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ERROR ESTIMATE OF APPROXIMATE SOLUTION FOR A QUASILINEAR PARABOLIC INTEGRODIFFERENTIAL EQUATION IN THE L_p – SPACE

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Summary. The Rothe-Galerkin method is used for discretization. The rate of convergence in $C(I, L_p(G))$ for the approximate solution of a quasilinear parabolic equation with a Volterra operator on the right-hand side is established.

Keywords: Rothe's method, Galerkin's method, error estimate.

AMS classification: 65M15, 65M20, 35K22.

1. INTRODUCTION

During the last years many authors have been studying evolution problems. The semidiscretization in time (also called Rothe's method or the method of lines) is one of frequently used methods of proof- see [6], [7], [11] with many other references. The existence, uniqueness and regularity of solution of such problems can be proved using a relatively simple technique and the L_2-theory.

The L_p-theory of elliptic Dirichlet's boundary value problems is based mainly on a suitable generalization of Garding's inequality and is built up similarly to the L_2-theory (see [12]). This fact can be exploited for solving the parabolic equations by Rothe's method. In this way the local existence of a solution of one equation with a locally Lipschitz continuous right-hand side is proved in [10].

Using the technique developed in [10] we derive the error estimate for a totally discrete approximate solution of one quasilinear parabolic integrodifferential problem in the L_p-space.

For a more profound analysis of solution of parabolic equations in the L_p-spaces we refer the reader to the following papers: [3], [9], [14], etc.

2. DEFINITIONS AND PRELIMINARIES

Let $G \subset R^N$ $(N \geqq 2)$ be a simply connected bounded domain with a Lipschitz continuous boundary ∂G; let I be a time interval $\langle 0, T\rangle$ $(T \in R_+)$; $Q_T = I \times G$;

let $\mathring{W}_p^1(G)$ be a standard Sobolev space (p > 2). The duality between $u \in L_p(G)$ and $v \in L_q(G)$ $(p^{-1} + q^{-1} = 1)$ is denoted by $\langle u, v \rangle$. The notation $\| \ \|_p$, $\| \ \|_{1,p}$ is used for the norms in $L_p(G)$, $\mathring{W}_p^1(G)$ respectively, where

$$\|u\|_{1,p}^p = \sum_{i=1}^N \|\partial u/\partial_{xi}\|_p^p .$$

The spaces $L_p(G)$, $\mathring{W}_p^1(G)$, $W_p^1(G)$ are denoted by L_p, $\mathring{W}_p^1$, W_p^1. In the following we work in the function spaces like $C(I, X)$, $L_p(I, X)$, $L_\infty(I, X)$, where X is a Banach space, the basic properties of which can be found in [8].

Let X, Y be Banach spaces. By $Lip(X, Y)$ we denote the set of all functions $g: X \to Y$ satisfying

$$\|g(u) - g(v)\|_Y \leqq C\|u - v\|_X \quad \forall u , \quad v \in X .$$

Definition 2.1. *The operator* $E: L_\infty(I, L_p) \to L_\infty(I, L_p)$ *is said to be a Volterra operator iff*

$$(u(s) = v(s) \ \ for\ a.e. \ \ s \in \langle 0, t \rangle , \ \ t \in I) \Rightarrow (E(u)(s) = E(v)(s) \ \ for\ a.e. \ \ s \in \langle 0, t \rangle) .$$

Let us fix a Volterra operator E: $Lip(I, L_p) \to Lip(I, L_p)$. Suppose

(2.2) $$A_0 u = - \sum_{i,k=1}^N \frac{\partial}{\partial x_k}\left(a_{ik}(t, x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^N a_i(t, x)\frac{\partial u}{\partial x_i} + a_0(t, x)\, u$$

(2.3) $$a(t; u, v) = \int_G \left[\sum_{i,k=1}^N a_{ik}(t, x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_k} + \sum_{i=1}^N a_i(t, x)\frac{\partial u}{\partial x_i} v + a_0(t, x)\, uv\right] \mathrm{d}x ,$$

(2.4) $$p(t; u, v) = \int_G p_0(t, x)\, uv \,\mathrm{d}x .$$

The aim of this paper is to derive the error estimate for the totally discrete approximate solution of the *PC*-1 problem.

