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FIRST KIND INTEGRAL EQUATIONS FOR THE NUMERICAL SOLUTION OF THE PLANE DIRICHLET PROBLEM*)

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Summary. We present, in a uniform manner, several integral equations of the first kind for the solution of the two-dimensional interior Dirichlet boundary value problem. We apply a general numerical collocation method to the various equations, and thereby we compare the various integral equations, and recommend two of them. We give a survey of the various numerical methods, and present a simple method for the numerical solution of the recommended integral equations.

Keywords: Integral equation of 1st kind, Dirichlet problem, collocation method.

1. INTRODUCTION

Boundary value problems, formulated in terms of partial differential equations and boundary conditions, can be solved using various methods:

1) By means of finite difference methods or finite element methods the boundary value problem is replaced approximately by a system of linear algebraic equations, which is solved numerically.

2) By means of certain analytical methods the boundary value problem is reformulated, without any approximation, as integral equations, which are replaced approximately by a system of linear algebraic equations, which is solved numerically; see the survey papers [C14], [C16], and the references given therein. It is much more common to reformulate as integral equations of the second kind than in terms of first kind integral equations. This difference is without doubt due to the fact, that the analytical and numerical treatment of second kind equations is easier than that for the first kind equations. On the other hand, first kind equations can have some advantages in that their unknowns can be of particular interest, e.g. because of their physical significance, or because they are well suited as input to some further calculations.

*) Partly based on material which were to be presented as an invited lecture at EQUADIFF 6, Brno, Czechoslovakia.

As a model for the boundary value problem we here exclusively consider a two-dimensional interior Dirichlet problem for the Laplace's equation for a simply connected domain D with closed, simple and smooth ***Boundary curve*** Γ^B, where the boundary value f is prescribed

$$\Delta\psi(P) = 0\,, \qquad P \in D\,, \tag{1-1a}$$

$$\psi(P) = f(P)\,, \quad P \in \Gamma^B\,. \tag{1-1b}$$

This system can be used for the solution of two-dimensional boundary value problems arising in various physical and technical applications, e.g., the classical conductor problem, conformal mappings, plane elasticity and viscous flow problems.

As mentioned above (1-1) is more often reformulated as an integral equation of the second kind [J1; § 2.6] than as an equation of the first kind. However, we will here exclusively reformulate (1-1) in terms of *first* kind equations. This is because they actually appear within applications (see, e.g., [H6; Part II] and several references mentioned in [C1], [H5], [L1]), and also because the unknown of the equations have special significance when applied to the problems just mentioned (see, e.g., [H6; § 2.2]).

When we reformulate (1-1) as integral equations of the first kind it turns out that several equations can be obtained, and it is therefore of interest to choose among the various equations. We present (§ 2) the various first kind equations together with results concerning existence and uniqueness of the solution. We apply (§ 3) the same general numerical method to the various integral equations and obtain various systems of linear algebraic equations; by means of the condition number of the corresponding rectangular matrices we carry out a numerical comparison of the various integral equations, and recommend two of them. Finally we present (§ 4) a survey of known methods and a simple method which can be applied to solve the recommended integral equations.

2. THE INTEGRAL EQUATIONS

In principle two types of methods are available for derivation of the first kind integral equations for the problem (1-1), viz. via single layer potentials, i.e., the *indirect* formulation, (§ 2.1) and via *Green's third identity*, i.e., the *direct* formulation, (§ 2.2). Besides the boundary curve Γ^B it is convenient to introduce an ***auxiliary curve*** Γ^A, which is placed *outside* Γ^B, cf. (§ 2.1.2) & (§ 2.2.2).

2.1. Single layer potentials

Solutions to Laplace's equation (1-1a) are expressed as

$$\tilde{\psi}(P) = \omega - \oint_{\Gamma^S} \ln|P - Q|\, \Omega(Q)\, \mathrm{d}s_Q\,, \tag{2-1}$$

where $|P - Q|$ is the distance between the two points P and Q, with Q placed on the *source curve* Γ^S, along which an integration is carried out with respect to the arc length s. The function $\Omega(Q)$ (source density) and the scalar ω are at disposal; when they are determined, see below, a function $\tilde{\psi}$ can be computed, and it may satisfy (1-1).

