Aplikace matematiky

Hans-Peter Gittel

On some properties of solutions of transonic potential flow problems

Aplikace matematiky, Vol. 34 (1989), No. 5, 402–416

Persistent URL: http://dml.cz/dmlcz/104368

34 (1989) APLIKACE MATEMATIKY No. 5, 402—416

ON SOME PROPERTIES OF SOLUTIONS OF TRANSONIC POTENTIAL FLOW PROBLEMS

HANS-PETER GITTEL

Dedicated to Professor Jindřich Nečas on the occasion of his sixtieth birthday

(Received May 23, 1988)

Summary. The paper deals with solutions of transonic potential flow problems handled in the weak form or as variational inequalities. Using suitable generalized methods, which are well known for elliptic partial differential equations of the second order, some properties of these solutions are derived. A maximum principle, a comparison principle and some conclusions from both ones can be established.

Keywords: transonic potential flow, weak formulation, variational inequality, maximum principle, comparison principle.

AMS Classification: 76H05.

1. INTRODUCTION

The *irrotational, steady and isentropic flow of a non-viscous, compressible fluid* in a bounded, simply connected domain $\Omega \subset \boldsymbol{R}^N$ $(N \geqq 2)$ is described by the equation for the *velocity potential* u ($\mathfrak{v} = \nabla u$ – gas velocity):

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\varrho(|\nabla u|^2)\,\frac{\partial u}{\partial x_i}\right) = 0\,. \tag{1.1}$$

Here ϱ denotes the *density*. For a *polytropic gas* it is given by

$$\varrho = \varrho(|\nabla u|^2) = \varrho_0\left(1 - \frac{|\nabla u|^2}{q_m}\right)^{1/(\varkappa-1)} \tag{1.2}$$

for $|\nabla u|^2 < q_m$ with constants $\varrho_0 > 0$, $\varkappa > 1$ (see e.g. [7] for the physical background), To formulate boundary value problems we assume that $\partial\Omega$ is Lipschitz-continuous and has the representation $\partial\Omega = S_1 \cup S_2 \cup S \cup \mathfrak{N}$ where S_1, S_2 and S are open subsets of $\partial\Omega$ and $\mu_{N-1}(\mathfrak{N}) = 0$, μ_{N-1} being the $(N-1)$-dimensional Lebesgue measure on $\partial\Omega$. We consider two cases of *boundary conditions* for u where $\mathfrak{n}$ is the outer normal to $\partial\Omega$:

Case 1: $$u = 0 \quad \text{on} \quad S_1\,,$$

$$\varrho(|\nabla u|^2)\frac{\partial u}{\partial n} = g \quad \text{on} \quad S_2 \cup S\,;$$

Case 2: $$\mu_{N-1}(S_1) = 0\,,$$

$$\varrho(|\nabla u|^2)\frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega, \text{ where } \textstyle\int_{\partial\Omega} g \, \mathrm{d}o = 0 \text{ is assumed.}$$

An example for Case 1 can be found in [6: p. 451]. As *weak formulations* of these boundary value problems we get

$$\textstyle\int_\Omega \varrho(|\nabla u|^2)\, \nabla u \nabla v \, \mathrm{d}x = \int_R g v \, \mathrm{d}o \quad \text{for all } v \in V\,, \tag{1.3}$$

with

$$V = V_{S_1} = \{v \in W^{1,2}(\Omega) \mid v = 0 \text{ on } S_1 \text{ in trace sense}\}\,, \tag{1.4}$$

$$R = S_2 \cup S$$

in Case 1 and

$$V = V_0 := \{v \in W^{1,2}(\Omega) \mid \textstyle\int_\Omega v \, \mathrm{d}x = 0\}\,, \tag{1.5}$$

$$R = \partial\Omega$$

in Case 2. In both cases V is a Hilbert space with the norm $\|v\| = (\int_\Omega |\nabla v|^2 \, \mathrm{d}x)^{1/2}$ and $g \in L^2(R)$ is assumed. A further generalization of (1.1) and (1.3) is the *variational inequality* for $u \in K$:

$$\textstyle\int_\Omega \varrho(|\nabla u|^2)\, \nabla u \, \nabla(u - v) \, \mathrm{d}x \leqq \int_R g(u - v) \, \mathrm{d}o \quad \text{for all} \quad v \in K\,, \tag{1.6}$$

where K is a non-empty closed convex subset of V, $K \subseteq G_a$, and

$$G_a := \{v \in V \mid |\nabla v|^2 \leqq a \text{ a.e. on } \Omega\}\,. \tag{1.7}$$

K may be given by a suitable *entropy condition* [1; 6; 9], e.g.

$$K = \{v \in G_a \mid -\textstyle\int_\Omega \nabla v \, \nabla h \, \mathrm{d}x \leqq M \int_\Omega h \, \mathrm{d}x \quad \text{for all} \quad h \in (C_0^\infty(\Omega))_+$$

with constants $M \geqq 0$, $a < q_m$, and

$$(C_0^\infty(\Omega))_+ = \{h \in C^\infty(\Omega) \mid \operatorname{supp} h \subset\subset \Omega,\, h \geqq 0\}\,.$$

It is well known that the partial differential operator in (1.1) is of the mixed type. If we consider *transonic flows* then *subsonic regions* (where (1.1) is elliptic) as well as *supersonic* ones (where (1.1) is hyperbolic) occur in Ω and the transitions between them are usually discontinuous. This fact causes many difficulties in the proof of existence of solutions for (1.3) and (1.6) and this problem has not yet been completely solved. Some results in this direction have been found by Feistauer, Mandel, Morawetz, Nečas and the author for the weak problem (1.3) [1–4; 8] and for the variational inequality (1.6) [6; 9]. Nevertheless, in this paper we want to study what properties the solutions of (1.3) or (1.6) must have. Using suitable generalized methods like the maximum and comparison principles which are well known for elliptic equations of the second order (see e.g. [5]) we get some of these properties. We point out that throughout the paper we need no entropy condition.

2. MAXIMUM PRINCIPLE

If we put

$$a_{ij}(x) = \varrho(|\nabla u(x)|^2)\, \delta_{ij} \,. \tag{2.1}$$

δ_{ij} the Kronecker symbol, for a solution $u \in G_a$ of (1.3), we can consider (1.3) as a linear elliptic partial differential equation with bounded measurable coefficients a_{ij}; $i, j = 1, \ldots, N$. The ellipticity holds because of

$$\sum_{i,j=1}^{N} a_{ij}(x)\, \xi_i \xi_j = \varrho(|\nabla u(x)|^2)\, |\xi|^2 \geqq 0 \,.$$

The last relation, and hence the following assertions are also valid for any density function $\varrho = \varrho(\nabla u)$ which is positive, continuous and bounded for $|\nabla u|^2 \leqq a$, with some $a \in (0, \infty)$, but may be different from (1.2).