PC-1. *To find u such that*

(i) $u \in L_\infty(I, \mathring{W}_p^1) \cap Lip(I, L_p)$,
$\partial_t u \in L_\infty(I, L_p)$

(ii) $u(0) = \alpha \in \mathring{W}_p^1$; $A_0\alpha$, $E(\alpha)(0), f(0, \circ, 0) \in L_p$; $p > 2$ *and*

(2.5) $$p(t; \partial_t u(t), v) + a(t; u(t), v) = \langle f(t, x, E(u)(t)), v \rangle \quad \forall v \in \mathring{W}_q^1, \ \ for\ a.e. \ \ t \in I,\ p^{-1} + q^{-1} = 1 .$$

Remark 2.6. The results obtained in Theorem 4.2 can be established for $p = 2$, too, and in this case we have a better error estimate for a more general problem (see [13], [14], [16], etc).

We consider the following conditions ($i, k = 1, \dots, N$; C and C_1 are positive constants):

(2.7) $$a_{ik} \in C(\bar{Q}_T)\,; \quad a_i, a_0, p_0 \in L_\infty(G)\,,$$

(2.8) $$C_1 \, |\xi|^2 \leqq \sum_{i,k=1}^{N} a_{ik}\xi_i\xi_k \leqq C|\xi|^2 \quad \forall \xi \in R^N\,,$$

(2.9) $$|f(t, x, \xi) - f(t', x, \xi')| \leqq C[|t - t'| + |t - t'|\,|\xi| + |\xi - \xi'|]$$
$$\forall \xi, \xi' \in R\,; \quad \forall t, t' \in I\,,$$

(2.10) $$|w(t, x) - w(t', x)| \leqq C|t - t'|$$
$$\text{for} \quad w = a_{ik}, a_i, a_0, p_0 \quad \text{and} \quad \forall t, t' \in I\,,$$

(2.11) $$C_1 \leqq p_0(t, x) \leqq C\,,$$

(2.12) $$\|E(z)(t) - E(z)(t')\|_p \leqq |t - t'|\; \Theta(\|z\|_{C(\langle 0,t\rangle, L_p)})\,(1 + \|\partial_t z\|_{L_\infty(\langle 0,t\rangle, L_p)})$$

$\forall t, t' \in I$; $t' \leqq t$; $\forall z \in \mathrm{Lip}\,(I, L_p)$; $\Theta \in C(R_+, R_+)$.

(The next condition for $N > 2$ is equivalent to the ellipticity of the operator A_0.)

(2.13) (the "*root condition*")
For every $\xi \in (\xi_1, \dots, \xi_{N-1}) \in R^{N-1}$ and every $y \in Q_T$ the polynomial P in τ $P(\tau; \xi, y) := \sum_{|s|=2} a_s(y)\, \xi^{s'}\tau^{s_n}$ where $s = (s', s_n)$ and

$$\sum_{|s|=2} a_s D^\alpha \equiv - \sum_{i,k=1}^{N} a_{ik} \frac{\partial^2}{\partial x_i\, \partial x_k}$$

has exactly one root with a positive imaginary part and one root with a negative imaginary part (see [12, Def. 1.3]).

The existence and uniqueness of the PC-1 solution follow from the semigroup theory of quasilinear parabolic equations (see [5, Th. 3.3.3]), or they can be proved by Rothe's method in the same way as in [10].

Remark 2.14. In the following C, ε, C_ε denote generic positive constants, where ε is a small constant and $C_\varepsilon = C(\varepsilon^{-1})$.

