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ON THE GENERALIZED RICCATI MATRIX DIFFERENTIAL EQUATION. EXACT, APPROXIMATE SOLUTIONS AND ERROR ESTIMATE

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Summary. In this paper explicit expressions for solutions of Cauchy problems and two-point boundary value problems concerned with the generalized Riccati matrix differential equation are given. These explicit expressions are computable in terms of the data and solutions of certain algebraic Riccati equations related to the problem. The interplay between the algebraic and the differential problems is used in order to obtain approximate solutions of the differential problem in terms of those of the algebraic one.

Keywords: Generalized Riccati matrix differential equation, Cauchy problem, two-point boundary value problem, algebraic Riccati equation.

1. INTRODUCTION

In recent papers [8], [9], Cauchy problems and boundary value problems concerned with the generalized matrix differential equation

$$\mathrm{d}/\mathrm{d}t\, X(t) = A + B\,X(t) - X(t)\,C - X(t)\,D\,X(t) \tag{1.1}$$

are treated, but solutions are given in terms of the entries of the matrix function

$$S(t) = \exp\left(\begin{bmatrix} C & D \\ A & B \end{bmatrix} t\right) = (S_{ij}(t))\,, \quad 1 \leqq i\,, \quad j \leqq 2$$

without the explicit knowledge of the entries $S_{ij}(t)$ in terms of the data. The aim of this paper is to present an explicit expression for the solutions of the problems

$$\mathrm{d}/\mathrm{d}t\, X(t) = A + B\,X(t) - X(t)\,C - X(t)\,D\,X(t)\,; \quad X(0) = P_0 \tag{1.2}$$

and

$$\begin{gathered} \mathrm{d}/\mathrm{d}t\, X(t) = A + B\,X(t) - X(t)\,C - X(t)\,D\,X(t)\,; \\ E\,X(b) - X(0)\,F = G\,; \quad 0 \leqq t \leqq b \end{gathered} \tag{1.3}$$

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where A, B, C, D, E, F, G, P_0 and $X(t)$ are square matrices from $\mathbb{C}^{nxn}$, and t lies on the real line. Solutions of problems (1.2) and (1.3) are given in terms of the data and solutions of generalized algebraic Riccati equations of the type

$$M + NX - XP - XQX = 0\,. \tag{1.4}$$

Different methods for solving algebraic equations of the type (1.4) may be found in the literature, for example in [1], [5], [7], [14], [15] and [16].

The interplay between the solutions of the problems (1.2) and (1.3) and the solutions of algebraic equations of the type (1.4) may be used for obtaining approximate solutions of problems (1.2) and (1.3), and error estimates of them, in terms of approximate solutions and error estimates for solutions of the algebraic problem. So, any approximate method for solving equations of the type (1.4) provides a method for obtaining approximate solutions of problems (1.2) and (1.3), and depending on the problem we can choose the most convenient resolution method for the equation (1.4), so that the error of the approximate solution of problems (1.2) and (1.3) be as little as possible.

Because of the interplay between the solutions of algebraic and differential problems, this paper may be regarded as a continuation of [10], [11] and [12]. The paper is organized as follows. Section 2 deals with the explicit expression of the solution of problem (1.2), as well as with finding approximate solutions and error estimates of them, in terms of the data and a solution of the algebraic equation

$$A + BX - XC - XDX = 0\,. \tag{1.5}$$

Also it is proved that this explicit expression of the solution of problem (1.2) is stable with respect to the Cauchy condition, and the variation of the solution with respect to the change of the Cauchy condition is presented. Section 3 deals with the explicit solution of the two-point boundary value problem (1.3). Sufficient conditions for its resolution and an explicit expression for solutions in terms of solutions of the equation (1.5) and a solution of a certain algebraic equation of the type (1.4) is given. Starting from approximate solutions of equation (1.5), approximate solutions of problem (1.3) are presented. Also, error estimates for the approximate solutions of problem (1.3) in terms of error estimates of the approximate solutions of problem (1.4) are given.