In this section we first consider the general case of a given symmetric matrix $(a_{ij}(x))$ with bounded measurable elements and

$$\sum_{i,j=1}^{N} a_{ij}(x)\, \xi_i \xi_j > 0 \tag{2.2}$$

for all $x \in \Omega$, $\xi = (\xi_1, \ldots, \xi_N) \in \boldsymbol{R}^N \setminus \{0\}$. For $u, v \in W^{1,2}(\Omega)$ we define the bilinear form

$$L(u, v) = \int_\Omega \sum_{i,j=1}^{N} a_{ij}(x)\, u_{x_i} v_{x_j}\, \mathrm{d}x \,. \tag{2.3}$$

Let R_1 be an open subset of $\partial\Omega$ with $\mu_{N-1}(R_1) > 0$. According to [5: p. 168] we say that $u \leqq 0$ on R_1 is satisfied if $u^+ = \max\{u, 0\} = 0$ on R_1 in the trace sense. Other definitions concerning the ordering on R_1 follow naturally: $u \leqq v$ on R_1 if $u - v \leqq 0$ on R_1;

$$\sup_{R_1} u = \inf\{k \mid u \leqq k \text{ on } R_1,\ k \in \boldsymbol{R}\} \,; \quad \inf_{R_1} u = -\sup_{R_1}(-u) \,.$$

Our results will be obtained by easy extensions of the usual arguments, However, in the standard literature they are not to be found explicitly for a differential operator in our special form (2.3). Therefore we present the simple proofs here, too.

Lemma 2.1. *Let $u \in W^{1,2}(\Omega)$ satisfy*

$$L(u, h) \leqq 0 \quad (L(u, h) \geqq 0) \quad \textit{for all} \quad h \geqq 0\,, \quad h \in V_{R_1}\ ^{1)}\,. \tag{2.4}$$

---

[1]) V_{R_1} is defined by (1.4) where S_1 has to be replaced by R_1. In the case $\mu_{N-1}(R_1) = 0$ we set $V_{R_1} = W^{1,2}(\Omega)$.

Then

(2.5)
$$\sup_{\Omega} u \leqq \sup_{R_1} u \quad (\inf_{\Omega} u \geqq \inf_{R_1} u)$$

if $\mu_{N-1}(R_1) > 0$. *In the case* $\mu_{N-1}(R_1) = 0$ *we have* $u = \text{const.}$ *a.e. on* Ω.

Proof. a) Let $c = \sup_{R_1} u$, $v = u - c$, $\Omega^+ = \{x \in \Omega \mid v(x) > 0\}$. Then we have

$$v^+ = \max\{v, 0\} = \begin{cases} v & \text{on} \quad \Omega^+ \\ 0 & \text{on} \quad \Omega \setminus \Omega^+ \end{cases}$$

$$\nabla v^+ = \begin{cases} \nabla v = \nabla u & \text{on} \quad \Omega^+ \\ 0 & \text{on} \quad \Omega \setminus \Omega^+ , \end{cases}$$

$v^+ = 0$ on R_1, and hence $v^+ \in V_{R_1}$. Putting $h = v^+$ in (2.4) and using (2.2) we obtain

$$0 \leqq L(v^+, v^+) = \int_\Omega \sum_{i,j=1}^N a_{ij} v^+_{x_i} v^+_{x_j} \, dx = \int_\Omega \sum_{i,j=1}^N a_{ij} u_{x_i} v^+_{x_j} \, dx \leqq 0 .$$

This inequality implies $\nabla v^+ = 0$ a.e. on Ω and $v^+ = \text{const.}$ By virtue of $v^+ = 0$ on R_1 we obtain $v^+ = 0$, which means $v = u - c \leqq 0$ a.e. on Ω.

b) Let $\mu_{N-1}(R_1) = 0$. For an arbitrary $\varphi \in C^\infty(\bar{\Omega})$ we set $h = \varphi - \min_{\bar{\Omega}} \varphi$ and have $h \in W^{1,2}(\Omega)$, $h \geqq 0$, $\nabla h = \nabla \varphi$. Then (2.3), (2.4) yield $L(u, \varphi) = L(u, h) \leqq 0$, and hence $L(u, \varphi) = 0$ for all $\varphi \in W^{1,2}(\Omega)$. Here we have used the density of $C^\infty(\bar{\Omega})$ in $W^{1,2}(\Omega)$ which holds because $\partial\Omega$ is assumed to be Lipschitz-continuous. We get $L(u, u) = 0$ and this implies the assertion.

The case in parentheses follows in a similar way. □

Remark 2.2. Suppose that $V = V_0$ and let $u \in V$ satisfy (1.3). Since in this case $\int_{\partial\Omega} g \, do = 0$ is assumed it is easy to see that the integral relation (1.3) is also valid for $v \in W^{1,2}(\Omega)$.

We set $R^+ = \{x \in R \subseteq \partial\Omega \mid g(x) > 0\}$ and $R^- = \{x \in R \mid g(x) < 0\}$ and assume that R^+, R^- are open subsets of $\partial\Omega$. This is fulfilled if g is piecewise continuous on R. Now, we can establish the following properties.

Theorem 2.3. *Let* $\mu_{N-1}(R) \neq 0$, $g \neq 0$ *and let* $u \in G_a$ *be a solution of* (1.3). *Then*

a) $\mu_{N-1}(R^+)$, $\mu_{N-1}(R^-)$ *are positive for* $V = V_0$.

b) *u cannot be a constant in any subdomain* $\Omega' \subseteq \Omega$ *for which* $\partial\Omega' \cap R^+$ *or* $\partial\Omega' \cap R^-$ *has a positive measure* μ_{N-1}.

c) *u has a nonnegative maximum M and a nonpositive minimum m on* $\bar{\Omega}$. *It achieves M on* $\overline{R^+ \cup S_1}$ *and m on* $\overline{R^- \cup S_1}$ *but does not achieve them in the interior of* Ω.

d) *Suppose that* $u = \text{const.}$ *on* $\partial\Omega'$ *for a subdomain* $\Omega' \subseteq \Omega$. *Then we have* $u = \text{const.}$ *on* Ω' *and* $\mu_{N-1}(\partial\Omega' \cap R^+) = \mu_{N-1}(\partial\Omega' \cap R^-) = 0$.

