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BIFURCATIONS OF GENERALIZED VON KÁRMÁN EQUATIONS FOR CIRCULAR VISCOELASTIC PLATES

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Summary. The paper deals with the analysis of generalized von Kármán equations which describe stability of a thin circular clamped viscoelastic plate of constant thickness under a uniform compressive load which is applied along its edge and depends on a real parameter, and gives results for the linearized problem of stability of viscoelastic plates. An exact definition of a bifurcation point for the generalized von Kármán equations is given. Then relations between the critical points of the linearized problem and the bifurcation points are analyzed.

Keywords: Von Kármán equations, viscoelastic plates, bifurcations.

AMS Classification: 35B32.

1. INTRODUCTION

We consider the axisymmetric deformation of a thin circular clamped viscoelastic plate of constant thickness under a uniform compressive load applied along its edge. We describe the behaviour of this plate by the generalized von Kármán equations which for our problem can be reduced to the system [2]

$$(1.1) \qquad (1 + \alpha D_t)\,[x^3 w'(x, t)]' = (1 + \beta D_t)\, x^3 w(x, t)\,[f(x, t) - \lambda]\,,$$

$$(1.2) \qquad (1 + \beta D_t)\,[x^3 f'(x, t)]' = -(1 + \alpha D_t)\, x^3 w^2(x, t)$$

$$x \in (0, 1)\,; \quad t \in (0, T\rangle\,; \quad T < \infty$$

where w is the space derivative of the transverse displacement of the plate, f is the space derivative of the stress function, λ is the positive parameter of proportionality of the given boundary loading, α, β are positive viscous parameters such that $\alpha > \beta$, D_t denotes the differentiation with respect to time, a prime denotes the differentiation with respect to the space variable. We consider the boundary conditions

$$(1.3) \qquad \left|w(x, t)|_{x=0}\right| < \infty \qquad t \in \langle 0, T\rangle$$

$$(1.4) \qquad \left|f(x, t)|_{x=0}\right| < \infty \qquad t \in \langle 0, T\rangle$$

(1.5) $\quad w(x, t)|_{x=1} = 0 \qquad t \in \langle 0, T \rangle$

(1.6) $\quad f(x, t)|_{x=1} = 0 \qquad t \in \langle 0, T \rangle$

and initial conditions

(1.7) $\quad w(x, t)|_{t=0^-} = 0 \qquad x \in \langle 0, 1 \rangle$

(1.8) $\quad f(x, t)|_{t=0^-} = 0 \qquad x \in \langle 0, 1 \rangle .$

The problem (1.1)–(1.8) can be reformulated to the operator form [2]

$$w(t) = \lambda \frac{\beta}{\alpha} L w(t) - C[w(t)] + \tag{1.9}$$

$$+ \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t \{w(\tau) + G[w(t), w^2(\tau)]\} K(t - \tau) \, d\tau ,$$

$$f(t) = \frac{\alpha}{\beta} B[w(t), w(t)] - \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t B[w(\tau), w(\tau)] K(t - \tau) \, d\tau \tag{1.10}$$

defined on the space $L_\infty(0, T; H)$ where H is the Hilbert space

$$H = \{u(x) \in W^{1,2}((0, 1); x^3) \mid u(1) = 0\}$$

with the inner product

$$\langle u, v \rangle = \int_0^1 x^3 u'(x) v'(x) \, dx \tag{1.11}$$

and the corresponding norm

$$\|u\|_H = [\langle u, u \rangle]^{1/2} . \tag{1.12}$$

The kernel K has the form

$$K(t - \tau) = \exp\left[-\frac{1}{\beta}(t - \tau)\right]$$

and

$$\langle Lu(t), \varphi \rangle = \int_0^1 x^3 u(x, t) \varphi(x) \, dx , \tag{1.13}$$

$$\langle B[u(t), v(t)], \varphi \rangle = \int_0^1 x^3 u(x, t) v(x, t) \varphi(x) \, dx , \tag{1.14}$$

$$C[u(t)] = B[u(t), B[u(t), u(t)]] , \tag{1.15}$$

$$G[u(t), u^2(\tau)] = B[u(t), B[u(\tau), u(\tau)]] \tag{1.16}$$

for a.e. $t \in \langle 0, T \rangle$ and for $\varphi \in H$, $u, v \in L_\infty(0, T; H)$. L is a linear bounded selfadjoint compact operator mapping H into itself for a.e. $t \in \langle 0, T \rangle$; B is a bilinear bounded symmetric compact operator defined on $H \times H$ with range in H for a.e. $t \in \langle 0, T \rangle$; C is a bounded compact operator mapping H into itself for a.e. $t \in \langle 0, T \rangle$.

