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# ITERATIVE SOLUTION OF EIGENVALUE PROBLEMS FOR NORMAL OPERATORS

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*Summary.* We will discuss Kellogg's iterations in eigenvalue problems for normal operators. A certain generalisation of the convergence theorem is shown.

*Keywords:* Eigenvalue problem, normal operator, Kellogg's iteration.

*AMS Classification:* 49G20, 47B15.

A certain variant of the iterative process of Kellogg's type for solving the eigenvalue problem is investigated. We will search a solution of the equation

(1) $$Tx = \lambda x$$

in a complex Hilbert space $\mathbf{X}$ for a normal operator. The previous results of Kolomý [3] and Marek [4] are used. It is shown that the convergence of a certain iterative process, which includes those studied earlier, is guaranteed under more general assumption on the properties of the operator. Some terms of the theory of spectral representation for normal operators are used [1].

Let $(\cdot, \cdot)$ denote the scalar product in $\mathbf{X}$, the norm being defined as usual, $\|\cdot\|_{\mathbf{X}} = (\cdot, \cdot)^{1/2}$. Let $[\mathbf{X}]$ be the space of linear bounded operators on $\mathbf{X}$, $\|T\|_{\mathbf{X}} = \sup_{\|x\|_{\mathbf{X}}=1} \|Tx\|$. If there is no danger of misunderstanding the indices will be omitted. Similarly, the braces $\{\cdot\}$ denote sequences as well as sets. Let $\Pi$ be the open complex plane, we denote the spectrum of the operator $T \in [\mathbf{X}]$ by $\sigma(T)$; the spectral radius of $T$ is denoted by $r(T)$. Let the spectral radius circle be the set of $\lambda \in \Pi$ for which $|\lambda| = r(T)$. We will denote it by the letter $\omega$. Further, for the operator $T \in [\mathbf{X}]$ we define its adjoint operator $T^*$ for which $(Tx, y) = (x, T^*y)$ holds for every $x, y \in \mathbf{X}$. The operator $T$ is said to be self-adjoint if $T = T^*$, and normal if $T^*T = TT^*$. Let $\mathrm{G(S)}$ be the set of functions which consists of all limits of uniformly convergent sequences of finite linear combinations of characteristic functions of

Borelian sets in $S \in \Pi$ such that their essential supremums are finite. Then for $f \in G(S)$

$$(2) \qquad f(T) = \int_S f(\lambda)\, E(d\lambda)$$

holds, where $E$ is the spectral measure of the normal operator $T$ [1].

Iterations are constructed similarly as in [4] by Marek:

$$(3) \qquad x^{(n+1)} = Tx^{(n)}\,, \quad x_n = \frac{x^{(n)}}{k_n}$$

$$\mu_r = \frac{(x^{(n+1)}, y_n)}{(x^{(n)}, z_n)}$$

where $x^{(0)} \in X$, the sequences $\{y_n\}$, $\{z_n\}$ of elements of $X$ and the number sequence $\{k_n\}$ are such that the denominators in (3) are not equal to zero and

$$(4) \qquad \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = y$$

where $y \in X$. The following theorem describes the behaviour of the sequences (3). Note that no assumptions about the neighbourhood of the spectral radius circle are made.

**Theorem.** *Let $T \in [X]$ be a normal operator and $x^{(0)} \in X$ a fixed vector such that*

$$(E(\omega)\, x^{(0)}, y) = (E(\{\mu_0\})\, x^{(0)}, y) \neq 0\,,$$

*where $\mu_0 \in \omega \cap \sigma(T)$ and $E$ is the spectral measure generated by $T$. Let (4) hold for $y_n$, $z_n$ and $y \in X$. Further, let $k_n$ be such that*

$$(5) \qquad \lim_{n\to\infty} \prod_{i=1}^{n} \mu_0 k_n^{-1} = \beta \neq 0\,, \quad |\beta| < \infty\,.$$

*We denote $x_0 = \beta\, E(\{\mu_0\})\, x^{(0)}$. Then*

$$(6) \qquad \lim_{n\to\infty} \mu_n = \mu_0 \quad \textit{and} \quad \lim_{n\to\infty} x_n = x_0\,,$$