Proof. a) Because of $\int_R g \, do = 0$ (Case 2), $g \neq 0$, neither $g \leqq 0$ nor $g \geqq 0$ is possible a.e. on R.

b) Suppose that $u = \text{const.}$ in a subdomain $\Omega' \subseteq \Omega$ with $\mu_{N-1}(\partial\Omega' \cap R^+) > 0$. We choose a $y \in \partial\Omega' \cap R^+$ and a sufficiently small ball $B = B(y)$ such that $B \cap \Omega \subseteq \subseteq \Omega'$ and $B \cap \partial\Omega \subseteq R^+$ (see Fig. 1). For an arbitrary $\varphi \in C_0^\infty(B)$ we put

$$v(x) = \begin{cases} \varphi(x), & x \in B \\ 0 & x \notin B . \end{cases}$$

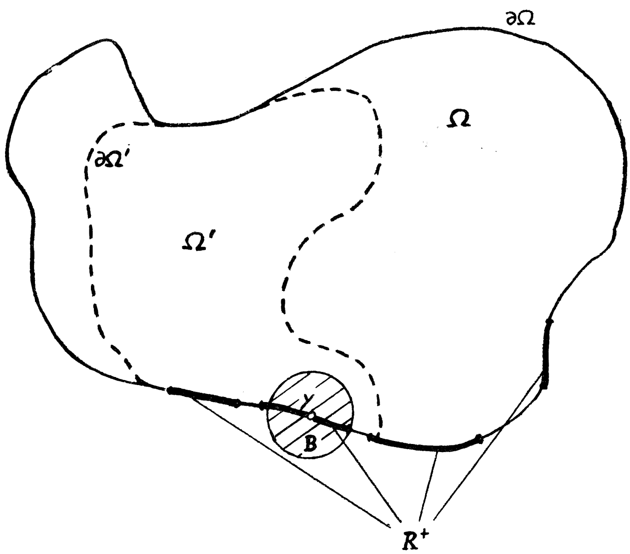

Fig. 1.

Then we have $v \in V_{S_1}$ and

$$0 = \int_{\Omega \cap B} \varrho(|\nabla u|^2) \nabla u \, \nabla\varphi \, dx = \int_\Omega \varrho(|\nabla u|^2) \nabla u \, \nabla v \, dx = \int_R g \, v \, do =$$
$$= \int_{R \cap B} g\varphi \, do$$

where (1.3) is used. This equality yields $g = 0$ on $R \cap B \subseteq R^+$ in contradiction to the definition of R^+. By replacing R^+ by R^- the other case follows.

c) By virtue of $u \in G_a$ we have $u \in W^{1,\infty}(\Omega)$ and u is a.e. equal to a function from $C^{0,1}(\bar{\Omega})$ which has a maximum and a minimum on $\bar{\Omega}$.

Now, we first apply Lemma 2.1 with $R_1 = R^+ \cup S_1$ and a_{ij} given by (2.1). From (1.3) together with Remark 2.2 we obtain

$$L(u, h) = \int_{R^+} gh \, do + \int_{R^-} gh \, do = \int_{R^-} gh \, do \leqq 0$$

for all $h \geqq 0$, $h \in V_{R_1}$, and hence $M = \max_{\bar{\Omega}} u = \max_{\overline{R^+ \cup S_1}} u$. Application of Lemma 2.1

with $R_1 = R^- \cup S_1$ yields $m = \min_{\bar{\Omega}} u = \min_{\overline{R^- \cup S_1}} u$. In the case $\mu_{N-1}(S_1) > 0$ we have $M \geqq \inf_{S_1} u = 0 = \sup_{S_1} u \geqq m$. In the other case the inequality $m\, \mu_N(\Omega) \leqq \int_\Omega u \,\mathrm{d}x \leqq \leqq M\, \mu_N(\Omega)$ and the condition in (1.5) yield $m \leqq 0 \leqq M$.

The rest of this assertion follows from the strong maximum principle for weak subsolutions [5: Theorem 8.19] together with part b) above.

d) We consider the equation (1.3) for $v \in C_0^\infty(\Omega')$, extended by 0 outside of Ω', and get $\int_{\Omega'} \varrho(|\nabla u|^2)\, \nabla u\, \nabla v \,\mathrm{d}x = 0$ for all $v \in W_0^{1,2}(\Omega')$. Replacing Ω by Ω' we can again apply Lemma 2.1 with $R_1 = \partial\Omega'$, $V_{R_1} = W_0^{1,2}(\Omega')$ and a_{ij} given by (2.1). Note that in this case no assumptions on $\partial\Omega'$ are needed in the proof of Lemma 2.1. The inequalities (2.5) immediately yield $u = \text{const.}$ and part b) above completes the proof. □

Corollary 2.4. *Let $g = 0$ or $\mu_{N-1}(R) = 0$. Then (1.3) has only the solution $u = 0$.*

Proof. We apply Lemma 2.1 with $R_1 = S_1$ and recall the corresponding definition of V. □

Remark 2.5. The subdomains in which $u = \text{const.}$ have a physical meaning. Namely, there the gas is at rest because of $\nabla u = 0$. Theorem 2.3b), d) deals with such subdomains. Moreover, part b) is a generalization of the following fact: For a u which is smooth in a neighbourhood of R^+ or R^-, (1.3) yields $\varrho(|\nabla u|^2)\, \partial u/\partial n = g$, $\partial u/\partial n > 0$ on R^+, and $\partial u/\partial n < 0$ on R^-, respectively. Hence, u cannot be a constant in this neighbourhood.

Theorem 2.6. *Let $V = V_{S_1}$, $g \leqq 0$ $(\geqq 0)$, $g \not\equiv 0$ and $u \in G_a$ be a solution of (1.3). Then we have $u < 0$ (>0) in the interior of Ω. Furthermore, for all points $x_0 \in S_1$ at which Ω satisfies an* interior sphere condition[2]) *and the outer normal derivative $\partial u/\partial n$ exists we have $\partial u/\partial n(x_0) > 0$ (<0).*

Proof. According to Theorem 2.3c) with $\mu_{N-1}(R^+) = 0$ we have $u(x) < \max_{\bar{S}_1} u = = 0$ for all $x \in \Omega$, analogously for the case in parantheses. The assertion concerning the sign of the normal derivative follows from [5: Lemma 3.4] which is easy to extend for weak subsolutions. □

Remark 2.7. The last theorem shows that on S_1 only flux outwards occurs, provided $g \leqq 0$; that means: only flux inwards on R, is supposed.