Solving the PC-1 problem we first perform discretization in time and then in space. We consider the following approximation of $\mathring{W}_p^1$ by finite dimensional subspaces V_λ ($V_\lambda \subset \mathring{W}_p^1, \lambda > 0$):

(2.15) $$\forall v \in \mathring{W}_p^1 \quad \exists \hat{v}_\lambda \in V_\lambda \quad \text{such that} \quad \hat{v}_\lambda \to v \quad \text{in} \quad \mathring{W}_p^1 \quad \text{for} \quad \lambda \to 0\,.$$

This approximation is frequently used in numerical approach (see [1, Th. 3.1.4], [4, p. 50], [2, Th. IV. 3.2], etc.). Thus for an initial function $\alpha \in \mathring{W}_p^1$ there exist $\alpha_\lambda \in V_\lambda$ such that

(2.16) $$\alpha_\lambda \to \alpha \quad \text{in} \quad \mathring{W}_p^1 \quad \text{for} \quad \lambda \to 0\,.$$

Let us divide the time interval I into n subintervals $\langle t_{i-1}, t_i\rangle$ for $i = 1, \ldots, n$ where $h = T/n$, $t_i = ih$. For a given function $w(t)$ the following notation is introduced

$$w_i = w(t_i)\,, \quad \delta w_i = (w_i - w_{i-1})/h \quad \text{for} \quad i = 1, \ldots, n\,.$$

There are several functions often occuring in this paper ($i = 1, \ldots, n$ and $\sigma = (n, \lambda)$):

(2.17) $$u_\sigma(t) \;\; = \alpha_\lambda \qquad t = 0$$
$$u^\lambda_{i-1} + (t - t_{i-1})\,\delta u^\lambda_i \qquad t \in (t_{i-1}; t_i\rangle\,,$$

(2.18) $$\bar{u}_\sigma(t) \;\; = \alpha_\lambda \qquad t = 0$$
$$u^\lambda_i \qquad t \in (t_{i-1}; t_i\rangle\,,$$

(2.19) $$\tilde{u}^\lambda_{i-1}(t) = \tilde{u}^\lambda_{i-1,n}(t) = \alpha_\lambda \qquad t \in \langle 0, h\rangle$$
$$u^\lambda_{j-1} + (t - t_j)\,\delta u^\lambda_j \qquad t \in \langle t_j; t_{j+1}\rangle\,,$$
$$u^\lambda_{i-1} \qquad t \in \langle t_i; T\rangle\,,$$

where $j = 1, \ldots, i - 1$,

(2.20) $$E_h(\tilde{u}_\sigma)(t) = E(\tilde{u}_\sigma)(t_i) \qquad t \in (t_{i-1}; t_i\rangle\,,$$

where $\tilde{u}_\sigma \equiv \tilde{u}^\lambda_{n-1}$,

(2.21) $$r_h(t, \xi) = r(t_i, \xi) \qquad t \in (t_{i-1}; t_i\rangle\,; \qquad r = p, a, f\,.$$

Let us consider the following discrete problem:

PD-1. *To find* $u^\lambda_i \in V_\lambda$ $(i = 1, \ldots, n)$ *such that*

$$u^\lambda_0 = \alpha_\lambda \in V_\lambda\,; \quad A_0\alpha_\lambda\,, \quad E(\alpha_\lambda)(0), f(0, \circ, 0) \in L_p$$

and (2.22) *is satisfied*:

(2.22) $$p(t_i, \delta u^\lambda_i, v) + a(t_i; u^\lambda_i, v) = \langle f(t_i, x, E(\tilde{u}^\lambda_{i-1})(t_i)), v\rangle$$
$$\forall v \in \mathscr{V}_\lambda = \operatorname{span}\{v \in \mathring{W}^1_q\colon \; \exists u \in V_\lambda,\, v = |u|^{p-2}\,u\}\; p^{-1} + q^{-1} = 1\,.$$

3. A PRIORI ESTIMATES

The technique of our proof requires application of the following inequalities:

(3.1) $$ab \leqq \varepsilon a^p + C_\varepsilon b^p \qquad p^{-1} + q^{-1} = 1\,,$$

(3.2) $$\prod_{i=1}^n a_i \leqq \sum_{i=1}^n p_i^{-1} a_i^{p_i} \qquad \sum_{i=1}^n p_i^{-1} = 1\,,$$

(3.3) $$\Big(\sum_{i=1}^n a_i\Big)^p \leqq n^{p-1} \sum_{i=1}^n a_i^p\,.$$

Lemma 3.4. *Let the assumptions of Theorem* 4.2 *be satisfied. Then the problem PD*-1 *has a uniquely determined solution* $u_i^\lambda \in V_\lambda$ $(i = 1, \ldots, n)$ *for all* $h \leqq h_0$.