Equations (1.9) and (1.10) are uncoupled in the sense that w can be determined independently of f. Thus it is sufficient to consider only (1.9) if we wish to determinate w.

2. THE LINEARIZED PROBLEM

For an analysis of the bifurcation problem we need to deal with the stability of the linearized problem of the viscoelastic plates, that is with the analysis of the linear equation

$$(2.1)\qquad (1 + \alpha D_t)\left[x^3 w'(x, t)\right]' + \lambda(1 + \beta D_t)\, x^3 w(x, t) = 0$$

$$x \in (0, 1)\,; \quad t \in (0, T\rangle\,; \quad T < \infty\,,$$

$$(2.2)\qquad \left|w(x, t)|_{x=0}\right| < \infty \qquad t \in \langle 0, T\rangle\,,$$

$$(2.3)\qquad w(x, t)|_{x=1} = 0 \qquad t \in \langle 0, T\rangle\,,$$

$$(2.4)\qquad w(x, t)|_{t=0^-} = 0 \qquad x \in \langle 0, 1\rangle\,.$$

In terms of our operator formulation the linearized problem leads to the operator equation

$$(2.5)\qquad w(t) = \lambda \frac{\beta}{\alpha} L w(t) + \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t w(\tau)\, K(t - \tau)\, \mathrm{d}\tau$$

defined on the space $L_\infty(0, T; H)$.

To analyze the linearized problem we consider perturbations of the initial condition and use the following concepts:

Definition 2.1. *Stability (of the solution of the given problem) is characterized by the fact that all perturbations tend to zero as time tends to infinity.*

Instability is characterized by the existence of perturbations which grow to infinity as timr tends to infinity.

Neutral stability is a limit case between stability and instability such that there exists a perturbation which, after being introduced, remains of constant amplitude in time.

Instant instability is a limit case between stability and instability such that there exists a perturbed solution which is finite for a finite time, but it tends to infinity for an arbitrary time as the parameter λ approaches from the left the limit point which we call the critical point.

In the proof of the next theorem we use the following results [3], [4].

Assertion 2.1. *The eigenvalues of the equation*

$$u = \varrho L u$$

are simple, fulfil the relation

$$J_1(\sqrt{\varrho_n}) = 0$$

where J_1 is the Bessel function of the first order, and form a sequence of discrete *numbers tending to* ∞. *The corresponding eigenfunctions are from the space* H *and they have the form*

$$u_n(x) = \frac{c_n}{x} J_1(\sqrt{(\varrho_n)}\, x) \quad x \in \langle 0, 1\rangle$$

where c_n *is a constant. Every* u_n *has exactly* $n - 1$ *simple internal zero points* ($s \in (0, 1)$ *is a simple internal zero point of* u_n *if* $u_n(s) = 0$ *and* $u_n'(s) \neq 0$).

Assertion 2.2. (Paley-Wiener theorem) *If there exists*

$$c = \lim_{t \to \infty} \int_0^t u(\tau) f(t - \tau)\, d\tau$$

and if $f(t)$ *is absolutely integrable, i.e.*

$$\int_0^\infty |f(\tau)|\, d\tau < \infty$$

and nonnegative then

$$c = \lim_{t \to \infty} u(t) \int_0^\infty f(\tau)\, d\tau\,.$$

Theorem 2.1. *The critical points* λ_n *of the linearized problem* (2.5) *coincide with the eigenvalues* λ_n^0 *of the problem* (2.5) *for time* $t = 0$. *At these points the solution of* (2.5) *is instantly unstable, and they form a sequence of discrete numbers tending to* ∞. *The eigenvalues* λ_n^∞ *of the problem* (2.5) *for time* $t = \infty$ *also form a sequence of discrete numbers tending to* ∞. *Between* λ_n^0 *and* λ_n^∞ *the relation*

$$\frac{\beta}{\alpha} \lambda_n^0 = \lambda_n^\infty \tag{2.6}$$

takes place.