2.1.1. Without auxiliary curve

In (2-1) we put $\Gamma^S := \Gamma^B$ and $\omega := 0$ and require that $\tilde{\psi}$ satisfy the boundary condition (1-1b) when $P \to \Gamma^B$. This leads to the integral equation [J1; Eq. 4.3.1]

$$-\oint_{\Gamma^B} \ln|P - Q|\, \Omega(Q)\, ds_Q = f(P)\,; \quad P, Q \in \Gamma^B\,, \tag{2-2}$$

with the unknown function $\Omega(Q)$. For certain curves Γ^B the solution $\Omega(Q)$ does not exist, nor is it unique [J1; § 4.3]. When in (2-1) we do not put $\omega := 0$, the equation (2-2) can be replaced by the following system (2-3) (cf. [H3], [S4]), in which we here have introduced the scalars W and K, because they are important for the numerical solution (§ 3),

$$-\oint_{\Gamma^B} \ln|P - Q|\, \Omega(Q)\, ds_Q + K\frac{\omega}{K} = f(P)\,; \quad P, Q \in \Gamma^B\,, \tag{2-3a}$$

$$W\oint_{\Gamma^B} \Omega(Q)\, ds_Q = W\alpha\,; \quad Q \in \Gamma^B\,. \tag{2-3b}$$

Here the unknown quantities are the function $\Omega(Q)$ and the scalar ω, while the known quantities are the function $f(P)$ and the scalar α. When f and α are given then Ω and ω exist and are unique [H3; Th. 3]. When Ω and ω have been determined, they can be inserted in (2-1) and the solution to (1-1) can be computed.

2.1.2. With auxiliary curve

In (2-1) we put $\Gamma^S := \Gamma^A$ and $\omega := 0$ and require that $\tilde{\psi}$ satisfy the boundary condition (1-1b) when $P \to \Gamma^B$. This leads to the integral equation

$$-\oint_{\Gamma^A} \ln|P - Q|\, \Omega(Q)\, ds_Q = f(P)\,; \quad P \in \Gamma^B\,, \quad Q \in \Gamma^A\,, \tag{2-4}$$

with the unknown function $\Omega(Q)$. It can be inserted in (2-1) and a function $\tilde{\psi}$ can be computed. The solution $\Omega(Q)$ may not exist [C10; Appendix A.1, Method *I*] and for certain curves Γ^A it is not unique [C10; Appendix A.2, Method I]. (Despite the deficiences connected with (2-4) it has been used in a number of cases: [B1], [D1], [M1], [M3]; see the references in [C10; § 3.1] for other problems than (1-1).) In

analogy with (2-2) & (2-3) the equation (2-4) can be replaced by the following system (2-5), in which we here have introduced the scalars W and K, because they are important for the numerical solution (§ 3),

$$(2\text{-}5a) \qquad -\oint_{\Gamma^A} \ln|P-Q|\,\Omega(Q)\,ds_Q + K\frac{\omega}{K} = f(P)\,; \quad P\in\Gamma^B\,, \quad Q\in\Gamma^A\,,$$

$$(2\text{-}5b) \qquad W\oint_{\Gamma^A} \Omega(Q)\,ds_Q = W\alpha\,; \quad Q\in\Gamma^A\,.$$

Here the unknown quantities are $\Omega(Q)$ and ω. It may be questioned whether Ω and ω exist and are unique. If Ω and ω have been determined, they can be inserted in (2-1) and a function $\tilde{\psi}$ can be computed.

2.2. Green's third identity

When ψ and $\partial\psi/\partial n_Q$ are both known on Γ^B, where $\partial/\partial n_Q$ denotes differentiation in the direction of the inward normal to Γ^B at the point Q, then *Green's third identity* [J1; § 4.4]

$$-\oint_{\Gamma^B} \frac{\partial}{\partial n_Q}\ln|P-Q|\,\psi(Q)\,ds_Q + \oint_{\Gamma^B} \ln|P-Q|\,\frac{\partial}{\partial n_Q}\psi(Q)\,ds_Q =$$

$$(2\text{-}6a) \qquad = 2\pi\,\psi(P)\,; \quad P \text{ inside } \Gamma^B$$

$$(2\text{-}6b) \qquad = \pi\;\psi(P)\,; \quad P \text{ on } \Gamma^B$$

$$(2\text{-}6c) \qquad = 0\;\psi(P)\,; \quad P \text{ outside } \Gamma^B\,,$$

can be used primarily to compute $\psi(P)$ for P inside Γ^B; for P on Γ^B or P outside Γ^B the value of the integral in (2-6) is still known. In the present case only ψ is known on Γ^B, cf. (1-1b), and $\partial\psi/\partial n_Q$ must be found before (2-6a) can provide the sought value of ψ inside Γ^B.