Proof. Let us rewrite (2.22) into the form

$$h^{-1}\, p(t_i; u_i^\lambda, v) + a(t_i; u_i^\lambda, v) = \langle f_i, v\rangle + h^{-1}\, p(t_i; u_{i-1}^\lambda, v)\,.$$

where $f_i = f(t_i, x, E(\tilde{u}_{i-1}^\lambda)(t_i))$..

Applying [12, Corollary 7.4] successively for $i = 1, \ldots, n$ the required result follows.

The following lemma contains some technical arguments useful for us.

Lemma 3.5. *Let the assumptions of Theorem* 4.2 *hold.*

(i) *If* $u \in W_p^1$, $p > 2$ *then*

$$v = |u|^{p-2}\, u \in \mathring{W}_q^1 \quad (p^{-1} + q^{-1} = 1)$$

and

$$\frac{\partial v}{\partial x_i} = (p-1)\, |u|^{p-2} \frac{\partial u}{\partial x_i}\,, \tag{3.6}$$

$$\|v\|_q = \|u\|_p^{p-1}\,. \tag{3.7}$$

(ii) *If* $u \in W_p^1$ *then*

$$a(t; u, |u|^{p-2}\, u) \geqq K_1 \|\, |u|^{(p-2)/2}\, u\|_{1,2}^2 - C\|u\|_p^p\,, \quad K_1 > 0\,. \tag{3.8}$$

(iii) *If* $w, u \in W_p^1$ *then for* $j = 1, \ldots, n$ *we have*

$$|\delta a(t_j; w, |u|^{p-2}\, u)| \leqq \varepsilon \|\, |u|^{(p-2)/2}\, u\|_{1,2}^2 + C_\varepsilon \|u\|_p^p + C(\|w\|_p^p + \|w\|_{1,p}^p)\,, \tag{3.9}$$

$$|\delta a(t_j; w, |u|^{p-2}\, u)\,|h \leqq \varepsilon \|\, |u|^{(p-2)/2}\, u\|_{1,2}^2 + C_\varepsilon \|u\|_p^p + {} + Ch^p(\|w\|_p^p + \|w\|_{1,p}^p)\,. \tag{3.10}$$

(iv) *If* $w, u \in W_p^1$ *then*

$$|a(t; w, |u|^{p-2}\, u)| \leqq \varepsilon \|\, |u|^{(p-2)/2}\, u\|_{1,2}^2 + C_\varepsilon \|u\|_p^p + {} + C(\|w\|_p^p + \|w\|_{1,p}^p)\,. \tag{3.11}$$

Proof. (i) The assertion is proved in [10].

(ii) We have

$$\textstyle\int_G a_0(t, x)\, u|u|^{p-2}\, u\, \mathrm{d}x \leqq C\|u\|_p^p\,.$$

The rest is proved in [10].

(iii) The relations (3.9) and (3.10) can be proved in the same way, so we show only the proof of (3.10). Using (3.2) for $n = 3$, $p_1 = p$, $p_2 = 2$, $p_3 = 2p/(p-2)$ and (3.1) we obtain

$$I_1 = \left| h \int_G \sum_{i,k=1}^N \delta\, a_{ik}(t_j, x) \frac{\partial w\, \partial}{\partial x_i\, \partial x_k} (|u|^{p-2}\, u)\, \mathrm{d}x \right| \leqq$$

$$\leqq C \sum_{i,k=1}^{N} \int_G \left[h \left| \frac{\partial w}{\partial x_i} \right| \right] \left| \frac{\partial}{\partial x_k} \left(|u|^{(p-2)/2} u \right) \right| |u|^{(p-2)/2} \, dx \leqq$$

$$\leqq \varepsilon \| |u|^{(p-2)/2} u \|_{1,2}^2 + C_\varepsilon \|u\|_p^p + C h^p \|w\|_{1,p}^p .$$