Proof. Using the Fourier method we solve the linearized problem (2.5), i.e. we look for the solution in the form

$$w(x, t) = u(x)\, v(t)\,. \tag{2.7}$$

If we insert (2.7) into (2.5) we get

$$u \left\{ v(t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t v(\tau)\, K(t - \tau)\, d\tau \right\} = \lambda \frac{\beta}{\alpha}\, v(t)\, Lu\,.$$

Hence

$$u = \mu \frac{\beta}{\alpha} Lu\,, \tag{2.8}$$

$$v(t)\left(1 - \frac{\lambda}{\mu}\right) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t v(\tau)\, K(t - \tau)\, d\tau = 0 \tag{2.9}$$

where μ is a real parameter. Denoting

$$\varrho = \mu \frac{\beta}{\alpha}$$

we obtain from (2.8)

$$u = \varrho L u . \tag{2.10}$$

This equation is identical with the operator form of the linearized equation for stationary von Kármán equations. According to Assertion 2.1, it has nontrivial solutions only if $\varrho = \varrho_n$ where ϱ_n is one of the eigenvalues of the operator L. These eigenvalues form a sequence of discrete numbers tending to ∞. We denote the corresponding eigenfunctions by u_n. Then the eigenvalues μ_n of the operator $(\beta/\alpha) L$ satisfy

$$\mu_n = \frac{\alpha}{\beta} \varrho_n . \tag{2.11}$$

The homogeneous Volterra integral equation (2.9) has only the trivial solution in $L_\infty((0, T))$ (see e.g. [5]). Therefore we consider a nonhomogeneous initial perturbation

$$w(x, t)\big|_{t=0} = \psi(x) = c\, u(x) \neq 0 \quad x \in \langle 0, 1) - M$$

where mes $M = 0$. Then, instead of (2.5), we get

$$w(t) - \lambda \frac{\beta}{\alpha} L w(t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t w(\tau) K(t - \tau) \, d\tau = \left(\psi - \frac{\beta}{\alpha} \lambda L \psi\right) \exp\left(-\frac{1}{\beta} t\right). \tag{2.12}$$

If we insert (2.7) into (2.12) we get (2.8). But (2.8) has nontrivial solutions only for μ_n satisfying (2.11). Hence for $n = 1, 2, \ldots$ we arrive at

$$v_n(t)\left(1 - \frac{\lambda}{\mu_n}\right) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t v_n(\tau) K(t - \tau) \, d\tau = c_n\left(1 - \frac{\lambda}{\mu_n}\right) \exp\left(-\frac{1}{\beta} t\right) \tag{2.13}$$

where c_n is the Fourier coefficient of the initial perturbation according to u_n. If $\lambda = \mu_n$ the equation (2.13) is a homogeneous Volterra integral equation of the first kind and has only the trivial solution (see e.g. [5]), and so the only solution of (2.12) is the trivial one. If $\lambda \neq \mu_n$ then the solution of (2.12) is

$$w(x, t) = \sum_{n=1}^{\infty} c_n\, u_n(x) \exp\left\{-\frac{1}{\beta} \frac{\frac{\beta}{\alpha} \mu_n - \lambda}{\mu_n - \lambda} t\right\}. \tag{2.14}$$