In order to clarify the presentation we recall some concepts and results that will be used in next sections. If A is a matrix in $\mathbb{C}^{nxn}$, we represent by $\|A\|$ its operator norm, defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

where $\| \ \|_2$ denotes the usual euclidean norm in $\mathbb{C}^n$. If A is an invertible matrix in $\mathbb{C}^{nxn}$, and B is a matrix in $\mathbb{C}^{nxn}$ such that $\|B - A\| < (\|A^{-1}\|)^{-1}$, then B is invertible and

(1.6) $$\|B^{-1} - A^{-1}\| < \|A^{-1}\| \, \|A - B\| \, \|B^{-1}\|$$

and for any pair of matrices C and D in $\mathbb{C}^{n \times n}$, one gets $\|C\,D] \leqq \|C\| \, \|D\|$, see [4], and [6].

Finally, if f is a differentiable matrix function acting on $\mathbb{C}^{n \times n}$, and A, B are matrices in $\mathbb{C}^{n \times n}$, then the mean value theorem, [3], p. 158, implies that

(1.7) $$\|f(A + B) - f(A)\| \leqq \|B\| \sup_{0 \leqq t \leqq 1} \|f^{(1)}(A + tB)\|$$

2. CAUCHY PROBLEMS: EXPLICIT, APPROXIMATE SOLUTIONS AND ERROR ESTIMATE

We begin this section with the Cauchy problem (1.2) under the existence hypothesis of a solution $\bar{X}$ of the algebraic equation (1.5).

Lemma 1. *Let us suppose $\bar{X}$ is a solution of equation (1.5), let U_0, B_0 and C_0 be defined by*

(2.1) $$U_0 = P_0 - \bar{X}\,; \quad B_0 = B - \bar{X}D\,; \quad C_0 = C + D\bar{X}\,.$$

Let J be a neighborhood of $t = 0$ on the positive real line such that

(2.2) *for all* $t \in J$, *the matrix* $I + \int_0^t \exp(-vC_0)\, D \exp(vB_0)\, dv U_0$ *is invertible.*

Then the only solution of the problem (1.2) on J is given by the expression

(2.3)
$$X(t) = \bar{X} + \exp(tB_0)\, U_0 (I + \int_0^t \exp(-vC_0)\, D \exp(vB_0)\, dv\, U_0)^{-1} \exp(-tC_0)\,.$$

Proof. Let us consider the change of variable

(2.4) $$U(t) = X(t) - \bar{X}\,.$$

Then the problem (1.2) is equivalent to

(2.5) $$d/dt\, U(t) = B_0\, U(t) - U(t)\, C_0 - U(t)\, D\, U(t)\,; \quad U(0) = U_0$$

whete B_0, C_0 and U_0 are given by (2.1). Now, let us consider the extended linear system

(2.6) $$d/dt \begin{bmatrix} V(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} C_0 & D \\ 0 & B_0 \end{bmatrix} \begin{bmatrix} V(t) \\ Z(t) \end{bmatrix}; \quad \begin{bmatrix} V(0) \\ Z(0) \end{bmatrix} = \begin{bmatrix} I \\ U_0 \end{bmatrix}.$$

Solving (2.6) we obtain that

(2.7) $$Z(t) = \exp(tB_0)\, U_0\,,$$
$$V(t) = \exp(tC_0)\,(I + \int_0^t \exp(-vC_0)\, D \exp(vB_0)\, dv\, U_0)\,,$$

see [2], chap. 1 for details.

Note that as $V(0) = I$, there exists a neighborhood J of $t = 0$ such that $V(t)$ is invertible in J, [6]. Thus, if we define the matrix function $U(t) = Z(t)\,(V(t)\,(V(t))^{-1}$ for $t \in J$, an easy computation yields (2.5), see [9], p. 18, for details. By [13], there exists only one solution, and by virtue of (2.4), (2.7), it is given by (2.3).