*where $x_0$ is the eigenvector of $T$ corresponding to the eigenvalue $\mu_0$.*

Proof. We begin with the proof of the convergence of $\mu_n$. Because $x^{(n+1)} = Tx^{(n)}$ we have

$$(x^{(n)}, y_n) = (T^n x^{(0)}, y_n)$$

due to (2) and by virtue of (3),

$$(x^{(n)}, y_n) = \int_S \lambda^n (E(d\lambda)\, x^{(0)}, y_n) = \mu_0^n \int_S (\lambda/\mu_0)^n\, (E(d\lambda)\, x^{(0)}, y_n)$$

holds, where S is such that $\sigma(\boldsymbol{T}) \subset$ S. Further, we have

$$\begin{aligned}
&\int_S (\lambda/\mu_0)^n (\boldsymbol{E}(\mathrm{d}\lambda)\, \boldsymbol{x}^{(0)}, \boldsymbol{y}_n) = \int_\omega (\lambda/\mu_0)^n (\boldsymbol{E}(\mathrm{d}\lambda)\, \boldsymbol{x}^{(0)}, \boldsymbol{y}_n) + \\
(7) \qquad &+ \int_{S-\omega} (\lambda/\mu_0)^n (\boldsymbol{E}(\mathrm{d}\lambda)\, \boldsymbol{x}^{(0)}, \boldsymbol{y}_n) = (\boldsymbol{E}(\{\mu_0\})\, \boldsymbol{x}^{(0)}, \boldsymbol{y}_n) + \\
&+ \int_{\omega-\{\mu_0\}} (\lambda/\mu_0)^n (\boldsymbol{E}(\mathrm{d}\lambda)\, \boldsymbol{x}^{(0)}, \boldsymbol{y}_n) + \int_{S-\omega} (\lambda/\mu_0)^n (\boldsymbol{E}(\mathrm{d}\lambda)\, \boldsymbol{x}^{(0)}, \boldsymbol{y}_n) .
\end{aligned}$$

The last but one term in (7) converges to zero, because $|(\lambda/\mu_0)^n| \leqq 1$ and $(\boldsymbol{E}(\omega - \{\mu_0\})\, \boldsymbol{x}^{(0)}, \boldsymbol{y}_n)$ converges to $(\boldsymbol{E}(\omega - \{\mu_0\})\, \boldsymbol{x}^{(0)}, \boldsymbol{y})$, which equals zero, too. The last term in (7) converges to zero due to the inequality $|(\lambda/\mu_0)^n| < 1$. The Lebesgue dominant convergence theorem has been used in both cases [1]. Hence

$$\lim_{n\to\infty} \frac{(\boldsymbol{x}^{(n+1)}, \boldsymbol{y}_n)}{(\boldsymbol{x}^{(n)}, \boldsymbol{y}_n)} = \lim_{n\to\infty} \mu_0 \frac{(\boldsymbol{E}(\{\mu_0\})\, \boldsymbol{x}^{(0)}, \boldsymbol{y}_n)}{(\boldsymbol{E}(\{\mu_0\})\, \boldsymbol{x}^{(0)}, \boldsymbol{y}_n)} = \mu_0 ,$$

where the last equality is obvious owing to the assumption that $(\boldsymbol{E}(\{\mu_0\})\, \boldsymbol{x}^{(0)}, \boldsymbol{y}) \neq 0$. To show the convergence of $\boldsymbol{x}_n$, we have by its definition

$$\boldsymbol{x}_n = \boldsymbol{x}^{(n)}/\mathrm{k}_n = \frac{\boldsymbol{T}^n \boldsymbol{x}^{(0)}}{\mathrm{k}_1 \ldots \mathrm{k}_n} = \frac{\boldsymbol{T}^n \boldsymbol{x}^{(0)}}{\mu_0^n} \prod_{i=1}^{n} \mu_0/\mathrm{k}_i .$$