2.2.1. Without auxiliary curve

In (2-6) we insert the boundary condition (1-1b) and obtain from (2-6b), i.e. for P on Γ^B, an integral equation [J1; Eq. 4.4.4]

$$(2\text{-}7a) \qquad -\oint_{\Gamma^B} \ln|P-Q|\,\Omega(Q)\,ds_Q = \mathcal{T}_B\{f\}\,; \quad P, Q\in\Gamma^B\,,$$

where

$$(2\text{-}7b) \qquad \mathcal{T}_B\{f\} = -\pi f(P) - \oint_{\Gamma^B} \frac{\partial}{\partial n_Q}\ln|P-Q|\,f(Q)\,ds_Q\,; \quad P, Q\in\Gamma^B\,,$$

is an integral transformation of f, where the curve on which P is placed is explicitly

indicated by the index B. The unknown of the integral equation is $\Omega(Q)$. A solution exists because $\partial\psi/\partial n_Q$ is a possible solution. However, $\Omega(Q)$ is not unique for all curves Γ^B (see [C2], [C4], [J1; § 4.5]), but the uniqueness is assured if $\Omega(Q)$ furthermore has to satisfy the following *supplementary* condition (2-7c) (see [C2; Eq. 7], [C4; Eq. 6.9], [J1; § 4.5]), in which we here have introduced the scalar W, because it is important for the numerical solution (§ 3),

$$W \oint_{\Gamma^B} \Omega(Q)\, ds_Q = 0\,; \quad Q \in \Gamma^B\,. \tag{2-7c}$$

The possible solution $\partial\psi/\partial n_Q$ also satisfies this condition.

A term ω, with $\omega = 0$, can be added to the left hand side of (2-7a) whereby we get the following system (2-8), in which we here have introduced the scalars W and K, because they are important for the numerical solution (§ 3),

$$-\oint_{\Gamma^B} \ln|P - Q|\, \Omega(Q)\, ds_Q + K\frac{\omega}{K} = \mathcal{T}_B\{f\}\,; \quad P, Q \in \Gamma^B\,, \tag{2-8a}$$

$$W \oint_{\Gamma^B} \Omega(Q)\, ds_Q = 0\,; \quad Q \in \Gamma^B\,, \tag{2-8b}$$

which has a strong resemblance with (2-3). A solution, Ω and ω, exist, because $\Omega(Q) = \partial\psi/\partial n_Q$ and $\omega = 0$ are possible; this solution is unique. When Ω has been determined, it can be inserted in (2-6a) and the solution to (1-1) can be computed.

2.2.2. With auxiliary curve

In (2-6) we insert the boundary condition (1-1b) and obtain from (2-6c), i.e. for P outside Γ^B, and P placed on Γ^A, an integral equation, called *Kupradze's Functional Equation*, [K1], [K2] and the references in [C5; Footnote 11], [C8], [C10; § 3.2, II 1.1] (see the references in [C10; § 3.2] for other problems than (1-1))

$$-\oint_{\Gamma^B} \ln|P - Q|\, \Omega(Q)\, ds_Q = \mathcal{T}_A\{f\}\,; \quad P \in \Gamma^A\,, \quad Q \in \Gamma^B\,, \tag{2-9a}$$

where

$$\mathcal{T}_A\{f\} = -\oint_{\Gamma^B} \frac{\partial}{\partial n_Q} \ln|P - Q|\, f(Q)\, ds_Q\,; \quad P \in \Gamma^A\,, \quad Q \in \Gamma^B\,, \tag{2-9b}$$

is an integral transformation of f, where the curve on which P is placed is explicitly indicated by the index A. The unknown of the integral equation is $\Omega(Q)$. A solution $\Omega(Q)$ exists because $\partial\psi/\partial n_Q$ is a possible solution. However, $\Omega(Q)$ is not unique for all curves Γ^A, Γ^B (see [C3], [C5]), but the uniqueness is assured if $\Omega(Q)$ furthermore has to satisfy the following *supplementary* condition (2-9c) (see [C3; Eq. 4], [C5;

Eq. 4.10] and [K2; p. 746, Footnote]), in which we here have introduced the scalar W, because it is important for the numerical solution (§ 3),

$$W \oint_{\Gamma^B} \Omega(Q)\, ds_Q = 0\,; \quad Q \in \Gamma^B\,. \tag{2-9c}$$

The possible solution $\Omega(Q) = \partial\psi/\partial n_Q$ also satisfies this condition (2-9c), which is identical with (2-7c), while (2-9a) is different from (2-7a).

A term ω can be added to the left hand side of (2-9a) whereby we get the following system (2-10), in which we here have introduced the scalars W and K, because they are important for the numerical solution (§ 3),

$$-\oint_{\Gamma^B} \ln|P - Q|\, \Omega(Q)\, ds_Q + K\frac{\omega}{K} = \mathscr{T}_A\{f\}\,; \quad P \in \Gamma^A\,, \quad Q \in \Gamma^B\,, \tag{2-10a}$$

$$W \oint_{\Gamma^B} \Omega(Q)\, ds_Q = 0\,; \quad Q \in \Gamma^B\,. \tag{2-10b}$$

This system has some resemblance with (2-8). A solution, Ω and ω, exist, because $\Omega(Q) = \partial\psi/\partial n_Q$ and $\omega = 0$ are possible; this solution is unique. When Ω has been determined, it can be inserted in (2-6a) and the solution to (1-1) can be computed.

