,r,D} + \gamma h^{-1} \|\psi - u - w + B\|^2_{0,r,\Gamma}\} =$$
$$= C K_h^2(u, w) .$$

Using the Friedrichs inequality (1.1) and the inequalities (3.3) and

$$\gamma h^{-1} \geqq 2 \quad \forall h \in (0, 1/2] ,$$

we may write

$$C \|e - w\|^2_{1,r,D} \leqq$$

$$\leqq a_0 |e - w|^2_{1,r,D} + \gamma h^{-1} \|e - w + B\|^2_{0,r,\Gamma} + 2\|B\|^2_{0,r,\Gamma} \leqq$$

$$\leqq C_1 K_h^2(u, w) + 2\|B\|^2_{0,r,\Gamma} ,$$

so that

(3.4)
$$C_2 \|e - w\|_{1,r,D} \leqq K_h(u, w) + \|B\|_{0,r,\Gamma} .$$

Since

$$\left\| \frac{\partial w_1}{\partial v_A} \right\|_{0,r,\Gamma} \leqq C\|w_1\|_{2,r,D} \leqq C_1\|u\|'_{3,r,D}$$

holds by virtue of (2.3), we have

$$\|B\|_{0,r,\Gamma} \leqq \gamma^{-2}h^2C_1\|u\|'_{3,r,D}$$

and (3.1) follows from (3.4).

The estimate (3.2) is an immediate consequence of (3.3).

Corrolary 3.1. *Let the assumptions of Theorem* 3.1 *be satisfied. Then there are constants* $C_i(\gamma)$, $i = 1, 2$, *such that*

$$\|v(\gamma) - u - w(\gamma)\|_{1,r,D} \leqq C_1(\gamma)\, h^2\|u\|'_{3,r,D}\,, \tag{3.5}$$

$$\left\| v(\gamma) - g - w(\gamma) + \gamma^{-1}h\,\frac{\partial w}{\partial v_A} \right\|_{0,r,\Gamma} \leqq C_2(\gamma)\, h^{5/2}\|u\|'_{3,r,D} \tag{3.6}$$

hold for $h \leqq 1/2$.

Proof. Obviously, we have $u_I + \gamma^{-1}h\phi \in \Sigma_h$ for any $\phi \in \Sigma_h$, so that

$$K_h(u, w) = \inf_{\phi\in\Sigma_h} \Big\{ |u_I + \gamma^{-1}h\phi - u - w|^2_{1,r,D} +$$

$$+ \gamma h^{-1} \left\| u_I + \gamma^{-1}h\phi - u - w + \gamma^{-1}h\,\frac{\partial w}{\partial v_A} \right\|^2_{0,r,\Gamma} \Big\}^{1/2} \leqq$$

$$\leqq \inf_{\phi\in\Sigma_h} C \Big\{ |u_I - u|^2_{1,r,D} + (\gamma^{-1}h)^2|\phi - w_1|^2_{1,r,D} +$$

$$+ \gamma h^{-1}\left[\|u_I - u\|^2_{0,r,\Gamma} + (\gamma^{-1}h)^2 \left\| \phi - w_1 + \frac{\partial w}{\partial v_A} \right\|^2_{0,r,\Gamma} \right] \Big\}^{1/2} \leqq$$

$$\leqq C\{H^2_\gamma(u - u_I) + (\gamma^{-1}h)^2 \inf_{\phi\in\Sigma_h} G^2_\gamma(\phi, w_1)\}^{1/2} \leqq$$

$$\leqq C_1H_\gamma(u - u_I) + C_1\gamma^{-1}h \inf_{\phi\in\Sigma_h} G_\gamma(\phi, w_1)\,.$$

Making use of Lemma 2.2 and (2.3), we obtain the estimate

$$K_h(u, w) \leqq Ch^2\|u\|_{3,r,D} + C_2\gamma^{-1}h^2(\|w_1\|_{2,r,D} + \|u\|_{2,r,D}) \leqq$$

$$\leqq C_3h^2\|u\|'_{3,r,D}\,.$$

Now (3.5) and (3.6) is a consequence of (3.1) and (3.2), respectively.