On the other hand, the equation (2.5) has for $t = 0$ the form

$$w(x, 0) = \lambda \frac{\beta}{\alpha} Lw(x, 0) .$$

A comparison with (2.8) gives

$$\lambda_n^0 = \mu_n . \tag{2.15}$$

Further, using Assertion 2.2 we obtain from (2.5) for time $t = \infty$

$$w(x, \infty) = \lambda Lw(x, \infty) .$$

Thus from (2.10) and (2.11) we have

$$\lambda_n^\infty = \frac{\beta}{\alpha} \mu_n . \tag{2.16}$$

From (2.15) and (2.16) we obtain the validity of (2.6). If we use (2.15) and (2.16) we get from (2.14)

$$w(x, t) = \sum_{n=1}^{\infty} c_n u_n(x) \exp\left\{-\frac{1}{\beta}\frac{\lambda_n^\infty - \lambda}{\lambda_n^0 - \lambda} t\right\} . \tag{2.17}$$

Thus, when we consider the perturbation of the initial condition in the form of the eigenfunction u_n, then for $\lambda < \lambda_n^\infty$ we have

$$\lim_{t\to\infty} w_n(x, t) = 0 \quad x \in \langle 0, 1\rangle$$

and the solution is stable. For $\lambda = \lambda_n^\infty$ we have

$$w_n(x, t) = c_n u_n(x) \quad x \in \langle 0, 1\rangle ; \quad t \in \langle 0, T\rangle$$

and the solution is neutrally stable. For $\lambda_n^\infty < \lambda < \lambda_n^0$ we have

$$\lim_{t\to\infty} w_n(x, t) = \pm \infty \quad x \in \langle 0, 1) - I_n$$

where I_n denotes the set of $n - 1$ internal zero points of the eigenfunction u_n (see Assertion 2.1). Then the solution is unstable. For $\lambda = \lambda_n^0$ the equation (2.12) has only the trivial solution because in this case (2.13) is a homogeneous Volterra integral equation of the first kind (see e.g. [5]) but

$$\lim_{\lambda\to\lambda^0{}_n-} w_n(x, t) = \pm\infty \quad t \in (0, T\rangle ; \quad x \in \langle 0, 1) - I_n ,$$

hence in this case the solution is instantly unstable. For $\lambda > \lambda_n^0$ we have

$$\lim_{t\to\infty} w_n(x, t) = 0 \quad x \in \langle 0, 1\rangle$$

and the solution is stable.

Thus from (2.17) we have for $n = 1, 2, \ldots$

$$\lim_{\lambda \to \lambda^0{}_n^-} w(x, t) = \pm\infty \quad t \in (0, T\rangle\,; \quad x \in \langle 0, 1) - I$$

where

$$I = \bigcup_{n=1}^{\infty} I_n\,,$$

which is a countable set. But on the other hand (2.17) implies that for $\lambda = \lambda_n^0$, $n = 1, 2, \ldots$ the solution $w(x, t)$ is finite for $x \in \langle 0, 1\rangle$; $t \in (0, T\rangle$. Then at the points λ_n^0, $n = 1, 2, \ldots$ the solution of (2.5) is instantly unstable and thus these points are critical points of (2.5).

Corollary 2.1. *If we consider the perturbation of the initial condition in the form of the eigenfunction u_n then the solution of (2.5) is neutrally stable at the point λ_n^∞ and instantly unstable at the point λ_n^0.*

3. BIFURCATION POINTS

Now we introduce a concept of a bifurcation point from the trivial solution for generalized von Kármán equations for time $t = 0$.

We now introduce the solution $w(x, t)$ as the function not only the space and time variable of but also as the function of the parameter λ, i.e. $w(x, t, \lambda)$.

Definition 3.1. *For a given λ the value*

$$w(x, 0, \lambda) \quad x \in \langle 0, 1\rangle$$

is called the starting point of $w(x, t, \lambda)$. This point is called the zero starting point if

$$w(x, 0, \lambda) = 0$$

for all $x \in \langle 0, 1\rangle$, and the nonzero starting point if

$$w(x, 0, \lambda) \neq 0$$

for $x \in \langle 0, 1\rangle - I$ where $\operatorname{mes} I = 0$.