3. NUMERICAL COMPARISON

Of the various first kind equations presented (§ 2) we consider here the systems (2-3) (2-5) (2-7) (2-8) (2-9) and (2-10). Except for (2-5) they are known to have a unique solution. For the numerical solution of the systems we present (§ 3.1) a general method which in all cases leads to systems of linear algebraic equations having a rectangular (and sometimes a square) matrix of coefficients. Partly based on the 2-norm condition number derived from the singular values of the corresponding matrix, suitable values for W and K are determined (§ 3.2), and the various systems of integral equations are discussed (§ 3.3), leading to the conclusion that the systems (2-3) and (2-7) are preferable. For these two systems a simpler numerical method is presented (§ 4.2).

3.1. **A general method**

It is common to all the above systems of equations that they contain products of the unknown function Ω and the known functions $\ln|P - Q|$ or 1 integrated along a curve, the *integration curve*, here denoted Γ, with $Q \in \Gamma$, and that the value of the integral is to be evaluated at points on a curve, the *collocation curve*, here denoted $\hat{\Gamma}$, with $P = \hat{Q} \in \hat{\Gamma}$. The curve $\hat{\Gamma}$ may coincide with Γ, or the curve $\hat{\Gamma}$ may be placed inside or outside Γ. The relation among the various curves are (with Γ^A outside Γ^B):

§ 2.1.1.: $\Gamma = \Gamma^B$, $\hat{\Gamma} = \Gamma^B$

§ 2.1.2.: $\Gamma = \Gamma^A$, $\hat{\Gamma} = \Gamma^B$

§ 2.2.1.: $\Gamma = \Gamma^B$, $\hat{\Gamma} = \Gamma^B$

§ 2.2.2.: $\Gamma = \Gamma^B$, $\hat{\Gamma} = \Gamma^A$.

The integral with the factor $\ln|P - Q|$ becomes

$$\tilde{\Psi}(\hat{Q}) = -\int_{\Gamma} \ln|\hat{Q} - Q|\, \Omega(Q)\, ds_Q\,, \quad \begin{cases} \hat{Q} \in \hat{\Gamma} \\ Q \in \Gamma \end{cases} \tag{3-1}$$

with source density $\Omega(Q)$. The value of Ω is sought at the collocation points $\hat{Q} := \hat{Q}_1$, $\hat{Q}_2, \ldots, \hat{Q}_N$, with $\hat{Q}_i \in \hat{\Gamma}$; $i = 1, 2, \ldots, N$. The curve Γ is divided into N small sections Γ_j; $j = 1, 2, \ldots, N$, by the interval points $Q_{1/2}, Q_{3/2}, \ldots, Q_{N-1/2}$ on Γ; i.e., $\Gamma_j = Q_{j-1/2}Q_{j+1/2}$. Within each section Γ_j a nodal point Q_j is chosen. The section Γ_j is approximated by the union of two straight lines $Q_{j-1/2}Q_j$ and $Q_jQ_{j+1/2}$, which is denoted by $\bar{\Gamma}_j$ and has the length $\bar{h}_j$. We assume the unknown to be a constant, denoted by $\bar{\Omega}_j$, along Γ_j. We are therefore led to integrate the kernel $-\ln|\hat{Q}_i - Q|$; $\hat{Q}_i \in \hat{\Gamma}$ with respect to the arc length s along Γ_j, with $Q \in \Gamma_j$. In general this integral cannot be worked out in closed form. Therefore the integration is performed with $\bar{Q}$ along $\bar{\Gamma}_j$ in order to get an approximation

$$\bar{\mathbf{A}}_{ij} := -\int_{\bar{\Gamma}_j} \ln|\hat{Q}_i - \bar{Q}|\, ds_{\bar{Q}}\,, \quad \begin{cases} \hat{Q}_i \in \hat{\Gamma} \\ \bar{Q} \in \bar{\Gamma}_j \end{cases} \begin{cases} i = 1, 2, \ldots, N \\ j = 1, 2, \ldots, N \end{cases}. \tag{3-2}$$

This integral can be expressed in closed form by means of the lengths involved [C17; Appendix B (with corrections)]. The elements $\{\bar{\mathbf{A}}_{ij}\}$ are combined to form the matrix $\bar{\mathbf{A}}$. The element $\bar{\mathbf{A}}_{ij}$ computed from (3-2) may be expressed as $\bar{\mathbf{A}}_{ij} = \bar{h}_j \bar{\mathbf{a}}_{ij}$, with $\bar{\mathbf{a}}_{ij} = -\ln|\hat{Q}_i - \bar{Q}_j^{(i)}|$; $\hat{Q}_i \in \hat{\Gamma}$, $\bar{Q}_j^{(i)} \in \bar{\Gamma}_j$, where $\bar{Q}_j^{(i)}$ is a point whose position depends in general on Q_i. However, the value of $\bar{a}_{ij}$ is not computed using $\bar{Q}_j^{(i)}$, but simply as

$$\bar{\mathbf{a}}_{ij} = \bar{\mathbf{A}}_{ij}(\bar{h}_j)^{-1}\,, \tag{3-3}$$

where $\bar{\mathbf{A}}_{ij}$ is determined from (3-2).