Further

$$I_2 = \left| h \int_G \sum_{i=1}^{N} \delta \, a_i(t_j, x) \frac{\partial w}{\partial x_i} |u|^{p-2} u \, dx \right| \leqq$$

$$\leqq C \sum_{i=1}^{N} \int_G \left| h \frac{\partial w}{\partial x_i} \right| |u|^{p-1} \, dx \leqq C h^p \|w\|_{1,p}^p + C \|u\|_p^p$$

and

$$I_3 = \left| h \int_G \delta \, a_0(t_j, x) \, w |u|^{p-2} u \, dx \right| \leqq C h^p \|w\|_p^p + C \|u\|_p^p .$$

The estimates of I_1, I_2, I_3 imply (3.10).

(iv) The proof proceeds in the same way as in (iii).

Lemma 3.12. *Let the assumptions of Theorem 4.2 be satisfied. Then there exist C, $h_0 > 0$ such that*

(3.13) $$\|u_j^\lambda\|_p \leqq C \quad \text{for} \quad h \leqq h_0 \quad \textit{and} \quad j = 1, \ldots, n .$$

Proof. Putting $v = |u_i^\lambda|^{p-2} u_i^\lambda h$ in (2.22) we get

$$p(t_i; u_i^\lambda, |u_i^\lambda|^{p-2} u_i^\lambda) + a(t_i; u_i^\lambda, |u_i^\lambda|^{p-2} u_i^\lambda) \, h =$$

$$= p(t_i; u_{i-1}^\lambda, |u_i^\lambda|^{p-2} u_i^\lambda) + \langle f_i, |u_i^\lambda|^{p-2} u_i^\lambda \rangle \, h .$$

Summing it up for $i = 1, \ldots, j$ we obtain by virtue of (3.8) and

(3.14) $$ab \leqq \frac{1}{p} a^p + \frac{p-1}{p} b^q \quad (p^{-1} + q^{-1} = 1)$$

the estimate

$$\|u_j^\lambda\|_p^p \leqq C[1 + \sum_{i=1}^{j} \max_{1 \leqq k \leqq i} \|u_k^\lambda\|_p^p \, h] .$$

Hence Gronwall's lemma ([7, Lemma 1.3.19] implies the required a priori estimates of $\|u_j^\lambda\|_p$.

Lemma 3.15. *Let the assumptions of Theorem 4.2 hold. If $u \in V_\lambda$ is a solution of*

$$a(t_j; u, v) = \langle F, v \rangle \quad \forall v \in \mathscr{V}_\lambda$$

then

$$\|u\|_{1,p} \leqq C(\|F\|_p + \|u\|_p) .$$

Proof. The lemma is a consequence of [12, Theorem 6.3].

Consequence 3.16. *Let the assumptions of Theorem* 4.2. *be satisfied. Then*

$$\|u_j^\lambda\|_{1,p} \leqq C[1 + \|\delta u_j^\lambda\|_p + \sum_{i=1}^{j} \|\delta u_k^\lambda\|_p\, h] \quad \forall j = 1, \ldots, n\,.$$

Proof. The proof follows from Lemma 3.15.