Definition 3.2. *A point $\lambda = \lambda_{cr}$ is a bifurcation point of the problem (1.1)–(1.8) from the trivial solution for time $t = 0$ if:*

a) $\exists\, \varepsilon > 0$ *such that for all $\lambda \in (\lambda_{cr}, \lambda_{cr} + \varepsilon)$ there exist nonzero starting points $w(x, 0, \lambda)$ of the nontrivial solutions $w(x, t, \lambda)$;*

b) $\lim_{\lambda \to \lambda_{cr}^+} w(x, 0, \lambda) = w(x, 0, \lambda_{cr}) = 0$;

c) the zero starting point $w(x, 0, \lambda_{cr})$ is the starting point of a nontrivial *solution.*

If the solutions from parts a) and c) are from the space $C^1(\langle 0, T\rangle; C^2(\langle 0, 1\rangle))$ (from the space of the classical solutions of the problem (1.1)–(1.8), see [2]) we call λ_{cr} a bifurcation point of type I, while if these solutions are solutions with small norm in the space $L_\infty(0, T; H)$ (i.e. sufficiently small for our next study of the problem (1.1)–(1.8)) we call it a bifurcation point of type II.

We will deal with the bifurcation points of type I in the paper "Analysis of Post-buckling Solutions of Generalized von Kármán Equations for Circular Viscoelastic Plates". Now we prove two important theorems about the bifurcation points of type II.

Theorem 3.1. *The bifurcation points of type I and also of type II of the generalized von Kármán equations with respect to the trivial solution for time $t = 0$ can occur only at the critical points of the linearized problem.*

Proof. We show that if λ_0 is not a critical point of the linearized problem, there exists an interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ such that the only solution with small norm in the space $L_\infty(0, T; H)$ of the problem (1.1)–(1.8) is the trivial solution. To this end, it is sufficient to use the operator formulation (1.9) of the problem. All norms in this proof are in $L_\infty(0, T; H)$:

$$\begin{aligned} (3.1)\qquad & \left\| w(t) - \lambda \frac{\beta}{\alpha} L w(t) + C[w(t)] - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t \{w(\tau) + \right. \\ & \left. + G[w(t), w^2(\tau)]\} K(t - \tau)\, d\tau \right\| \geqq \left\| w(t) - \lambda_0 \frac{\beta}{\alpha} L w(t) - \right. \\ & \left. - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t w(\tau) K(t - \tau)\, d\tau \right\| - |\lambda - \lambda_0| \frac{\beta}{\alpha} \|L w(t)\| - \\ & - \|C[w(t)]\| - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \left\| \int_0^t G[w(t), w^2(\tau)] K(t - \tau)\, d\tau \right\| . \end{aligned}$$

Now as λ_0 is not a critical point of the linearized problem, there exists a positive constant k_1 independent of w and such that

$$(3.2)\qquad \left\| w(t) - \lambda_0 \frac{\beta}{\alpha} L w(t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t w(\tau) K(t - \tau)\, d\tau \right\| \geqq k_1 \|w(t)\| .$$

As L is a bounded operator, there exists a positive constant k_2 independent of w and such that

$$(3.3)\qquad \|L w(t)\| \leqq k_2 \|w(t)\| .$$

From (1.14)–(1.16) we obtain existence of a positive constant k_3 independent of w and such that

$$(3.4)\qquad \|C[w(t)]\| \leqq k_3 \|w(t)\|^3$$

and

(3.5) $$\|\int_0^t G[w(t), w^2(\tau)]\, K(t-\tau)\,\mathrm{d}\tau\| \leqq \beta k_3 \|w(t)\|^3 .$$

Thus from (3.1)–(3.5) by choosing $|\lambda - \lambda_0|$ and $\|w(t)\|$ sufficiently small we obtain existence of a positive constant k_4 independent of w and such that

$$\left\| w(t) - \lambda \frac{\beta}{\alpha} L w(t) + C[w(t)] - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t \{w(\tau) + \right.$$

$$\left. + G[w(t), w^2(\tau)]\} K(t-\tau)\,\mathrm{d}\tau \right\| \geqq k_4 \|w(t)\| .$$

Thus λ_0 cannot be a bifurcation point of the problem (1.1)–(1.8).

Theorem 3.2. *The critical points of the linearized problem are bifurcation points of type II for the problem* (1.1)–(1.8).