The integrals in (2-2) (2-3a) (2-4) (2-5a) (2-7a) (2-8a) (2-9a) and (2-10a) are approximated by the sums

$$\sum_{j=1}^{N} \bar{\mathbf{A}}_{ij}\bar{\Omega}_j\,, \tag{3-4a}$$

and similarly the integrals in (2-3b) (2-5b) (2-7c) (2-8b) (2-9c) and (2-10b) are approximated by the sum

$$\sum_{j=1}^{N} \bar{h}_j\bar{\Omega}_j\,. \tag{3-4b}$$

Hereby it is possible to replace the various integral equations by systems of linear

algebraic equations with a square or rectangular matrix $\bar{\mathbf{B}}$ and with suitable unknowns:

Equations (2-2) and (2-4):

Matrix

$$\bar{\mathbf{B}} := \bar{\mathbf{A}}\,, \tag{3-5a}$$

unknowns $\{\bar{\Omega}_j\}_1^N$.

Equations (2-7) and (2-9):

Matrix

$$\bar{\mathbf{B}} := \left[\begin{array}{c|c} \multicolumn{2}{c}{\bar{\mathbf{A}}} \\ \hline W\bar{h}_1 \ldots W\bar{h}_N & \end{array}\right], \tag{3-5b}$$

unknowns $\{\bar{\Omega}_j\}_1^N$.

Equations (2-3) (2-5) (2-8) and (2-10):

Matrix

$$\bar{\mathbf{B}} := \left[\begin{array}{c|c} \multicolumn{1}{c}{\bar{\mathbf{A}}} & \begin{matrix} K \\ \vdots \\ K \end{matrix} \\ \hline W\bar{h}_1 \ldots W\bar{h}_N & 0 \end{array}\right], \tag{3-5c}$$

unknowns $\{\bar{\Omega}_j\}_1^N$ and ω/K.

However, it is advantageous to introduce new unknowns

$$\bar{\Omega}_j^* := \bar{\Omega}_j \bar{h}_j \tag{3-6}$$

whereby the matrix $\bar{\mathbf{a}}$, cf. (3-3), is introduced instead of $\bar{\mathbf{A}}$. This replacement corresponds to a column-scaling[1]) of the first N columns of the matrices $\bar{\mathbf{B}}$ in (3-5). Compared with $\bar{\mathbf{A}}$ the matrix $\bar{\mathbf{a}}$ is less non-symmetric because it does not contain the factors $\{\bar{h}_j\}$.

Hereby it is again possible to replace the various integral equations by systems of linear algebraic equations with a square or rectangular matrix $\bar{\mathbf{b}}$ and with suitable unknowns:

Equations (2-2) and (2-4):

Matrix

$$\bar{\mathbf{b}} := \bar{\mathbf{a}}\,, \tag{3-7a}$$

unknowns $\{\bar{\Omega}_j^*\}_1^N$.

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Equations (2-7) and (2-9):

Matrix

$$\bar{\mathfrak{b}} := \left[\begin{array}{c|c} & \bar{\mathfrak{a}} \\ \hline W \ldots W & \end{array}\right], \tag{3-7b}$$

unknowns $\{\bar{\Omega}_j^*\}_1^N$.

Equations (2-3) (2-5) (2-8) and (2-10):

Matrix

$$\bar{\mathfrak{b}} := \left[\begin{array}{c|c} & K \\ \bar{\mathfrak{a}} & \vdots \\ & K \\ \hline W \ldots W & 0 \end{array}\right], \tag{3-7c}$$

unknowns $\{\bar{\Omega}_j^*\}_1^N$ and ω/K.

The computation of the right hand side of the equations (2-7a) (2-8a) (2-9a) and (2-10a) is not discussed here; for (2-9a) and (2-10a), see [C5], [C7], [C8].