Lemma 3.17. *Let the assumptions of Theorem* 4.2 *hold. Then there exist* $C, h_0 > 0$ *such that*

$$\|\delta u_j^\lambda\|_p \leqq C \quad \textit{for} \quad j = 1, \ldots, n \quad \textit{and} \quad h \leqq h_0\,. \tag{3.18}$$

Proof. First we estimate $\|\delta u_1^\lambda\|_p$. Let us put $v = |\delta u_1^\lambda|^{p-2}\, \delta u_1^\lambda$ in (2.22) for $i = 1$. We have

$$p(t_1; \delta u_1^\lambda, |\delta u_1^\lambda|^{p-2}\, \delta u_1^\lambda) + a(t_1; \delta u_1^\lambda, |\delta u_1^\lambda|^{p-2}\, \delta u_1^\lambda)\, h =$$
$$= -a(0; u_0^\lambda, |\delta u_1^\lambda|^{p-2}\, \delta u_1^\lambda) - \delta\, a(t_1; u_0^\lambda, |\delta u_1^\lambda|^{p-2}\, \delta u_1^\lambda)\, h + \langle f_1, |\delta u_1^\lambda|^{p-2}\, \delta u_1^\lambda\rangle\,.$$

Using (3.1), (3.8), (3.9) we obtain

$$\|\delta u_1^\lambda\|_p + \|\, |\delta u_1^\lambda|^{(p-2)/2}\, \delta u_1^\lambda\|_{1,2}^2\, h \leqq C \quad \text{for} \quad h \leqq h_0\,. \tag{3.19}$$

Now we estimate $\|\delta u_j^\lambda\|_p$ for $j \geqq 2$. After subtracting (2.22) for i, $i - 1$, setting $v = |\delta u_i^\lambda|^{p-2}\, \delta u_i^\lambda$ and summing up for $i = 2, \ldots, j$ we can write

$$\sum_{i=2}^{j} [p(t_i; \delta u_i^\lambda, |\delta u_i^\lambda|^{p-2}\, \delta u_i^\lambda) - p(t_{i-1}; \delta u_{i-1}^\lambda, |\delta u_i^\lambda|^{p-2}\, \delta u_i^\lambda) + \tag{3.20}$$
$$+ a(t_{i-1}; \delta u_i^\lambda, |\delta u_i^\lambda|^{p-2}\, \delta u_i^\lambda)\, h + \delta\, a(t_i; u_i^\lambda, |\delta u_i^\lambda|^{p-2}\, \delta u_i^\lambda)\, h] =$$
$$= \sum_{i=2}^{j} \langle f_i - f_{i-1}, |\delta u_i^\lambda|^{p-2}\, \delta u_i^\lambda\rangle\,.$$

From (3.8), (3.9), (3.14) and Consequence 3.16 we get

$$\|\delta u_j^\lambda\|_p^p + \sum_{i=2}^{j} \|\, |\delta u_i^\lambda|^{(p-2)/2}\, \delta u_i^\lambda\|_{1,2}^2\, h \leqq \tag{3.21}$$
$$\leqq C[1 + \|\delta u_1^\lambda\|_p^p + \sum_{i=2}^{j} \max_{1 \leqq k \leqq i} \|\delta u_k^\lambda\|_p^p\, h]\,.$$

Hence (3.19), (3.21) and Gronwall's lemma imply the required result.

From the a priori estimates of Lemmas 3.12, 3.15 and Consequence 2.16 we deduce

$$\|u_\sigma(t)\|_p + \|u_\sigma(t)\|_{1,p} + \|\tilde{u}_\sigma(t)\|_p + \|\partial_t\, \tilde{u}_\sigma(t)\|_p \leqq C\,, \tag{3.22}$$

$$\|\partial_t u_\sigma(t)\|_p + \|\bar{u}_\sigma(t)\|_p + \|\bar{u}_\sigma(t)\|_{1,p} \leqq C\,, \tag{3.23}$$

$$\|u_\sigma(t) - \bar{u}_\sigma(t)\|_p + \|u_\sigma(t) - \bar{u}_\sigma(t - h)\|_p \leqq Ch\,, \tag{3.24}$$

$$\|u_\sigma(t) - u_\sigma(t')\|_p \leqq C|t - t'|\,, \tag{3.25}$$

$$\|\tilde{u}_\sigma(t) - u_\sigma(t)\|_p \leqq Ch \tag{3.26}$$

for $t, t' \in I$ (C is independent of σ).