For the sake of clarity of the proof of the following result we introduce an easy algebraic relationship satisfied by any matrices L_n, S_n, T_n, L, S and T in $\mathbb{C}^{mxm}$. Note that

$$(2.8) \qquad L_n S_n T_n - LST = L_n(S_n - S)\,T_n + L_n S(T_n - T) + (L_n - L)\,ST\,.$$

Lemma 1 provides an explicit expression of the solution of problem (1.2) when there exists a solution of the algebraic equation (1.5). The next result shows that a good approximation of the solution $\bar{X}$ of equation (1.5) provides a good approximation of the solution of problem (1.2), and also an estimate of the approximation error of the solution of (1.2) in terms of the data and the approximation error of the solution of (1.5) is given.

Let us suppose that $\{Z_n\}_{n\geq 1}$ is a sequence of matrices norm-convergent to a solution $\bar{X}$ of equation (1.5). Then it is clear from Lemma 1 that for each real number $t \in J$ such that hypothesis (2.2) is satisfied, the sequence of matrix functions $X_n(t)$ defined by

(2.9)

$$X_n(t) = Z_n + \exp(tB_n)\,U_n(I + \textstyle\int_0^t \exp(-vC_n)\,D\exp(vB_n)\,dv\,U_n)^{-1}\exp(-tC_n)$$

where

$$B_n = B - Z_n D\,; \quad C_n = C + DZ_n\,; \quad U_n = P_0 - Z_n\,,$$

is pointwise convergent to the only solution $X(t)$ of (1.2), given by (2.3). Let us also suppose that the matrix function $W(t)$ defined by

$$(2.10) \qquad W(t) = \textstyle\int_0^t \exp(-vC_0)\,D\exp(vB_0)\,dv\,U_0$$

satisfies

$$(2.11) \qquad \|W(t)\| = d(t) < 1\,, \quad \text{for all} \quad t \in J\,.$$

Now, considering the matrices

(2.12)

$$L_n = \exp(tB_n)\,U_n\,; \quad S_n = (I + \textstyle\int_0^t \exp(-vC_n)\,D\exp(vB_n)\,dv\,U_n)^{-1}\,;$$
$$T_n = \exp(-tC_n)$$
$$L = \exp(tB_0)\,U_0\,; \quad S = (I + \textstyle\int_0^t \exp(-vC_0)\,D\exp(vB_0)\,dv\,U_0)^{-1}\,;$$
$$T = \exp(-tC_0)$$

we obtain that from (2.3), (2·8), (2·9) and (2.12), that

$$(2.13) \qquad \|X_n(t) - X(t)\| \leqq \|L_n\|\,\|S_n - S\|\,\|T_n\| + \|L_n\|\,\|S\|\,\|T_n - T\| + \\ + \|L_n - L\|\,\|S\|\,\|T\|\,.$$

Let ε be a positive number. Then from the norm convergence of Z_n to $\bar{X}$ it is clear that B_n, C_n and U_n, defined by (2.9), are norm-convergent to B_0, C_0 and U_0, respectively. Let t be a fixed real number satisfying (2.11). Then there exists a positive integer n_0 (depending on t and ε) such that for $n > n_0$ the following conditions are satisfied

$$\text{(2.14)} \qquad \left\|\int_0^t \exp(-vC_n)\, D \exp(vB_n)\, \mathrm{d}v\, U_n\right\| < (1 + d(t))/2\,,$$
$$\|B_n\| < \|B_0\| + \varepsilon\,; \quad \|C_n\| < \|C_0\| + \varepsilon\,; \quad \|U_n\| < \|U_0\| + \varepsilon\,; \quad n > n_0\,.$$

Considering (2.12), (2.11) and taking norms for $n > n_0$ we obtain that

$$\text{(2.15)} \qquad \|L_n\| \leqq (\|U_0\| + \varepsilon) \exp(t(\|B_0\| + \varepsilon))\,; \quad \|T_n\| \leqq \exp(t(\|C_0\| + \varepsilon))$$
$$\|T\| \leqq \exp(t\|C_0\|)\,; \quad \|S\| \leqq (1 - \delta(t))^{-1}\,.$$