3.2. Determination of W and K

The purpose now is to choose among the various systems of integral equations presented above (§ 2), and this is done by carrying out a comparison of the various systems of linear algebraic equations having the matrices (3-5) or (3-7). Before that can be done suitable values of W and K have to be determined. The determination of W and K, and the choice among the various matrices, is performed by applying the concept *condition number*, $\bar{\kappa}$, *of a matrix*, which is defined in terms of the singular values $\{\bar{\sigma}_i\}$ of the matrix, as [Y1; p. 766 & p. 811]

$$\bar{\kappa} = \max_i \{\bar{\sigma}_i\} / \min_i \{\bar{\sigma}_i\} . \tag{3-8}$$

The determination of suitable values of W for (3-5b) and of W and K for (3-5c) is lengthy (in particular for (3-5c)). The results obtained can be found in the following references:

Matrix (3-5b), Equation (2-7): [C11]

Matrix (3-5b), Equation (2-9): [C8]

Matrix (3-5c), Equation (2-3): [C13] [C15] [W2]

Matrix (3-5c), Equation (2-5): [C12]

The analysis for the matrices (3-5b & c) is here carried over to the matrices (3-7b & c) with some modifications and the results obtained are presented below. It is here necessary to apply the concept of the *external conformal radius* of a curve C,

which we denote ECR$\{C\}$, [L2; p. 172]. We introduce

(3-9a, b) $$d = \mathrm{ECR}\{\Gamma\}\,, \qquad \hat{d} = \mathrm{ECR}\{\hat{\Gamma}\}$$

(3-10a, b) $$R = \mathrm{MAX}(d, \hat{d})\,, \quad r = \mathrm{MIN}(d, \hat{d})$$

(3-11) $$\gamma = R/r \geqq 1\,,$$

and we find:

Matrix (3-7a): $\bar{\kappa} = \infty$ for $R \approx 1$, indicating that the corresponding systems of linear algebraic equations do not have a unique solution, because the corresponding integral equations do not have a unique solution.

Matrix (3-7b): For both curves Γ and $\hat{\Gamma}$ a geometrical scaling is performed by multiplying all length by the same positive factor, in order to obtain that $R \approx 1$. Suitable values for W are

(3-12) $$W = \frac{N^{1/2}}{2} \cdot \frac{1}{\nu\gamma^{\nu}}\,; \quad 1 \leqq \nu \leqq \frac{N}{2}$$

leading to a minimal value of $\bar{\kappa}$, which can be approximated (cf. [C15; Appendix A]) by the function

(3-13a) $$\kappa = \frac{N}{2}\gamma^{(N/2)-1} \cdot \mathrm{F}(R, \gamma)$$

where

(3-13b) $$1 \leqq \mathrm{F}(R, \gamma) \leqq \sqrt{(1 + (2\gamma \ln R)^2)}$$

for $R \approx 1$.

Matrix (3-7c): As above, a geometrical scaling is performed, so that $R \approx 1$. Suitable values for W and K are

(3-14a, b) $$W = K\,, \quad W = \frac{N^{1/2}}{2\gamma}$$

leading to a minimal value of $\bar{\kappa}$, which can be approximated (cf. [C15; Appendix A]) by the function

(3-15) $$\kappa = \frac{N}{2}\gamma^{(N/2)-1} \cdot (1 + \gamma|R - 1|)$$

for $R \approx 1$.

Some of the above results, which are stated here for general curves Γ and $\hat{\Gamma}$, are obtained by generalizing results derived for two concentric circles C and $\hat{C}$, with radii c and $\hat{c}$, respectively, in which case $d = c$ and $\hat{d} = \hat{c}$. In particular we have used the fact the integral operator leading to the matrix $\bar{\mathbf{A}}$ has eigenvalues which contain a factor c (cf. [C12; Eq. 3-5]), while the column-scaled matrix $\bar{\mathbf{a}}$ can be considered as being derived from an integral operator with eigenvalues

(3-16a) $$\lambda_0 = -N \ln R\,,$$

(3-16b) $$\lambda_m = \frac{N}{2m} \cdot \frac{1}{\gamma^m}; \quad m = 1, 2, \ldots,$$

where R and γ are defined in (3-10), (3-11).

3.3. Discussion of the various equations

We are now able to choose among the various integral equations, which is done by choosing among the various systems of linear algebraic equations having the matrices (3-5) and (3-7):

1) The matrices (3-5) may be more non-symmetric than (3-7), because they contain the factors $\{\bar{h}_j\}$ and the systems of linear algebraic equations corresponding to (3-7) can probably be solved more accurately than the systems of linear algebraic equations corresponding to (3-5). Therefore (3-5) are left out of consideration.

2) The matrices (3-5a) and (3-7a) have $\bar{\kappa} = \infty$, for a certain geometry characterized by $\mathrm{MAX}(d, \hat{d}) \simeq 1$ (cf. (3-16a) and (3-10a); [C10; § 7]), indicating that the corresponding systems of linear algebraic equations do not have a unique solution. Therefore (3-5a) and (3-7a) are left out of consideration.