Let us rewrite (2.22) into the form

$$(4.1)\qquad p_h(t; \partial_t u_\sigma(t), v) + a_h(t; \bar{u}_\sigma(t), v) = \langle f_h(t, x, E_h(\tilde{u}_\sigma)(t)), v\rangle \quad \text{for} \quad t \in I\,, \quad v \in \mathscr{V}_\lambda\,.$$

If $u \in \mathring{W}_p^1$ is a PC-1 solution and $\hat{u}_\lambda \in V_\lambda$ (for $\lambda > 0$) is its approximation in the sense of (2.15), then the following theorem holds.

Theorem 4.2. *Let* (2.2)–(2.4), (2.7)–(2.13), (2.15), (2.16) *be satisfied. Suppose that E is a Volterra operator such that* $E \in \mathrm{Lip}\,(C(I, L_p), C(I, L_p))$. *Then for all* $t \in I$

$$(4.3)\qquad \max_{0 \leqq s \leqq t} \|u_\sigma(s) - u(s)\|_p^p \leqq C\big[h^{p/2} + \|\alpha_\lambda - \alpha\|_p^p + \textstyle\int_0^t \big[\|u - \hat{u}_\lambda\|_p^p + \|u - \hat{u}_\lambda\|_p^{p/2} + \|u - \hat{u}_\lambda\|_{1,p}^p\big]\,\mathrm{d}s\,.$$

Proof. Let us subtract (4.1) from (2.5) for $v = |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2}(\hat{u}_\lambda - \bar{u}_\sigma)$ and then integrate it over $(0, t)$. We have

$$(4.4)\qquad \int_0^t \big[p(s; \partial_s u, |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2}(\hat{u}_\lambda - \bar{u}_\sigma)) - p_h(s; \partial_s u_\sigma, |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2}(\hat{u}_\lambda - \bar{u}_\sigma)) + a(s; u, |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2}(\hat{u}_\lambda - \bar{u}_\sigma)) - a_h(s; \bar{u}_\sigma, |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2}(\hat{u}_\lambda - \bar{u}_\sigma))\big]\,\mathrm{d}s = \int_0^t \langle f(s, x, E(u)(s)) - f_h(s, x, E_h(\tilde{u}_\sigma)(s)), |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2}(\hat{u}_\lambda - \bar{u}_\sigma)\rangle\,\mathrm{d}s\,.$$

Using (3.10), (3.11), (3.22)–(3.26) we obtain

$$(4.5)\qquad \int_0^t \big[p(s; \partial_s(u - u_\sigma), |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2}(\hat{u}_\lambda - \bar{u}_\sigma)) + a(s; \hat{u}_\lambda - \bar{u}_\sigma, |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2}(\hat{u}_\lambda - \bar{u}_\sigma))\big]\,\mathrm{d}s \leqq \int_0^t \langle f(s, x, E(u)(s)) - f(s, x, E(u_\sigma)(s)), |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2}(\hat{u}_\lambda - \bar{u}_\sigma)\rangle\,\mathrm{d}s + \int_0^t \big[\varepsilon\| |\hat{u}_\lambda - \bar{u}_\sigma|^{(p-2)/2}(\hat{u}_\lambda - \bar{u}_\sigma)\|_{1,2}^2 + C_\varepsilon\big[h^p + \|u - \hat{u}_\lambda\|_p^p + \max_{\langle 0,s\rangle} \|u - u_\sigma\|_p^p + \|u - \hat{u}_\lambda\|_{1,p}^p\big]\big]\,\mathrm{d}s\,.$$