By application of the mean value theorem to the expressions

$$L_n - L = \exp(tB_n)\, U_n - \exp(tB_0)\, U_0\,; \quad T_n - T = \exp(-tC_n) - \exp(-tC_0)$$

it follows that

(2.16)
$$\|T_n - T\| \leqq t^2 \exp(t(\|C_0\| + \varepsilon))\, \|C_n - C_0\| \leqq t^2 \|D\|\, \|Z_n - \bar{X}\| \exp(t(\|C_0) + \varepsilon)),$$
$$\|L_n - L\| \leqq \{(\|U_0\| + \varepsilon)\, t^2 \|D\| \exp(t(\|B_0\| + \varepsilon) + \exp(t\|B_0\|\}\, \|Z_n - \bar{X}\|\,.$$

From (1.6) and (2.12), (2.14) for $n > n_0$ one gets

$$\text{(2.17)} \qquad \|S_n - S\| \leqq$$
$$\leqq \|S\|\, \|S_n\|\, \left\|\int_0^t ((\exp(-vC_n)\, D \exp(vB_n)\, U_n - \exp(vC_0)\, D \exp(vB_0)\, U_0)\, \mathrm{d}v\right\| \leqq$$
$$\leqq \|S_n\|\, \|S\| \int_0^t \|\exp(-vC_n)\, D(\exp(vB_n) - \exp(vB_0))\, U_n\|\, \mathrm{d}v +$$
$$+ \int_0^t \exp(-vC_n)\, D \exp(vB_0)\, (U_n - U_0)\|\, \mathrm{d}v +$$
$$+ \|S_n\|\, \|S\| \int_0^t \|\exp(-vC_n) - \exp(-vC_0))\, D \exp(vB_0)\, U_0\|\, \mathrm{d}v\,.$$

From (1.7) and (2.17), (2.14) one gets

$$\text{(2.18)} \qquad \|S_n - S\| \leqq$$
$$\leqq 2\|D\|\, (1 - \delta(t))^{-2} \left(\int_0^t \exp(v(\|C_0\| + \|B_0\| + 2\varepsilon)\, v^2\, \mathrm{d}v\right) \|D\|\, (\|U_0\| + \varepsilon)\,.$$
$$.\, \|Z_n - \bar{X} + 2\|D\|\, \|Z_n - \bar{X}\|\, (1 - \delta(t))^{-2} \left(\int_0^t (\exp(v(\|C_0\| + \|B_0\| + \varepsilon)\,.\right.$$
$$\|D\|\, \mathrm{d}v + 2\|D\|^2\, (1 - \delta(t))^{-2}\, \|U_0\|\, \left(\int_0^t v^2 \exp(v(\|C_0\| + \|B_0\| + \varepsilon))\,.\right.$$
$$.\, \mathrm{d}v)\, \|Z_n - \bar{X}\|\,.$$

Let us denote by γ and ϱ the positive constants defined by

$$\text{(2.19)} \qquad \gamma = \|C_0\| + \|B_0\| + 2\varepsilon\,; \quad \varrho = \|C_0\| + \|B_0\| + \varepsilon\,.$$

From (2.17)–(2.19) it follows that

$$\text{(2.20)} \qquad \|S_n - S\| \leqq$$
$$\leqq 2(1 - \delta(t))^{-2}\, \|D\|\, \|Z_n - \bar{X}\|\, (\|D\|\, (\|U_0\| + \varepsilon) \int_0^t v^2 \exp(\gamma v)\, \mathrm{d}v +$$
$$+ \int_0^t \exp(\varrho v)\, \mathrm{d}v) + 2(1 - \delta(t))^{-2}\, \|D\|^2\, \|U_0\|\, \|Z_n - \bar{X}\| \int_0^t v^2 \exp(v\varrho)\, \mathrm{d}v\,.$$