[2]) That means: there exists a ball $B \subset \Omega$ with $x_0 \in \partial B$. This condition is fulfilled if e.g. $\partial\Omega \in C^2$ [5: pp. 27, 32].

3. COMPARISON PRINCIPLE

If we want to compare two solutions u_1, u_2 of (1.3) or (1.6) we have to study the difference between them. From the relation (1.3) or (1.6) we can derive the inequality

$$\int_\Omega (\varrho(|\nabla u_1|^2)\, \nabla u_1 - \varrho(|\nabla u_2|^2)\, \nabla u_2)\, \nabla(u_1 - u_2)\, \mathrm{d}x \leqq 0\,.$$

An estimate from below is required. Therefore we estimate the integrand. For $p_1, p_2 \in \boldsymbol{R}^N$ we define the function

$$\begin{aligned}(3.1) \qquad F(p_1, p_2) &:= (\varrho(p_1^2)\, p_1 - \varrho(p_2^2)\, p_2) \cdot (p_1 - p_2) = \\ &= f(1) - f(0) = \int_0^1 f'(t)\, \mathrm{d}t\end{aligned}$$

where $f(t) = \varrho(r^2(t))\, r(t)\, (p_1 - p_2)$, $r(t) = p_2 + t(p_1 - p_2)$. Let

$$(3.2) \qquad H(q) := \varrho(q) + 2q\, \varrho'(q) \quad \text{be defined for} \quad q \in [0, a]\,.$$

Simple computation shows that

$$f'(t) \geqq 2\varrho'(r^2(t))\, r^2(t)\, (p_1 - p_2)^2 + \varrho(r^2(t))\, (p_1 - p_2)^2 = H(r^2(t))\, (p_1 - p_2)^2$$

if $\varrho'(q) < 0$ is assumed. Hence, we have

$$(3.3) \qquad F(p_1, p_2) = F(p_2, p_1) \geqq (p_1 - p_2)^2 \int_0^1 H(r^2(t))\, \mathrm{d}t\,.$$

Remark 3.1. For a polytropic gas (1.2) we obtain the formula

$$(3.4) \qquad H(q) = \varrho(q) \frac{\varkappa + 1}{\varkappa - 1} \cdot \frac{q_c - q}{q_m - q} \quad \text{with} \quad q_c := \frac{\varkappa - 1}{\varkappa + 1}\, q_m\,.$$

It is easy to see that

$$(3.5) \qquad H(q) > 0\, (<0) \quad \text{iff} \quad q < q_c\, (> q_c)\,.$$

For $|\nabla u|^2 < q_c\, (> q_c)$ the partial differential equation (1.1) is elliptic (hyperbolic). This function H plays an important role not only for the type of the differential operator but also for our estimates. Thus, we will examine it carefully.

From now on the following assumptions are imposed on $\varrho = \varrho(q)$:

i) $\varrho \in C^2([0, q_m))$,

ii) $\varrho(q) > 0$, $\varrho'(q) < 0$ for all $q \in [0, q_m)$,

iii) H defined by (3.2) satisfies (3.5) with some $q_c \in (0, q_m)$,

iv) $H'(q) < 0$ for all $q \in [q_c, (1 + d_0)\, q_c]$ for some given $d_0 \in \left(0, \dfrac{q_m - q_c}{q_c}\right)$.

If ϱ is defined by (1.2) all these assumptions are satisfied. To verify iv) we calculate

$$(3.6) \qquad H'(q) = \varrho(q) \frac{\varkappa + 1}{(\varkappa - 1)^2} \cdot \frac{q - 3q_c}{(q - q_m)^2}$$

from (3.4). Hence, iv) is fulfilled if we choose $d_0 < \min\{2/(\varkappa - 1), 2\}$. Using i)–iv) we obtain the following estimates for F.

Lemma 3.2. *Let $d \in [0, d_0]$. Then there exists a constant $b = b(d) > 0$ such that*

$$F(p_1, p_2) \geqq (b(p_1 - p_2)^2 + H((1 + d)\, q_c))\,(p_1 - p_2)^2 \tag{3.7}$$

for all $p_1^2, p_2^2 \leqq (1 + d)\, q_c$.

Proof. a) For a sufficiently small $\varepsilon > 0$ we have $H'(q) < 0$ for all $q \in I_\varepsilon :=$ $:= [(1 - \varepsilon)\, q_c, (1 + d)\, q_c]$. Thus, the mean value theorem yields

$$H((1 + d)\, q_c) - H(q) \leqq M((1 + d)\, q_c - q) \quad \text{with} \quad M = \max_{I_\varepsilon} H' < 0$$

On $J_\varepsilon := [0, (1 - \varepsilon)\, q_c]$ the function H has a minimum $m > 0$ by virtue of (3.5), and for $q \in J_\varepsilon$ we have

$$H(q) \geqq m \geqq A + (m - A)\frac{(1 + d)\, q_c - q}{(1 + d)\, q_c}$$

with $A := H((1 + d)\, q_c) \leqq 0$. If we put

$$B = \min\left\{-M, \frac{m - A}{(1 + d)\, q_c}\right\} > 0$$

these two inequalities imply

$$H(q) \geqq B((1 + d)\, q_c - q) + A \quad \text{for} \quad q \in [0, (1 + d)\, q_c]\,. \tag{3.8}$$

b) Because of the convexity of the function $r^2(t)$ on $[0, 1]$ we have

$$r^2(t) \leqq tr^2(1) + (1 - t)\, r^2(0) = tp_1^2 + (1 - t)\, p_2^2 \leqq (1 + d)\, q_c\,, \tag{3.9}$$

and (3.8) yields $H(r^2(t)) \geqq B((1 + d)\, q_c - r^2(t)) + A$. Using (3.3) and carrying out the integration we finally obtain

$$F(p_1, p_2) \geqq (p_1 - p_2)^2\,(B((1 + d)\, q_c - \tfrac{1}{2}p_2^2 - \tfrac{1}{2}p_1^2 + \tfrac{1}{6}(p_1 - p_2)^2) + A) \geqq$$
$$\geqq \frac{B}{6}\,|p_1 - p_2|^4 + A)p_1 - p_2)^2\,. \qquad \square$$