3) The matrices (3-5b) and (3-7b) are not square indicating that the corresponding systems of linear algebraic equations are not so simple to solve as if the matrix of the systems were square. Therefore (3-5b) and (3-7b) are left out of consideration.

4) We are now left with the matrix (3-7c), which is derived from integral equations *with* a term ω. They are now to be discussed.

When the curves Γ and $\hat{\Gamma}$ coincide then $-\ln|\hat{Q} - Q| \to \infty$ when $Q \to \hat{Q}$. This unbounded kernel may make it tempting to use non-coinciding curves so that $-\ln|\hat{Q} - Q|$ is bounded, so that standard routines/-methods for handling bounded integrands could be invoked. However, if Γ and $\hat{\Gamma}$ are close to each other, but not coinciding, the function $-\ln|\hat{Q} - Q|$ is "peaked" making it difficult to apply a general purpose integration routine. Therefore it could apparently seem better to have the curves Γ and $\hat{\Gamma}$ well separated. But this separation leads to other difficulties, which arise from the fact that the condition number of the matrices (3-7b) and (3-7c) increases drastically when the two curves are moved away from each other, as can be seen from (3-13) and (3-15), where $\gamma > 1$ indicates separated curves.

When the condition number of a matrix is increased the error of the computed solution of the corresponding system of linear algebraic equations may be increased. When the distance between the two curves is increased it is possible that the error of other parts of the computation will decrease. It is therefore not impossible that an optimal accuracy is obtained for a certain small distance between the curves. However, here we chose to strive at the smallest possible condition number.

Using the criterion that a small condition number is preferable, we can conclude that it is best to use $\gamma = 1$, i.e., to use the equations where Γ and $\hat{\Gamma}$ coincide, i.e., to use the equations where an auxiliary curve Γ^A, outside Γ^B, is *not* used. This means,

e.g., that *Kupradze's Functional Equations* (§ 2.2.2) are *not* to be recommended. Nor are to be recommended the equations based on single layer potentials (§ 2.1.2); this also because uniqueness and existence of the solution may be questioned.

The conclusion of the above reasoning is that we only consider the systems *without* an auxiliary curve Γ^A and *with* an extra scalar unknown ω; i.e., only the systems (2-3) and (2-8). The numerical effort needed for solving the two systems is nearly the same. Therefore the choice between the two systems is primarily based upon which unknown quantities are best suited for the subsequent computation, or which unknown quantities have the right physical significance, cf. § 1.

4. NUMERICAL METHODS

In § 3.3 it is recommended to use such integral equations where the auxiliary curve Γ^A is not used, i.e., $\Gamma = \hat{\Gamma}$. The method of § 3.1 for constructing the matrix $\bar{\mathbf{a}}$ is applicable both when $\Gamma \neq \hat{\Gamma}$ and when $\Gamma = \hat{\Gamma}$. Below we mention some methods for solving the equations with matrices (2-3) and (2-8) in the case when $\Gamma = \hat{\Gamma}$: a survey of some of the methods (§ 4.1) and, as an example, a very simple method (§ 4.2).

4.1. Survey of some known methods

The methods can be divided into (1) collocation methods, (2) Galerkin methods, and (3) other methods. However, within the framework of pseudo-differential operators the variational formulation provides a unifying analysis of Galerkin – and collocation methods. A detailed analysis which pertains to the first kind integral equations in question (and to other equations) is carried out in [A6], [A7], [S1], while [W2] gives a very detailed and thorough survey of boundary integral equations on smooth closed surfaces or curves with numerous references. Methods for solving the various integral equations of potential theory are available: A collection is deposited at VINITI, 1985 [M4]; several are mentioned in [M5], and some of them may apply to the first kind equations considered here.

1) Collocation methods are convenient to use, because of their simplicity (the method of § 3.1 is of this type). They have been used in many cases, and some of the first applications are referred to in [A1], [C1]. A variant of the method, which is very easy to use, has been proposed in [C1] and compared with a popular variant, and found to be the best of the two methods. With some modifications the easy method is used in § 4.2. No genuine analysis of error or convergence was performed in [C1]. Formulas similar to the easy ones in [C1] have been derived in [A1], [A2] where also an analysis of error and convergence is carried out. A collocation method with interpolation is presented and analyzed in [V1], [V2].

2) Galerkin methods can give accurate results at the expence of higher computational cost. They have been used in a number of cases and some of the first applications are referred to in [H4]. The method has been analyzed with respect to error and convergence [H4], stability and optimal choice of the mesh size [H2]. Galerkin methods can be formulated as a Galerkin-Bubnow method (with equal test and trial spaces) [M2], and as a Galerkin-Petrov method (with different test and trial spaces) [A5], [R2]. They can also be formulated without appeal to coercivity leading to a method which can be used to handle corners, cusps, or open arcs [S2], [S3]. Galerkin methods produce systems of linear algebraic equations, which can be analyzed by singular value decomposition, whereby the integral equations can be analyzed [H1].