From (2.13), (2.15), (2.16) and (2.20), it follows that

$$(2.21)\qquad \|X_n(t) - X(t)\| \leqq$$
$$\leqq \{2(\|U_0\| + \varepsilon)\,\|D\|\,(1 - \delta(t))^{-2} \exp(t\varrho)\,[\|D\|\,(\|U_0\| + \varepsilon) \int_0^t v^2 \exp(\gamma v)\,dv +$$
$$+ \int_0^t \exp(\varrho v)\,dv + \|D\|\,\|U_0\| \int_0^t v^2 \exp(\gamma v)\,dv)] + \exp(t\varrho)\,t^2\,\|D\|\,(1 - \delta(t))^{-1} +$$
$$+ (1 - \delta(t))^{-1} \exp(t\|C_0\|)\,((\|U_0\| + \varepsilon)\,t^2\|D\| \exp(t(\|B_0\| + \varepsilon)) + \exp(t\,\|B_0\|)\} \,.$$
$$.\, \|Z_n - \overline{X}\| \leqq \{2 \exp(t\gamma)\,(1 - \delta(t))^{-1}\,\|D\|\,(\|U_0\| + \varepsilon)\,[\int_0^t \exp(\varrho)\,dv +$$
$$+ \varepsilon\|D\| \int_0^t v^2 \exp(v\gamma)\,dv + \|D\|\,\|U_0\| \int_0^t v^2(\exp(v\gamma) + \exp(v\varrho))\,dv] +$$
$$+ t^2\|D\|\,(\|U_0\| + \varepsilon)\,(\exp(t\varrho) + \exp(t\gamma)) + \exp(t(\|B_0\| +$$
$$+ \|C_0\|)\}\,(1 - \delta(t))^{-1}\,\|Z_n - \overline{X}\| \,.$$

Thus, under the notation of Lemma 1, the following result is proved.

Theorem 1. *Let us consider problem* (1.2), *let* $W(t)$ *be defined by* (2.10), *let us suppose that* (2.11) *is satisfied in a neighborhood* $J = [0, r]$ *of* $t = 0$, *let* ε *be a positive number, let* γ, ϱ *be defined by* (2.19), *and let* n_0 *be such that* (2.14) *is satisfied for* $n > n_0$. *Then the error of the approximation* $X_n(t)$ *defined by* (2.9) *is given by* (2.21).

3. BOUNDARY VALUE PROBLEMS: EXACT, APPROXIMATE SOLUTIONS AND ERROR ESTIMATE

The next results concern the resolution problem (1.3), under the existence hypothesis of a solution $\overline{X}$ of equation (1.5). This problem (1.3) has been studied in [8], for the time varying finite-dimensional case, and in [9], for the time invariant infinite dimensional case, but in both papers, the solution of the problem is given in terms of the entries $S_{ij}(t)$ of the matrix function $S(t)$ defined as

$$S(t) = \exp\left(t\begin{bmatrix} C & D \\ A & B \end{bmatrix}\right).$$

In this section, a computable expression for solutions of problem (1.3), in terms of the data and solutions of algebraic Riccati equations of the type (1.4) is given.

Theorem 3. *Let* $\overline{X}$ *be a solution of equation* (1.5), *and let us consider the matrices* B_0, C_0 *and* U_0 *defined by* (2.1). *Let us consider the matrices* M, N, P *and* Q *defined by the expressions*

$$(3.1)\qquad K = G - E\overline{X} + \overline{X}F\,; \quad M = -K \exp(bC_0)\,;$$
$$Q = F \int_0^b \exp((b - v)\,C_0)\,D \exp(vB_0)\,dv$$
$$N = E \exp(bB_0) - K \int_0^b \exp((b - v)\,C_0)\,D \exp(vB_0)\,dv; \quad P = F \exp(bC_0)$$

and let us suppose that there exists a solution Y of equation (1.4), *where the coefficients are given by* (3.1), *such that*

$$\text{(3.2)} \qquad \textit{For all} \quad t \in [0, b],$$
$$\textit{the matrix} \quad I + \int_0^t \exp(-vC_0)\, D \exp(vB_0)\, dvY, \quad \textit{is invertible}$$

Then a solution of problem (1.3) *is given by*

$$\text{(3.3)} \quad X(t) = \overline{X} + \exp(tB_0)\, Y(I + \int_0^t \exp(-vC_0)\, D \exp(vB_0)\, dv\, Y)^{-1} \exp(-tC_0)$$

Proof. Let us consider the Cauchy problem (1.2) for any matrix P_0 in $\mathbb{C}^{n x n}$, and let us consider the change of variable (2.4). Taking into account Lemma 1, it follows that the matrix function $X(t)$ given by (2.3) is the solution of problem (1.2) with $X(0) = P_0$. Now we are interested in finding the value of P_0 so that $X(t)$ given by (2.3) be a solution of (1.3).