Corollary 3.3. *Let $H'(q) < 0$ on $I_1 = [0, (1 + d)\, q_c]$. Then we can choose $\varepsilon = 1$ in part a) of the proof above obtaining $b = -\frac{1}{6}\max_{I_1} H'$.*

Note that in (3.7) the constant $A = A(d) = H((1 + d)\, q_c)$ is negative for $d > 0$. But for $d = 0$ we have $H(q_c) = 0$ by virtue of (3.5), and hence

$$F(p_1, p_2) \geqq b(0)\,|p_1 - p_2|^4 \quad \text{for all} \quad p_1^2, p_2^2 \leqq q_c\,. \tag{3.10}$$

Lemma 3.4 *Let $p_1^2 \leqq (1 - d)\, q_c$ for some $d \in (0, 1]$. Then there exists a $\delta = \delta(d) > 0$ such that (3.10) is valid for all $p_2^2 \leqq (1 + \delta)\, q_c$.*

Proof. a) First, let $p_2^2 \leqq q_c$. Hence, (3.9) yields $r^2(t) \leqq (1 - td)\, q_c \leqq q_c$. According to (3.8) with $d = 0$, we obtain

$$\int_0^1 H(r^2(t))\,dt \geqq B \int_0^1 (q_c - r^2(t))\,dt =$$
$$= B(q_c - \tfrac{1}{2}p_2^2 - \tfrac{1}{2}p_1^2 + \tfrac{1}{6}(p_1 - p_2)^2) \geqq B(\tfrac{1}{2}dq_c + \tfrac{1}{6}(p_1 - p_2)^2)\,.$$

b) The function $K(p_1, p_2) := \int_0^1 H(r^2(t))\,dt - \frac{1}{6}B(p_1 - p_2)^2$ is uniformly continuous for $p_1^2 \leqq (1 - d)\, q_c$, $p_2^2 \leqq (1 + d_0)\, q_c$. By virtue of a) we have $K(p_1, \bar{p}_2) \geqq \geqq \frac{1}{2}Bdq_c > 0$ for $\bar{p}_2^2 = q_c$. Consequently, there exists an $\varepsilon = \varepsilon(d) > 0$ which is independent of p_1, p_2, such that $K(p_1, p_2) > 0$ for all p_2 with $|p_2 - \bar{p}_2| < \varepsilon$. Thus we obtain $K(p_1, p_2) \geqq 0$ for all $p_1^2 \leqq (1 - d)\, q_c$, $p_2^2 \leqq (1 + \delta)\, q_c$ if $\delta = \delta(d) > 0$ is suitably chosen. The inequality (3.3) yields the assertion. □

Lemma 3.5. *Let H be convex on $[0, (1 + d_0)\, q_c]$. Then (3.10), with $b(0) = -\frac{1}{6} H'(q_c)$, is valid for all $p_1^2 \leqq (1 - d)\, q_c$, $p_2^2 \leqq (1 + d)\, q_c$, $d \in (0, d_0]$.*

Proof. Using the convexity and $H(q_c) = 0$ we get $H(q) = H(q) - H(q_c) \geqq \geqq H'(q_c)\,(q - q_c)$ for $q \in I_1$. From (3.9) $r^2(t) \in I_1$ follows, and hence (3.3) yields

$$F(p_1, p_2) \geqq -H'(q_c)\,(p_1 - p_2)^2\,(q_c - \tfrac{1}{2}p_2^2 - \tfrac{1}{2}p_1^2 + \tfrac{1}{6}(p_1 - p_2)^2) \geqq$$
$$\geqq -\tfrac{1}{6} H'(q_c)\,|p_1 - p_2|^4\,.$$
□

The question arises which Lemmas hold for H in the case of a polytropic gas. The answer depends on $\varkappa$. From (3.6) we obtain $H'(q) < 0$, but $H''(q) \geqq 0$ for $1 < \varkappa \leqq 2$ and $H''(q) < 0$ for $\varkappa > 2$. Hence, Lemma 3.2 with Corollary 3.3 and Lemma 3.4 are valid for both cases of $\varkappa$ but with different $b(d)$. On the other hand, Lemma 3.5, which is a strengthened variant of Lemma 3.4, is only valid for $\varkappa \leqq 2$ (note that $\varkappa = 1.4$ for air).

Application of inequality (3.10) yields the following comparison principle.

Theorem 3.6. *Let $u_1, u_2 \in W^{1,2}(\Omega)$ satisfy $u_1 \leqq u_2$ on R_1 and*

$$Q(u_1, h) := \int_\Omega \varrho(|\nabla u_1|^2)\, \nabla u_1\, \nabla h\, dx \leqq Q(u_2, h) \tag{3.11}$$

for all $h \geqq 0$, $h \in V_{R_1}$. Moreover, we suppose that $|\nabla u_1|^2 \leqq (1 - d)\, q_c$, $|\nabla u_2|^2 \leqq \leqq (1 + \delta)\, q_c$ or $|\nabla u_2|^2 \leqq (1 - d)\, q_c$, $|\nabla u_1|^2 \leqq (1 + \delta)\, q_c$ a.e. on Ω for some $d \in [0, 1]$, where $\delta = \delta(d)$ is the function gives in Lemma 3.4 and $\delta(0) = 0$. Then $u_1 \leqq u_2$ on Ω if $\mu_{N-1}(R_1) > 0$. In the case $\mu_{N-1}(R_1) = 0$ we have $u_1 - u_2 =$ const. on Ω.

Proof. a) For $\mu_{N-1}(R_1) > 0$ we put $w = u_1 - u_2$, $\Omega^+ = \{x \in \Omega \mid w(x) > 0\}$ and have $w^+ \in V_{R_1}$. Using the definition (3.1), and (3.11) with $h = w^+$, we obtain

$$0 \geqq \int_\Omega (\varrho(|\nabla u_1|^2)\, \nabla u_1 - \varrho(|\nabla u_2|^2)\, \nabla u_2)\, \nabla w^+\, dx = \int_{\Omega^+} F(\nabla u_1, \nabla u_2)\, dx\,.$$

Application of Lemma 3.4 (or Lemma 3.2 in the case $d = 0$) with $p_1 = \nabla u_1(x)$, $p_2 = \nabla u_2(x)$ yields

$$0 \geqq b(0) \int_{\Omega^+} |\nabla u_1 - \nabla u_2|^4 \, dx = b(0) \int_\Omega |\nabla w^+|^4 \, dx .$$