Proof. Let

(3.6) $$Q[w(t)] = w(t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t w(\tau)\, K(t-\tau)\,\mathrm{d}\tau ,$$

then we can rewrite (1.9) in the form

(3.7) $$Q[w(t)] - \lambda \frac{\beta}{\alpha} L w(t) + B[w(t), Q[B[w(t), w(t)]]] = 0 .$$

Let $u_n(x)$ be the eigenfunction corresponding to λ_n, which according to Assertion 2.1 belongs to the space H. Let the subspace spanned by u_n be denoted by $[u_n]$ and its orthogonal complement in H by H_n. For a.e. $t \in \langle 0, T \rangle$ let P_n be the projector of H onto H_n.

Thus the totality of solutions of (3.7) in the neighbourhood of λ_n can be obtained for a.e. $t \in \langle 0, T_n \rangle$, where T_n is sufficiently small, by solving the equations

(3.8) $$P_n \left\{ Q[w(t)] - \lambda \frac{\beta}{\alpha} L w(t) + B[w(t), Q[B[w(t), w(t)]]] \right\} = 0 ,$$

(3.9) $$\left\langle Q[w(t)] - \lambda \frac{\beta}{\alpha} L w(t) + B[w(t), Q[B[w(t), w(t)]]] ,\, u_n \right\rangle = 0 .$$

Any solution w of (3.8) and (3.9) can be written in the form

(3.10) $$w(x, t) = y(x, t) + \varepsilon_n(t)\, u_n(x)$$

where $y \in L_\infty(0, T_n; H_n)$, $\varepsilon_n \in L_\infty(0, T_n)$.

Inserting (3.10) into (3.8) and (3.9) we get

$$(3.11)\qquad y(t) = \left(\lambda \frac{\beta}{\alpha} L - Q\right)^{-1} P_n\{B[w(t), Q[B[w(t), w(t)]]]\},$$

$$(3.12)\qquad \left(\frac{\lambda}{\lambda_n} - Q\right) \varepsilon_n(t) = \langle B[w(t), Q[B[w(t), w(t)]]], u_n\rangle .$$

The operator $(\lambda(\beta/\alpha) L - Q)^{-1}$ from (3.11) is defined for functions in the space $L_\infty(0, T_n; H_n)$ also for $\lambda = \lambda_n$.

Now we show that (3.11) is uniquely solvable for y in the term of ε_n and that the estimate

$$(3.13)\qquad \|y\|_{L_\infty(0,T_n;H)} \leqq k\|\varepsilon_n\|^3_{L_\infty(0,T_n)}$$

holds where k is a positive constant indepedent of ε_n. We assume that

$$\|y\|_{L_\infty(0,T_n;H)} \quad \text{and} \quad \|\varepsilon_n\|_{L_\infty(0,T_n)}$$

are small compared to 1 and that

$$(3.14)\qquad \|y\|_{L_\infty(0,T_n;H)} \leqq \|\varepsilon_n\|_{L_\infty(0,T_n)} .$$

We denote

$$T_\varepsilon[y(t)] = \left(\lambda \frac{\beta}{\alpha} L - Q\right)^{-1} P_n\{B[y(t) + \varepsilon_n(t)\, u_n ,$$
$$Q[B[y(t) + \varepsilon_n(t) u_n,\ y(t) + \varepsilon_n(t)\, u_n]]]\} .$$

Let $\bar{y} \in L_\infty(0, T_n; H_n)$ with

$$(3.15)\qquad \|\bar{y}\|_{L_\infty(0,T_n;H)} \leqq \|\varepsilon_n\|_{L_\infty(0,T_n)}$$

and let

$$\bar{w}(x, t) = \bar{y}(x, t) + \varepsilon_n(t)\, u_n(x) .$$

Then using (3.14), (3.15) and (1.14) we have

$$\|T_\varepsilon[y] - T_\varepsilon[\bar{y}]\|_{L_\infty(0,T_n;H)} \leqq k_1 \|\varepsilon_n\|^2_{L_\infty(0,T_n)} \|y - \bar{y}\|_{L_\infty(0,T_n;H)}$$

where k_1 is a positive constant independent of w and $\bar{w}$. Thus for sufficiently small $\|\varepsilon_n\|_{L_\infty(0,T_n)}$ it is possible to find a positive $\gamma < 1$ such that

$$\|T_\varepsilon[y] - T_\varepsilon[\bar{y}]\|_{L_\infty(0,T_n;H)} \leqq \gamma \|y - \bar{y}\|_{L_\infty(0,T_n;H)} .$$