3) The convenience of the collocation method and the accuracy of the Galerkin method are combined in the Galerkin-collocation method, derived in [H5] and used in several cases in [H6]. The matrix hereby obtained [H6; Appendix] has also the form (3-7c), and the system can be solved by using Gaussian elimination (after the modifications of [C15] have been performed), but the system can also be solved by using iterative techniques after a suitable preconditioning [R1]. Comparison between the collocation method and the Galerkin-collocation method has been carried out [A3], [A4], [H5; p. 126]. Because the integral equations are defined on a closed curve it is possible to derive an accurate and fast numerical scheme using Fourier series [L1].

4.2. A simple collocation method

For the construction of the matrix $\bar{\mathbf{a}}$ a simple method can be derived, cf. [C1], [C15; Eq. 2-7]:

The arc length s in the integrals is replaced by a regular 1-periodic parametric representation, with a parameter t, where t^* and t correspond to P and Q, respectively. The distance ϱ is introduced, and it is defined by $\varrho(t^*, t) := |P - Q|$. New unknown functions are introduced, cf. (3-6),

$$\tilde{\Omega}(t) := \Omega(Q)\frac{\mathrm{d}s}{\mathrm{d}t}, \quad \tilde{\Omega}^*(t) := \frac{1}{N}\tilde{\Omega}(t). \tag{4-1a, b}$$

Using equally spaced collocation- and integration points,

$$t^*: \; t_i := (i-1)/N; \quad i = 1, 2, \ldots, N, \tag{4-2a}$$

$$t: \; t_j := (j-1)/N; \quad j = 1, 2, \ldots, N, \tag{4-2b}$$

we can replace the left hand side of (2-3) and (2-8) by a system of linear algebraic equations having a matrix of the same form as (3-7c). The elements of $\bar{\mathbf{a}}$ are computed as follows, provided N is even,

$$\bar{\mathbf{a}}_{ij} = -\ln \varrho(t_i, t_j); \quad i \neq j \tag{4-3a}$$

(4-3b) $$\bar{\mathbf{a}}_{ii} = -\left(\ln \frac{(\mathrm{d}s/\mathrm{d}t)_i}{2\pi N} + \phi(N)\right)$$

where

(4-3c) $$\phi(N) = \ln\left[\pi N \left(\frac{(N/2)^{N/2}}{(N/2)!}\right)^2\right] + \frac{1}{3N} - N\,.$$

Following [C15] the geometry of the boundary curve Γ^B is scaled so that $d = \mathrm{ECR}\{\Gamma^B\}$ is near one. The values of W and K to be used are, cf. (3-14),

(4-4a, b) $$W = K = \tfrac{1}{2}N^{1/2}$$

giving the condition number $\bar{\kappa}$, which can be approximated (cf. [C15; Appendix A]) by the function

(4-5) $$\kappa = \frac{N}{2}(1 + |d - 1|)\,.$$

5. CONCLUSION

We have investigated several integral equations of the first kind for the solution of the two-dimensional Dirichlet boundary value problem, and we recommend the equations (2-3) and (2-8). For the numerical solution we propose a simple collocation method (§ 4.2).

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Souhrn

INTEGRÁLNÍ ROVNICE PRVNÍHO DRUHU PRO NUMERICKÉ ŘEŠENÍ ROVINNÉHO DIRICHLETOVA PROBLÉMU

SØREN CHRISTIANSEN

Autor uvádí v jednotné formě několik integrálních rovnic prvního druhu pro řešení dvourozměrné vnitřní Dirichletovy okrajové úlohy. Obecná numerická kolokační metoda je aplikována na různé rovnice, při čemž jsou porovnávány různé integrální rovnice a dvě z nich jsou doporučeny. Je podán přehled různých numerických metod a uvedena jednoduchá metoda numerického řešení doporučených integrálních rovnic.

Резюме

ИНТЕГРАЛЬНЫЕ УРАВНЕНИЯ ПЕРВОГО РОДА ДЛЯ ЧИСЛЕННОГО РБШЕНИЯ ЗАДАЧИ ДИРИХЛЕ В ПЛОСКОСТИ

SØREN CHRISTIANSEN

Автор приводит в единой форме несколько интегральных уравнений первого рода для решения двумерной внутренней краевой задачи Дирихле. Общий численный метод коллокации

применяется к различным уравнениям, причем сравниваются различные интегральные уравнения и два из них рекомендуются. Приводится обзор различных численных методов и предлагается простой метод численного решения рекомендованных интегральных уравнений.

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