This inequality implies $\nabla w^+ = 0$ a.e. on Ω and finally $w \leqq 0$ (see the end of part a) in the proof of Lemma 2.1).

b) Let $\mu_{N-1}(R_1) = 0$. First, by the same argument as in part b) of the proof of Lemma 2.1 the relation $Q(u_1, \varphi) = Q(u_2, \varphi)$ for all $\varphi \in W^{1,2}(\Omega)$ follows. We proceed as above. Putting $\varphi = u_1 - u_2$ we obtain $0 = \int_\Omega F(\nabla u_1, \nabla u_2) \, dx \geqq \geqq b(0) \int_\Omega |\nabla u_1 - \nabla u_2|^4 \, dx$, and hence $u_1 - u_2 = \text{const.}$ on Ω. □

Corollary 3.7. *If, in addition, H is assumed convex then the last theorem holds with $\delta(d) = d$ for $d \leqq \min\{1, d_0\}$ because Lemma 3.5 can be used.*

Theorem 3.6 is a generalization of the well known comparison principle for smooth functions u_1, u_2 [5: p. 207] where the quasilinear differential operator must be elliptic with respect to only one of the two functions. From our comparison principle a uniqueness result for the weak problem (1.3) follows immediately.

Theorem 3.8. *Let $u \in V$ be a solution of (1.3), and $|\nabla u|^2 \leqq (1 - d) q_c$ a.e. on Ω for some $d \in [0, d_0]$. Then there is no other solution of (1.3) in G_a with $a = (1 + \delta(d)) q_c$. In particular, problem (1.3) has at most one solution in G_{q_c}.*

Proof. Suppose that $v \in G_a$ is another solution of (1.3). Then we have $Q(u, h) = Q(v, h)$ for all $h \geqq 0$, $h \in V_{S_1}$ where Remark 2.2 has to be used in the case $V = V_0$. Application of Theorem 3.6 with $R_1 = S_1$, and $u_1 = u$, $u_2 = v$ or vice versa, yields $u = v$ on Ω provided $\mu_{N-1}(R_1) > 0$. In the other case we get $u = v + \text{const.}$ By virtue of the condition in (1.5) this constant must be 0. □

4. SOME CONCLUSIONS AND ESTIMATES FOR THE DIFEFRENCE OF TWO SOLUTIONS

Throughout this section we assume that in addition to i)–iv) H is a convex function on $[0, (1 + d_0) q_c]$ (that means: $\varkappa \leqq 2$ for a polytropic gas (1.2)). Let $d \in [0, d_0]$ be a given number. Then we put $G(d) = G_a$ with $a = (1 + d) q_c$.

Definition 4.1. Let $u \in W^{1,2}(\Omega)$. Then $\Omega_e(u) = \{x \in \Omega \mid |\nabla u|^2 \leqq (1 - d) q_c\}$ denotes the *elliptic or subsonic region* with respect to u in Ω, $\Omega_t(u) = \{x \in \Omega \mid (1 - d) q_c < |\nabla u|^2 \leqq (1 + d) q_c\}$ the *transonic region*, and $\Omega_h(u) = \{x \in \Omega \mid (1 + d) q_c < |\nabla u|^2\}$ the *hyperbolic or supersonic region*. u is called *elliptic or subsonic* in Ω if $\mu_N(\Omega \setminus \Omega_e(u)) = 0$, and *transonic* if $\mu_N(\Omega \setminus \Omega_e)u)) = \mu_N(\Omega_t(u)) > 0$.

Remark 4. 2. Let $d > 0$. Then the differential operator of (1.1) is elliptic with respect to u in $\Omega_e(u)$, and hyperbolic in $\Omega_h(u)$.

From Theorem 3.6 some information on the transonic region with respect to a solution of (1.3) follows.

Theorem 4.3. *Let $u \in G(d)$ be a solution of (1.3). Then a subset of $\Omega_t(u)$ cannot be contained in any subdomain $\Omega' \subseteq \Omega$ for which the Dirichlet problem*

$$\int_{\Omega'} \varrho(|\nabla w|^2) \nabla w \nabla v \, dx = 0 \quad \text{for all} \quad v \in W^{1,2}(\Omega'), \tag{4.1}$$
$$w - u = 0 \quad \text{on} \quad \partial\Omega',$$

has a subsonic solution w in Ω'.

Proof. We consider the equation (1.3) for $v \in C_0^\infty(\Omega')$, extended by 0 outside of Ω', and obtain that u is a solution of (4.1). Recalling that $|\nabla u|^2 \leqq (1 + d) q_c$, $|\nabla w|^2 \leqq (1 - d) q_c$ a.e. on Ω' we can apply Theorem 3.6, Corollary 3.7 if we replace Ω by Ω' and put $R_1 = \partial\Omega'$, $V_{R_1} = W_0^{1,2}(\Omega')$. This yields $u = w$ on Ω' and the assertion follows. □

Remark 4.4. Roughly speaking, the assertion of Theorem 3.8 is the following one: If there exists a subsonic solution of (1.3) then we have no transonic one.

Now, we proceed to study the difference of two solutions of the variational inequality (1.6). Lemma 3.2 and Lemma 3.5 allow us to establish a general estimate.

Theorem 4.5. *Let $u_l \in K \subseteq G(d)$ be a solution of (1.6) with $g = g_l$, $l = 1, 2$. Then*

$$\int_\Omega |\nabla u_1 - \nabla u_2|^4 \, dx \leqq \tag{4.2}$$
$$\leqq c_1 \int_{\Omega_t} |\nabla u_1 - \nabla u_2|^2 \, dx + c_2 \|u_1 - u_2\|_{W^{1,2}(\Omega)} \|g_1 - g_2\|_{L^2(R)}$$

with

$$\Omega_t = \Omega_t(u_1) \cap \Omega_t(u_2), \tag{4.3}$$

and with positive constants $c_2 = c_2(\Omega)$; $c_1 = c_1(d) \to 0 + 0$ if $d \to 0 + 0$.