Thus the solution of

$$(3.16)\qquad T_\varepsilon[y] = y$$

exists and is unique, i.e. if $\|\varepsilon_n\|_{L_\infty(0,T_n)}$ is sufficiently small, y is uniquely determined by ε_n. Moreover, the solution of (3.16) with (3.14) satisfies

$$\|y\|_{L_\infty(0,T_n;H)} = \|T_\varepsilon[y] - T_0[y] + T_0[y]\|_{L_\infty(0,T_n;H)} \leqq k\|\varepsilon_n\|^3_{L_\infty(0,T_n;H)} .$$

Now we deal with (3.12). First we show that for $\|\varepsilon_n\|_{L_\infty(0,T_n)}$ sufficiently small the solutions of (3.12) can be completely described for a.e. $t \in \langle 0, T_n \rangle$ by studying the solutions of a simpler equation

$$(3.17) \qquad \left(\frac{\lambda}{\lambda_n} - Q\right) \varepsilon_n(t) = \langle B[\varepsilon_n(t)\, u_n, Q[B[\varepsilon_n(t)\, u_n, \varepsilon_n(t)\, u_n]]], u_n \rangle .$$

To this end we use the following inequality for a.e. $t \in \langle 0, T_n \rangle$:

$$|\langle B[w(t), Q[B[w(t), w(t)]]] -$$
$$- B[\varepsilon_n(t)\, u_n, Q[B[\varepsilon_n(t)\, u_n, \varepsilon_n(t)\, u_n]]], u_n \rangle| \leqq k_2 \|\varepsilon_n\|^3_{L_\infty(0,T_n)},$$

where k_2 is a positive constant independent of w. Thus for $\|\varepsilon_n\|_{L_\infty(0,T_n)}$ sufficiently small, (3.12) can be written in the form

$$\left(\frac{\lambda}{\lambda_n} - Q\right) \varepsilon_n(t) = \langle B[\varepsilon_n(t)\, u_n, Q[B[\varepsilon_n(t)\, u_n, \varepsilon_n(t)\, u_n]]] +$$
$$+ B[w(t), Q[B[w(t), w(t)]]] - B[\varepsilon_n(t)\, u_n, Q[B[\varepsilon_n(t)\, u_n, \varepsilon_n(t)\, u_n]]], u_n \rangle =$$
$$= \langle B[\varepsilon_n(t)\, u_n, Q[B[\varepsilon_n(t)\, u_n, \varepsilon_n(t)\, u_n]]], u_n \rangle \{1 + o(1)\} .$$

Then (3.17) can be rewritten in the form

$$(3.18) \qquad \left(\frac{\lambda}{\lambda_n} - Q\right) \varepsilon_n(t) = \varepsilon_n(t)\, Q[\varepsilon_n^2(t)] \langle C[u_n], u_n \rangle = d_n\, \varepsilon_n(t)\, Q[\varepsilon_n^2(t)]$$

where

$$d_n = \langle C[u_n], u_n \rangle = \langle B[u_n, u_n], B[u_n, u_n] \rangle = \|B[u_n, u_n]\|_H^2 > 0 .$$

Now we analyze (3.18). Using (3.6) we have

$$(3.19) \qquad \varepsilon_n^3(t) = \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \varepsilon_n(t) \int_0^t \varepsilon_n^2(\tau)\, K(t - \tau)\, d\tau +$$
$$+ \frac{1}{d_n}\frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t \varepsilon_n(\tau)\, K(t - \tau)\, d\tau + \frac{1}{d_n}\left(\frac{\lambda}{\lambda_n} - 1\right) \varepsilon_n(t) .$$

From [1] it is obvious that there exists $\delta > 0$ such that for $\lambda \in (\lambda_n - \delta, \lambda_n)$ (3.19) has only the trivial solution, for $\lambda \in (\lambda_n, \lambda_n + \delta)$ in addition to the trivial solution there exist exactly two nontrivial symmetric solutions of (3.19) starting from the nonzero starting points, for $\lambda = \lambda_n$ in addition to the trivial solution there exist at least two nontrivial symmetric solutions of (3.19) starting from the zero starting point. All these solutions are from the space $C^\infty((0, T_n \rangle)$.