Owing to

$$\| |x|^{p-2}x - |y|^{p-2}y\| \leqq (p - 1)(\|x\|_p + \|y\|_p)^{p-2}\|x - y\|_p$$

for $p^{-1} + q^{-1} = 1$ (see [10]) and

$$p(t; \partial_t x, |x|^{p-2}x) = p^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}t}\,p(t; x, |x|^{p-2}x) - p^{(1)}(t; x, |x|^{p-2}x)\right]$$

where $p^{(1)}(t; x, y) = \partial_t p(t; x, y)$ we conclude

$$(4.6)\qquad \int_0^t \Big[p^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\,p(s; u - u_\sigma, |u - u_\sigma|^{p-2}(u - u_\sigma)) +$$

$$+ a(s; \hat{u}_\lambda - \bar{u}_\sigma, |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2} (\hat{u}_\lambda - \bar{u}_\sigma)) \Big] \, ds \leqq$$

$$\leqq \int_0^t \langle f(s, x, E(u)(s)) - f(s, x, E(u_\sigma)(s)), |\hat{u}_\lambda - \bar{u}_\sigma|^{p-2} (\hat{u}_\lambda - \bar{u}_\sigma) \rangle \, ds +$$
$$+ \int_0^t [\varepsilon \| |\hat{u}_\lambda - \bar{u}_\sigma|^{(p-2)/2} (\hat{u}_\lambda - \bar{u}_\sigma) \|_{1,2}^2 + C_\varepsilon [h^{p/2} + \|u - \hat{u}_\lambda\|_p^p +$$
$$+ \max_{\langle 0,s \rangle} \|u - u_\sigma\|_p^p + \|u - \hat{u}_\lambda\|_{1,p}^p + \|u - \hat{u}_\lambda\|_p^{p/2}]] \, ds \, .$$

In virtue of (3.1), (3.8) (for sufficiently small ε) we get

$$\|u(t) - u_\sigma(t)\|_p^p + \int_0^t \| |\hat{u}_\lambda - \bar{u}_\sigma|^{(p-2)/2} (\hat{u}_\lambda - \bar{u}_\sigma) \|_{1,2}^2 \, ds \leqq \tag{4.7}$$
$$\leqq C[h^{p/2} + \|\alpha - \alpha_\lambda\|_p^p + \int_0^t [\max_{\langle 0,s \rangle} \|u - u_\sigma\|_p^p +$$
$$+ \|u - \hat{u}_\lambda\|_p^p + \|u - \hat{u}_\lambda\|_{1,p}^p + \|u - \hat{u}_\lambda\|_p^{p/2}] \, ds] \, .$$

The assertion of Theorem 4.2 follows from Gronwall's lemma and (4.7).

Remark 4.8. For frequently used approximations of $\mathring{W}_p^1$ (for suitable ∂G) we have

$$\|u - \hat{u}_\lambda\|_{j,p} \leqq C \lambda^{2-j} \|u\|_{2,p} \quad \forall u \in \mathring{W}_p^1 \cap W_p^2 \, ; \quad j = 0, 1 \, ,$$

i.e. if u is a PC-1 solution such that $u \in \mathring{W}_p^1 \cap W_p^2$, we can obtain the following estimate (from the Theorem 4.2):

$$\max_{\langle 0,t \rangle} \|u - u_\sigma\|_p^p \leqq C[h^{p/2} + \lambda^p] \, . \tag{4.9}$$

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Súhrn

ODHAD CHYBY PRIBLIŽNÉHO RIEŠENIA KVAZILINEÁRNEJ PARABOLICKEJ INTEGRODIFERENCIÁLNEJ ROVNICE V L_p-PRIESTORE

MARIÁN SLODIČKA

Pri diskretizácii problému je použitá Rotheho i Galerkinova metóda. Je určený rád konvergencie v $C(I, L_p(G))$ približného riešenia kvázilineárnej parabolickej rovnice s Volterovským operátorom v pravej strane.

Резюме

ОЦЕНКА ОШИБКИ ПРИБЛИЖЕННОГО РЕШЕНИЯ КВАЗИЛИНЕЙНОГО ПАРАБОЛИЧЕСКОГО ИНТЕГРОДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ В L_p-ПРОСТРАНСТВЕ

MARIÁN SLODIČKA

При дискретизации задачи использован метод Роте-Галеркина. В пространстве $C(I, L_p(G)$ определена скорость сходимости приближенного решения квазилинейного параболического уравнения с оператором типа Вольтерра в правой стороне.

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