Proof. a) If $g = g_1$ we put $v = u_2$ in (1.6), and if $g = g_2$ we put $v = u_1$. We obtain two inequalities which added to (3.1) yield

$$\int_\Omega F(\nabla u_1, \nabla u_2) \, dx \leqq \int_R (g_1 - g_2)(u_1 - u_2) \, do. \tag{4.4}$$

We split up $\Omega = \Omega_e(u_1) \cup \Omega_e(u_2) \cup \Omega_t$. On $\Omega_e(u_l)$ we can apply Lemma 3.5 and according to (3.10) we have

$$F(\nabla u_1, \nabla u_2) \geqq b(0) |\nabla u_1 - \nabla u_2|^4 .$$

On Ω_t Lemma 3.2 gives the estimate

$$F(\nabla u_1, \nabla u_2) \geqq b(d) |\nabla u_1 - \nabla u_2|^4 + A(d) |\nabla u_1 - \nabla u_2|^2 .$$

Since H is convex for $q \in [0, (1 + d_0) q_c]$ we have $H'(q) \leqq H'((1 + d_0) q_c) =: c_0 < 0$. Hence, using Corollary 3.3 we find $b(d) = -\frac{1}{6} \max_{I_1} H' \geqq -c_0/6$. Finally, we obtain

$$\int_\Omega F(\nabla u_1, \nabla u_2)\,\mathrm{d}x \geqq -\tfrac{1}{6}c_0 \int_\Omega |\nabla u_1 - \nabla u_2|^4\,\mathrm{d}x + \\ + A(d) \int_{\Omega_t} |\nabla u_1 - \nabla u_2|^2\,\mathrm{d}x\,.$$

b) We can estimate the right hand side of (4.4) in the following way:

$$\left|\int_R (g_1 - g_2)(u_1 - u_2)\,\mathrm{d}o\right| \leqq \|g_1 - g_2\|_{L^2(R)} \|u_1 - u_2\|_{L^2(R)} \leqq \\ \leqq c\|g_1 - g_2\|_{L^2(R)} \|u_1 - u_2\|_{W^{1,2}(\Omega)}$$

with $c = c(\Omega) > 0$. The last two inequalities together with (4.4) imply (4.2) where

$$c_2 = -\frac{6c}{c_0}\,; \quad c_1 = \frac{6A(d)}{c_0} = \frac{6}{c_0} H((1 + d)\,q_c) \to \frac{6}{c_0} H(q_c) = 0 \quad \text{if } d \to 0\,. \qquad \square$$

From the estimate (4.2) we are able to derive a result similar to Theorem 3.8, Remark 4.4.

Theorem 4.6. *If there exists a subsonic solution $u \in K$ of (1.6) then we have no other solution in $K \subseteq G(d)$. In particular, the variational inequality (1.6) has at most one solution in $K \subseteq G_q$.*

Proof. Suppose that $v \in K$ is another solution of (1.6). Then from (4.2) with $g_1 = g_2 = g$ we obtain

$$\int_\Omega |\nabla u - \nabla v|^4\,\mathrm{d}x \leqq c_1(d) \int_{\Omega_t} |\nabla u - \nabla v|^2\,\mathrm{d}x = 0\,.$$

Here we have used that $\mu_N(\Omega_t) \leqq \mu_N(\Omega_t(u)) = 0$ because u is assumed to be subsonic. Hence, $\nabla(u - v) = 0$ a.e. on Ω and finally $u = v$ by virtue of the definitions (1.4), (1.5) of V. $\square$

If there exist two different solutions of (1.6) with the same boundary data g we can also use (4.2) to obtain some conclusions.

Theorem 4.7. *Let $u_1, u_2 \in K \subseteq G(d)$ be two solutions of (1.6), and let Ω_t be defined by (4.3). Then*

a) *$u_1 \neq u_2$ is only possible if $\mu_N(\Omega_t) > 0$.*

b) *If $u_1 - u_2 = \text{const.}$ on Ω_t then $u_1 = u_2$ on Ω,*

c) $\|\nabla u_1 - \nabla u_2\|^2_{L^2(\Omega_t)} \leqq c_1(d)\,\mu_N(\Omega_t)$.

Proof. a), b) follow immediately from the inequality (4.2) with $g_1 = g_2 = g$.

c) The Schwarz inequality and (4.2) yield

$$\left(\int_{\Omega_t} |\nabla u_1 - \nabla u_2|^2\,\mathrm{d}x\right)^2 \leqq \mu_N(\Omega_t) \int_{\Omega_t} |\nabla u_1 - \nabla u_2|^4\,\mathrm{d}x \leqq \\ \leqq \mu_N(\Omega_t)\,c_1(d) \int_{\Omega_t} |\nabla u_1 - \nabla u_2|^2\,\mathrm{d}x\,. \qquad \square$$

Corollary 4.8. *By virtue of $H(q_c) = 0$ we have*

$$c_1(d) = \frac{6}{c_0}\Big(H((1 + d)\, q_c) - H(q_c)\Big) = \frac{6}{c_0}\, dq_c\, H'((1 + \vartheta d)\, q_c)\,, \quad \vartheta \in (0, 1)\,.$$

Hence, the right hand side of the inequality in c) of the above theorem is $O(d)$ if $d \to 0$. Note that $\mu_N(\Omega_t)$ remains bounded only if $d \to 0$.

Inequality (4.2) and Theorem 4.7 show that Ω_t, defined by (4.3), is the crucial subset of Ω when we study the difference of two solutions u_1, u_2. The behaviour of $u_1 - u_2$ on Ω_t determines in a sense its behaviour on the whole of Ω. Now, let us consider varying boundary data.

Theorem 4.9. *Let $u, u_0 \in K \subseteq G(d)$ be solutions of (1.6) with boundary data g, and g_0, respectively. Moreover, suppose that $\|g - g_0\|_{L^2(R)} \leqq \varepsilon$ and $\mu_N(\Omega_t(u_0)) \leqq \varepsilon^2$ for some $\varepsilon > 0$. Then $\|u - u_0\|_{W^{1,2}(\Omega)} \leqq C\varepsilon^{1/3}$ with $C = C(d, \Omega) > 0$.*

Proof. We put $u_1 = u$, $u_2 = u_0$, $g_1 = g$ and $g_2 = g_0$ in (4.2), and obtain

$$\begin{aligned} \int_\Omega |\nabla u - \nabla u_0|^4 \,\mathrm{d}x \leqq\; & c_1\Big(\int_{\Omega_t} (|\nabla u| + |\nabla u_0|)^2 \,\mathrm{d}x\Big)^{1/2} \Big(\int_\Omega |\nabla u - \nabla u_0|^2 \,\mathrm{d}x\Big)^{1/2} + \\ & + c_2\|g - g_0\|_{L^2(R)}\, \|u - u_0\|_{W^{1,2}(\Omega)} \leqq \\ & \leqq (2\, c_1((1 + d)\, q_c)^{1/2} + c_2)\, \varepsilon \|u - u_0\|_{W^{1,2}(\Omega)}\,. \end{aligned}$$