Note that the boundary value condition appearing in (1.3), for the variable $X(t)$, is equivalent to the boundary value condition

$$\text{(3.4)} \qquad E\, U(b) - U(0)\, F = G - E\overline{X} + \overline{X}F$$

where $U(t) = Z(t)\, (V(t))^{-1}$, and $Z(t)$ and $V(t)$ are defined by (2.7) for all $t \in [0, b]$, and $U_0 = P_0 - \overline{X}$. By imposing that $U(t)$ satisfies the boundary condition (3.4), it follows that U_0 must verify the condition

$$\text{(3.5)} \quad E(\exp(bB_0)\, U_0(I + \int_0^b \exp(-vC_0)\, D \exp(vB_0)\, dvU_0)^{-1} \exp(-bC_0)) - \\ - U_0F = K$$

where K is given by (3.1). By postmultiplying (3.5) by the matrix

$$\exp(bC_0)\,(I + \int_0^b \exp(-vC_0)\, D \exp(vB_0)\, dv\, U_0),$$

it follows that U_0 must verify the equation

$$(E \exp(bB_0) + K \exp(bC_0) \int_0^b \exp(vC_0)\, D \exp(vB_0)\, dv)\, U_0 - \\ - U_0F \exp(bC_0) - U_0(F \exp(bC_0) \int_0^b \exp(-vC_0)\, D \exp(vB_0)\, dv = K \exp(bC_0).$$

Hence, and from (3.1), it follows that U_0 must satisfy the equation (1.4), where M, N, P and Q are given by (3.1). Conversely, if Y is a solution of (1.4)–(3.1) and we consider the problem (1.2) with $X(0) = \overline{X} + Y$, placing Y as $U(0)$, the solution of this Cauchy problem, given by Lemma 1, satisfies the boundary condition (3.4). Note that by the invertibility of $V(t)$, the condition that $U(0)$ is a solution of (1.4) to (3.1) is necessary and sufficient for $U(t)$ to be a solution of (2.5), (3.4), or equivalently, for $X(t)$ to be a solution of (1.3).

Corollary 1. *Let* $\{Z_n\}_{n \geq 1}$ *be a sequence of approximations that converges to a solution Y of equation* (1.4) *with coefficients given by* (3.1), $\overline{X}$ *being a solution of equation* (1.5) *and let* U_0, B_0, C_0 *be defined by* (2.1). *Let* $W(t)$ *be defined by* (2.10), *and* *let us suppose that for all* $t \in [0, b]$, *the condition* $a(t) = \|Y\| \, \|W(t)\| < h < 1$ *is satisfied. Then the sequence of matrix functions defined by*

(3.6)

$$X_n(t) = \bar{X} + \exp(tB_0)\left(Z_n + \textstyle\int_0^t \exp(-vC_0)\, D \exp(vB_0)\, \mathrm{d}v Z_n\right)^{-1} \exp(-tC_0)$$

is pointwise convergent to the solution $X(t)$ *of problem* (1.3), *defined by* (3.3). *The error* $X_n(t) - X(t)$ *is norm-bounded by*

$$2 \exp\left(t(\|B_0\| + \|C_0\|)\right)(1 - h)^{-1} \|Z_n - Y\| \quad \textit{for} \quad n > n_0 \tag{3.7}$$

where n_0 *is chosen such that* $\|Z_n - Y\| < (1 - h)(2h)^{-1} \|Y\|$ *if* $Y \neq 0$ *and* $n > n_0$, *and if* $Y = 0$, *then we take* n_0 *such that for* $n > n_0$ *the condition* $\|W(t)\| \, \|Z_n\| < 1$ *is satisfied.*

Proof. The result is an easy consequence of theorems 1 and 3.