Similarly as in the first part of this proof

$$\frac{1}{\mu_0^n} \boldsymbol{T}^n \boldsymbol{x}^{(0)} = \int_S (\lambda \mu_0^{-1})^n \boldsymbol{E}(\mathrm{d}\lambda)\, \boldsymbol{x}^{(0)}$$

and using $\beta$ of the assumptions, we obtain that

$$\lim_{n\to\infty} \boldsymbol{x}_n = \lim_{n\to\infty} \beta \int_S (\lambda \mu_0^{-1})^n \boldsymbol{E}(\mathrm{d}\lambda)\, \boldsymbol{x}^{(0)} = \beta \boldsymbol{E}(\{\mu_0\})\, \boldsymbol{x}^{(0)} = \boldsymbol{x}_0$$

due to the Lebesgue dominant convergence theorem again. Finally, we show that $\mu_0$ is an eigenvalue with the corresponding eigenvector $\boldsymbol{x}_0$. We have

$$\begin{aligned}
(\boldsymbol{T} - \mu_0 \boldsymbol{I})\, \boldsymbol{x}_0 &= \beta(\boldsymbol{T} - \mu_0 \boldsymbol{I})\, \boldsymbol{E}(\{\mu_0\})\, \boldsymbol{x}^{(0)} = \\
&= \beta \lim_{n\to\infty} \int_S (\lambda \mu_0^{-1})^n (\lambda - \mu_0)\, \boldsymbol{E}(\mathrm{d}\lambda)\, \boldsymbol{x}^{(0)}
\end{aligned}$$

and as $(\lambda \mu_0^{-1})^n (\lambda - \mu_0)$ is bounded and tends to zero if $n$ tends to infinity, $\boldsymbol{T}\boldsymbol{x}_0 = \mu_0 \boldsymbol{x}_0$ and the assertion of the theorem has been proved.

Finally, we list some concluding remarks. If the assumption $(\boldsymbol{E}(\omega)\, \boldsymbol{x}^{(0)}, \boldsymbol{y}) = (\boldsymbol{E}(\{\mu_0\})\, \boldsymbol{x}^{(0)}, \boldsymbol{y})$ is not satisfied, the iterative process defined by (3) need not converge. We have investigated eigenvalue problems for normal operators and obtained nearly the same qualitative results as in [3], where self-adjoint operators were considered. We have not assumed that the points of the spectrum lying on the spectral radius circle are isolated as in [4] or [2], but our results are similar to those there.

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## Souhrn

## ITERAČNÍ ŘEŠENÍ PROBLÉMŮ VLASTNÍCH ČÍSEL PRO NORMÁLNÍ OPERÁTORY

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V práci je řešen problém vlastních čísel rovnice $Tx = \lambda x$ iteračními procesy typu Kellogga pro normální operátory v Hilbertově prostoru. Iterační proces je definován podobně jako v [4]. Je ukázáno, že když na spektrální kružnici se nachází právě jeden bod spektra, jehož „váha" je větší než ostatních, tj. $(E(\omega)\, x^{(0)}, x^{(0)}) = (E(\{\mu_0\})\, x^{(0)}, x^{(0)}) \neq 0$, pak posloupnosti (3) konvergují k vlastnímu číslu a jemu odpovídajícímu vlastnímu vektoru.



## Резюме

## ИТЕРАЦИОННОЕ РЕШЕНИЕ ПРОБЛЕМЫ СОБСВЕННЫХ ЗНАЧЕНИЙ ДЛЯ НОРМАЛЬНЫХ ОПЕРАТОРОВ

TOMÁŠ KOJECKÝ

В работе итерационным процессом типа Келлога решается проблема собственных значений уравнения $Tx = \lambda x$ для нормальных операторов в пространстве Гилберта. Итерационный процесс построен как в [4]. Показано, что если на спектральной окружности оператора находится только одна точка его спектра, „вес" которой больше „веса" остальных, $(E(\omega)\, x^{(0)}, x^{(0)}) = (E(\{\mu_0\})\, x^{(0)}, x^{(0)}) \neq 0$, то последовательности (3) сходятся к собственному значению и соответствующему собственному вектору.



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