Here we have used that $\mu_N(\Omega_t) \leqq \mu_N(\Omega_t(u_0)) \leqq \varepsilon^2$ by virtue of (4.3). The definition of V and the Schwarz inequality imply

$$\begin{aligned} \|u - u_0\|^4_{W^{1,2}(\Omega)} &\leqq c_3\Big(\int_\Omega |\nabla u - \nabla u_0|^2 \,\mathrm{d}x\Big)^2 \leqq \\ &\leqq c_3\, \mu_N(\Omega) \int_\Omega |\nabla u - \nabla u_0|^4 \,\mathrm{d}x \leqq c_4 \varepsilon \|u - u_0\|_{W^{1,2}(\Omega)}\,. \end{aligned}$$ □

Corollary 4.10. *If u_0 is subsonic then under the assumptions of the above theorem we have $\|u - u_0\|_{W^{1,2}(\Omega)} \leqq C\|g - g_0\|^{1/3}_{L^2(R)}$. That means: the solutions u of (1.6) depend continuously on the boundary data g at g_0.*

The last corollary shows that the subsonic solutions of (1.6) are stable with respect to varying boundary data in the class of all possible solutions from $K \subseteq G(d)$. Let us give some concluding remarks: The assertions of this section can similarly derived for non-convex H (that means: $\varkappa > 2$ for a polytropic gas (1.2)). Thereby the function $\delta(d)$ given in Lemma 3.4 has to be used. Most of our theorems were formulated for solutions of the variational inequality (1.6). It is clear that they are also valid for the solutions of the weak problem (1.3) because each solution of (1.3) from $G(d)$ is a solution of (1.6) with $K = G(d)$. Moreover, for the solutions of (1.3) we are able to prove a modification of Theorem 4.7b).

Theorem 4.11. *Let $u_1, u_2 \in G(d)$ be two solutions of (1.3), and let the boundary $\partial\Omega_t$ of Ω_t, defined by (4.3), be Lipschitz-continuous. If $u_1 = u_2$ on $\partial\Omega_t \cap \Omega$ then $u_1 = u_2$ on $\Omega \setminus \Omega_t$.*

Proof. Let $\Omega' = \Omega \setminus \Omega_t$ and $V' = \{v \in W^{1,2}(\Omega') \mid v = 0$ on $\partial\Omega' \cap (\Omega \cup S_1)$ in trace sense$\}$. For $v \in V'$, extended by 0 outside of Ω', the relation (1.3) together with Remark 2.2 implies

$$\int_{\Omega'} \varrho(|\nabla u_l|^2) \nabla u_l \nabla v \, dx = \int_{R \cap \partial\Omega'} gv \, do \,, \quad l = 1, 2 \,.$$

Since $u_1 - u_2 \in V'$ it is possible to put $v = u_1 - u_2$, and subtraction of the two relations yields $\int_{\Omega'} F(\nabla u_1, \nabla u_2) \, dx = 0$. Recalling that $\Omega' = \Omega_e(u_1) \cup \Omega_e(u_2)$ we can apply Lemma 3.5 on Ω'. From (3.10) we obtain $F(\nabla u_1, \nabla u_2) \geqq b(0) |\nabla u_1 - \nabla u_2|^4$, hence $\nabla(u_1 - u_2) = 0$ a.e. on Ω'. By virtue of $u_1 - u_2 = 0$ on $\partial\Omega' \cap \Omega = \partial\Omega_t \cap \Omega$ we have $u_1 = u_2$ on Ω'. □

Acknowledgement. The paper was written during a stay at Charles University in Prague. Thanks are due to all colleagues in the Mathematical Institute of the Charles University who helped me to prepare it.

References

[1] *M. Feistauer, J. Nečas:* On the solvability of transonic potential flow problems. Z. Anal. Anw. 4 (1985), 305–329.

[2] *M. Feistauer, J. Nečas:* On the solution of transonic flows with weak shocks. Comment. Math. Univ. Carolinae, 27 (1986), 791–804.

[3] *M. Feistauer, J. Nečas:* Viscosity method in a transonic flow. Comm. Partial Differential, Equations (to appear).

[4] *M. Feistauer, J. Nečas:* Remarks on the solvability of transonic flow problems. Manuscr. Math. (to appear).

[5] *D. Gilbarg, N. S. Trudinger:* Elliptic partial differential equations of second order. Springer-Verlag, Berlin, 1977.

[6] *H.-P. Gittel:* Studies on transonic flow problems by nonlinear variational inequalities. Z. Anal. Anw., 6 (1987), 449–458.

[7] *L. D. Landau, E. M. Lifschitz:* Lehrbuch der Theoretischen Physik, Bd. VI: Hydrodynamik. Akademie-Verlag, Berlin, 1966.

[8] *C. Morawetz:* On a weak solution for a transonic flow problem. Comm. Pure Appl. Math., 38 (1985), 797–818.

[9] *J. Nečas, J. Mandel, M. Feistauer:* Entropy regularization of the transonic potential flow problem. Comment. Math. Univ. Carolinae, 25 (1984), 431–443.

Souhrn

O NĚKTERÝCH VLASTNOSTECH ŘEŠENÍ PROBLÉMU TRANSONICKÉHO POTENCIÁLNÍHO PROUDĚNÍ

HANS-PETER GITTEL

V článku se studují řešení problému transonického potenciálního proudění ve slabé formě nebo ve tvaru variační nerovnosti. S použitím zobecněných metod, dobře známých pro eliptické parciální diferenciální rovnice druhého řádu jsou odvozeny některé vlastnosti těchto řešení. Dále je dokázán princip maxima, srovnávací princip a některé jejich důsledky.

Резюме

О НЕКОТОРЫХ СВОЙСТВАХ РЕШЕНИЯ ЗАДАЧИ СВЕРХЗВУКОВОГО ПОТЕНЦИОНАЛЬНОГО ПОТОКА

HANS-PETER GITTEL

В статье изучаются решения задачи сверхзвукового потенциального потока в слабой форме или в форме вариационного неравенства. При помощи обобщенных методов, хорошо известных для эллиптических уравнений в частных производных второго порядка, выведены некоторые свойства этих решений и доказаны принцип максимума, принцип сравнения и некоторые их следствия.

Author's address: Dr. *Hans-Peter Gittel*, Sektion Mathematik der Karl-Marx-Universität, Karl-Marx-Platz 10, Leipzig, DDR 7010.