Now we are going to consider a class of examples for solving the problem (1.3) by application of Th. 3, and a particular way for the resolution of the generalized Riccati equation (1.4) with coefficients given by (3.1).

Example 1. Let P, Q, M and N be the coefficient matrices defined by (3.1), and let R and H be defined by the expressions

$$H = \begin{bmatrix} P & Q \\ M & N \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$$

such that $HR = RJ_H$,

$$J_H = \begin{bmatrix} J_1 & * \\ 0 & J_2 \end{bmatrix}$$

where J_H is the Jordan canonical form of H. If $\bar{X}$ is a solution of (1.5), B_0, C_0 and U_0 are defined by (2.1) abd

(3.8) for all $t \in [0, b]$,

the matrix $R_1 + \int_0^t \exp(-vC_0)\, D \exp(vB_0)\, \mathrm{d}v R_3$ is invertible,

then a solution of problem (1.3) is given by

$$X(t) = \bar{X} + \exp(tB_0)\, R_3\left(R_1 + \textstyle\int_0^t \exp(-vC_0)\, D \exp(vB_0)\, \mathrm{d}v R_3\right)^{-1} \exp(-tC_0).$$

In fact, from the hypothesis (3.8), taking $t = 0$, it follows that R_1 is invertible. From [14], $Y = R_3 R_1^{-1}$ is a solution of (1.4), its coefficients being given by (3.1). Now the result is a consequence of Th. 3.

CONCLUSIONS

This paper presents a method for computing explicit solutions of Cauchy problems and two-point boundary value problems concerned with the generalized Riccati matrix differential equation (1.1). The method is based on the existence of solutions of algebraic Riccati type matrix equations related to the problem, and the expression for the solution of the differential problems is expressed in terms of the solutions of the corresponding algebraic problems.

The interplay between the solution of the algebraic and the differential problem allows us to obtain approximate solutions of problems (1.2) and (1.3), and an error estimate of them, in terms of approximate solutions of equations of the type (1.4) and their corresponding error estimates. Also, it is proved that the expression for the solution of the Cauchy problem (1.2) is stable with regard to a small change of the Cauchy condition.

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Souhrn

O ZOBECNĚNÉ RICCATIOVĚ MATICOVÉ DIFERENCIÁLNÍ ROVNICI. EXAKTNÍ A PŘIBLIŽNÁ ŘEŠENÍ, ODHAD CHYBY

LUCAS JÓDAR, E. NAVARRO

V práci jsou uvedeny explicitní formule pro řešení Cauchyovy úlohy a dvoubodové okrajové úlohy pro zobecněnou Riccatiovu maticovou diferenciální rovnici. Tyto výrazy lze vypočítat pomocí dat a řešení jistých algebraických Riccatiových rovnic souvisejících s danou úlohou. Vzájemné vztahy mezi algebraickou a diferenciální rovnicí jsou užity k nalezení přibližného řešení diferenciálního problému pomocí řešení problému algebraického.

Резюме

ОБ ОБОБЩЕННОМ МАТРИЧНОМ ДИФФЕРЕНЦИАЛЬНОМ УРАВНЕНИИ РИККАТИ. ТОЧНЫЕ И ПРИБЛИЖЕННЫЕ РЕШЕНИЯ, ОЦЕНКА ПОГРЕШНОСТИ

LUCAS JÓDAR, E. NAVARRO

В работе приведены явные формулы для решения задачи Коши и двуточечной краевой задачи для обобщенного матричного дифференциального уравнения Риккати. Эти выражения могут быть вычислены при помощи данных и решений некоторых алгебраических уравнений Риккати, связанных с данной задачей. Взаимные связи между алгебраическим и дифференциальным уравнениями использованы для определения приближенного решения дифференциальной задачи при помощи алгебраической задачи.

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