4. EXTRAPOLATIONS AND A PRIORI ERROR ESTIMATES

Let γ_0 and γ_1 be two real numbers such that $1 \leqq \gamma_0 < \gamma_1$. Then it is readily seen for

$$a_0 = \gamma_0/(\gamma_0 - \gamma_1)\,, \quad a_1 = 1 - a_0\,,$$

that

(4.1) $$\sum_{i=0}^{1} a_i \gamma_i^{-1} = 0 .$$

Let us define the following extrapolate of approximate solutions

(4.2) $$u_h = \sum_{i=0}^{1} a_i v(\gamma_i)$$

and the extrapolate of boundary flux approximations

(4.3) $$e_h = - \sum_{i=0}^{1} a_i \gamma_i \, h^{-1}(v(\gamma_i) - g) ,$$

where $v(\gamma_i)$ is the approximate solution by penalty method, defined in (2.9) for $\gamma = \gamma_i$.

Theorem 4.1. *Let the solution of* (2.1) $u \in W_r^{3,2}(D)$ *and* $1/r(\partial^2 u/\partial r\, \partial z) \in L_r^2(D)$. *Then constants* C_1, C_2 *exist, depending on the parameters* γ_0, γ_1 *but not on* h *and* u, *such that*

$$\|u - u_h\|_{1,r,D} \leqq C_1 h^2 \|u\|'_{3,r,D} ,$$

$$\left\| \frac{\partial u}{\partial v_A} - e_h \right\|_{0,r,\Gamma} \leqq C_2 h^{3/2} \|u\|'_{3,r,D} .$$

Proof. By means of (4.1) and (3.5) we may write

$$\|u - u_h\|_{1,r,D} = \left\| \sum_{i=0}^{1} a_i \{u - v(\gamma_i) + \gamma_i^{-1} h w_1\} \right\|_{1,r,D} \leqq$$

$$\leqq \max_{i=0,1} |a_i| \sum_{i=0}^{1} \|w(\gamma_i) - e(\gamma_i)\|_{1,r,D} \leqq C_1(\gamma_0, \gamma_1)\, h^2 \|u\|'_{3,r,D} .$$

On the basis of (3.6) and recalling that

$$\gamma_i^{-1} h \frac{\partial u}{\partial v_A} = - w(\gamma_i) \quad \text{on} \quad \Gamma ,$$

we obtain

$$\left\| \frac{\partial u}{\partial v_A} - e_h \right\|_{0,r,\Gamma} =$$

$$= \left\| \sum_{i=0}^{1} a_i \left\{ \gamma_i h^{-1}(v(\gamma_i) - g + \frac{\partial u}{\partial v_A} + \gamma_i^{-1} h \frac{\partial w_1}{\partial v_A} \right\} \right\|_{0,r,\Gamma} \leqq$$

$$\leqq \max_{i=0,1} |a_i| \sum_{i=0}^{1} \gamma_i h^{-1} \left\| v(\gamma_i - g - w(\gamma_i) + \gamma_i^{-1} h \frac{\partial w(\gamma_i)}{\partial v_A} \right\|_{0,r,\Gamma} \leqq$$

$$\leqq C_2(\gamma_0, \gamma_1)\, h^{3/2} \|u\|'_{3,r,D} .$$

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Souhrn

METODA PENALTY A EXTRAPOLACE PRO OSOVĚ SYMETRICKÉ ELIPTICKÉ ÚLOHY S DIRICHLETOVOU OKRAJOVOU PODMÍNKOU

IVAN HLAVÁČEK

Osově symetrická eliptická úloha druhého řádu se řeší v prostoru konečných prvků na trojúhelnících s případně zakřivenou stranou a to tak, že nehomogenní okrajová podmínka je splněna pouze přibližně ve smyslu penalty. Na základě dvou přibližných řešení, která se liší pouze váhou u penaltního členu, jsou definovány extrapolace řešení, resp. vnějšího toku (tj. derivace podle konormály). Za předpokladu, že přesné řešení je dostatečně regulární, jsou odvozeny apriorní odhady chyby extrapolace.

Резюме

МЕТОД ШТРАФА И ЭКСТРАПОЛЯЦИИ ДЛЯ ОСЕСИММЕТРИЧЕСКИХ ЭЛЛИПТИЧЕСКИХ ЗАДАЧ С КРАЕВЫМ УСЛОВИЕМ ДИРИХЛЕ

IVAN HLAVÁČEK

Осесиметрическая задача второго порядка решается в пространстве конечных элементов, причем краевое условие удовлетворяется в смысле штрафа. На основе двух приближенных решений, которые отличаются только весом штрафного члена, определены экстраполяции для решения и для внешнего тока. Предполагая, что точное решение достаточно регулярно, выведены априорные оценки для ошибок экстраполяции.

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