Now we show that for $\lambda = \lambda_n$ the equation (3.19) has in addition to the trivial solution exactly two nontrivial solutions. For (3.19) with $\lambda = \lambda_n$ and for nontrivial solutions ε_n we have

$$(3.20)\qquad \varepsilon_n^3(t) = \left(1 - \frac{\beta}{\alpha}\right)\left\{1 - \exp\left(-\frac{1}{\beta}t\right)\right\}\varepsilon_n^2(\xi_1)\,\varepsilon_n(t) + $$
$$+ \frac{1}{d_n}\left(1 - \frac{\beta}{\alpha}\right)\left\{1 - \exp\left(-\frac{1}{\beta}t\right)\right\}\varepsilon_n(\xi_2)$$

where $\xi_1, \xi_2 \in (0, t\rangle$. Because $\varepsilon_n(t)$ is nonzero for $t \in (0, T_n\rangle$ (see [1]), there exist nonzero constants m_1 and m_2 such that

$$\varepsilon_n^2(\xi_1) = m_1\,\varepsilon_n^2(t)\,,$$
$$\varepsilon_n(\xi_2) = m_2\,\varepsilon_n(t)\,.$$

Then (3.20) implies

$$\varepsilon_n^3(t)\left\{1 - \left(1 - \frac{\beta}{\alpha}\right)\left[1 - \exp\left(-\frac{1}{\beta}t\right)\right]m_1\right\} =$$
$$= \varepsilon_n(t)\,\frac{1}{d_n}\frac{m_2}{m_1}\left(1 - \frac{\beta}{\alpha}\right)\left\{1 - \exp\left(-\frac{1}{\beta}t\right)\right\}$$

and

$$\varepsilon_n^2(t) = \frac{1}{d_n}\frac{m_2}{m_1}\,\frac{\left(1 - \frac{\beta}{\alpha}\right)\left\{1 - \exp\left(-\frac{1}{\beta}t\right)\right\}}{1 - \left(1 - \frac{\beta}{\alpha}\right)\left\{1 - \exp\left(-\frac{1}{\beta}t\right)\right\}m_1}\,.$$

Thus (3.20) has at most two nontrivial solutions but according to [1] it has exactly two nontrivial solutions. This is completes the proof of the theorem.

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Súhrn

BIFURKÁCIE ZOVŠEOBECNENÝCH VON KÁRMÁNOVYCH ROVNÍC PRE KRUHOVÉ VÄZKOPRUŽNÉ DOSKY

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Článok sa zaoberá analýzou zovšeobecnených von Kármánovych rovníc popisujúcich stabilitu tenkej kruhovej väzkopružnej dosky na okraji upevnenej a radiálne symetricky zaťaženej. V článku je zavedený pojem bifurkácie pre zovšeobecnené von Kármánove rovnice a sú skúmané vzťahy medzi kritickými bodmi linearizovanej úlohy a bodmi bifurkácii.

Резюме

БИФУРКАЦИИ ОБОБЩЕННЫХ УРАВНЕНИЙ ФОН КАРМАНА ДЛЯ КРУГЛЫХ ВЯЗКОУПРУГИХ ПЛАСТИНОК

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Рассматриваются обобщенные уравнения фон Кармана для осесимметричного изгиба тонкой кругой жестко защемленной вязкопупругой пластинки постоянной толщины подвергающейся по своему контуру действию равномерных сжимающих усилий, интенсивность которых пропорциональна вещественному параметру. Определяется точка бифуркации для обобщенных уравнений фон Кармана. Исследуются соотношения между критическими точками линеаризованной задачи и точками бифуркации.

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